PACIFIC JOURNAL OF MATHEMATICS
Vol. 26, No. 3, 1968

FULL CO-ORDINALS OF RETS

ALFRED B. MANASTER

Recursive analogues of cardinal and ordinal numbers have been developed by considering only subsets of the natural numbers and considering only one-to-one partial recursive functions as the maps or correspondences between sets. The recursive analogue of a cardinal is called a recursive equivalence type (RET) and that of an ordinal is called a co-ordinal. Using the RETs and the co-ordinals analogues of Cantor's number classes are defined and considered in this paper. The degree of indecomposability of an RET is seen to determine the set of classical ordinals represented in the RET's co-ordinal number class. If the RET is infinite this set of ordinals is always an initial segment (not necessarily proper) of Cantor's second number class.

The basic reference for RETs is Dekker-Myhill [3]. The basic reference for co-ordinals is Crossley [1]. If ξ and η are subsets of $E (E = \{0, 1, 2, \cdots\})$, η is called recursively equivalent to ξ if there exists a one-to-one partial recursive function f whose domain includes ξ such that the f-image of ξ is η. The class of all sets recursively equivalent to ξ is called the recursive equivalence type (RET) of ξ and will be denoted by $\langle\xi\rangle$. If $<_\xi$ is a well-ordering of ξ and $<_\eta$ is a well-ordering of η, then $(\xi, <_\xi)$ is called recursively isotonic to $(\eta, <_\eta)$ if there exists a one-to-one partial recursive function f whose domain includes ξ and such that f is an order isomorphism of $(\xi, <_\xi)$ onto $(\eta, <_\eta)$. The class of all-orderings $(\eta, <_\eta)$ recursively isotonic to $(\xi, <_\xi)$ is called the co-ordinal of $(\xi, <_\xi)$ and will be denoted by $\langle\xi, <_\xi\rangle$. If $Y = \langle\xi, <_\xi\rangle$ is a co-ordinal, the classical order type of $(\xi, <_\xi)$ is a countable ordinal which will be referred to as the order type of Y and denoted $|Y|$ (Cf. Definition IV. 2.1 of Crossley [1]).

Addition is defined for RETs and co-ordinals in the following manner. The subsets ξ_1 and ξ_2 of E are called RE separable if there exists a pair of disjoint recursively enumerable (RE) sets, θ_1 and θ_2, such that $\xi_1 \subseteqq \theta_1$ and $\xi_2 \subseteqq \theta_2$. Assume that ξ_1 and ξ_2 are RE separable, $<_i$ is a well-ordering of ξ_i for $i = 1, 2$, and $X_i = \langle\xi_i\rangle$, $Y_i = \langle\xi_i, <_i\rangle$ for $i = 1, 2$. Then $X_1 + X_2 = \langle\xi_1 \cup \xi_2\rangle$ and

$$Y_1 + Y_2 = \langle\xi_1 \cup \xi_2, <_1 \cup <_2 \cup (\xi_1 \times \xi_2)\rangle .$$

Using the definitions of addition, partial orderings $\leqq$ have been defined in both the RETs and the co-ordinals. For RETs X and X_1 define $X_1 \leqq X$ if and only if there is a RET X_2 such that $X_1 + X_2 = X$. Analogously, for co-ordinals Y_1 and Y define $Y_1 \leqq Y$ if and only if there is a co-ordinal Y_2 such that $Y_1 + Y_2 = Y$.

Let $Y_1 = \langle \xi_1, <_1 \rangle$ and $Y_2 = \langle \xi_2, <_2 \rangle$ be co-ordinals. Y_2 is called an initial segment of Y_1 if $(\xi_2, <_2)$ is recursively isotonic to an initial segment of $(\xi_1, <_1)$. Definition X. 4.3 of Crossley [2] may now be rephrased as follows. A co-ordinal Y is *full* if every initial segment of Y is a predecessor of Y in the sense of $\leqq$. Note that if $(\xi_2, <_2)$ is an initial segment of $(\xi_1, <_1)$, then $\langle \xi_2, <_2 \rangle \leqq \langle \xi_1, <_1 \rangle$ if and only if ξ_2 and $\xi_1 - \xi_2$ are RE separable. Thus $\langle \xi_1, <_1 \rangle$ is full if and only if every initial segment of $(\xi_1, <_1)$ is RE separable from its complement in ξ_1. Example IV. 5.1 of Crossley [1] shows the existence of co-ordinals which are not full. The existence of many full co-ordinals is proved in IV. 5.4 of Crossley [1].

There is a natural sense in which the field of a co-ordinal is an RET. To see this consider $\langle \xi, <_\xi \rangle$; if $(\xi, <_\xi)$ is recursively isotonic to $(\eta, <_\eta)$ then ξ is recursively equivalent to η and $\langle \xi \rangle = \langle \eta \rangle$, conversely if ξ is recursively equivalent to η then there is an ordering $<_\eta$ of η such that $\langle \eta, <_\eta \rangle = \langle \xi, <_\xi \rangle$. This observation justifies the following definition.

DEFINITION. The field of the co-ordinal $Y = \langle \xi, <_\xi \rangle$ is the RET $X = \langle \xi \rangle$.

In this paper we consider the question of determining the order types of full co-ordinals with a given field X. For each RET X let $\mathscr{F}(X)$ be the set of full co-ordinals Y whose field is X. Let

$$\| \mathscr{F}(X) \| = \{ |Y| : Y \in \mathscr{F}(X) \}$$

be the set of order types of full co-ordinals with field X.

It will be shown that for each infinite RET X either $\| \mathscr{F}(X) \| = [\omega, \omega_1)$ where ω_1 is the first uncountable ordinal or there is a countable positive ordinal α and a finite $n > 0$ such that $\| \mathscr{F}(X) \| = [\omega, \omega^\alpha(n+1))$. ($[\alpha, \beta) = \{\gamma : \alpha \leqq \gamma < \beta\}$. c is the cardinality of the continuum.) For each positive ordinal α and each finite $n > 0$ there exist c RETs X such that $\| \mathscr{F}(X) \| = [\omega, \omega^\alpha(n+1))$. It will also be shown that if the RET X is not an isol (See Chapter IV of Dekker-Myhill [3]) then $\| \mathscr{F}(X) \| = [\omega, \omega_1)$.

A hierarchy structure of the RETs similar to that in Manaster [4] will be useful in demonstrating the results stated above. Note that although the terms and the symbols are similar to those in Definition 0.1 of [4], the definition is slightly different.

DEFINITION. $I_0 = \{X : X \text{ is finite}\}$. For each positive countable ordinal α define

$$P_\alpha = \left\{X : X = Y + Z \Rightarrow Y \in \bigcup_{\beta<\alpha} I_\beta \vee Z \in \bigcup_{\beta<\alpha} I_\beta\right\}$$

and

$$I_\alpha = \{X\text{: there is an } n \text{ and RETs } X_1, \cdots, X_n \text{ such that each } X_i \in P_\alpha \text{ and } X = X_1 + \cdots + X_n\} .$$

Elements of P_α are called *α-order indecomposable.* Elements of $\bigcup_{\beta<\alpha} I_\beta$ are called *α-small.* Elements of P_α which are not α-small are called *strictly α-order indecomposable.*

In spite of the difference between this definition and definition 0.1 of [4], the two notions of α-order indecomposability are similar enough that most of the results of [4] are also correct for this definition of α-order indecomposability. If Definition 0.1 of [4] is modified by defining $S_\alpha = \{X: X = 0\}$, then the two definitions of P_α and I_α are the same. Replacing some occurrences of $P_\alpha \cap S_\alpha(P_\alpha - S_\alpha)$ with occurrences of $\bigcup_{\beta<\alpha} I_\beta(P_\alpha - \bigcup_{\beta<\alpha} I_\beta$ respectively), all results of § 1 of [4] remain valid except Lemma 1.1 and Theorem 1.4. In particular P_α is closed under predecessor (Lemma 1.5) so that I_α is the ideal generated by P_α. Moreover the arguments used in Construction I of § 2 and the first part of § 3 (through Theorem 3.2) are still valid under the present interpretation and show the existence of c strictly α-order indecomposable isols for each countable ordinal α.

The main result of this paper is the following theorem.

THEOREM. *Let α be a positive countable ordinal. If X is a sum of n strictly α-order indecomposables, $\| \mathscr{F}(X) \| = [\omega, \omega^\alpha(n + 1))$. If $X \notin \bigcup_{\alpha<\omega_1} I_\alpha$, then $\| \mathscr{F}(X) \| = [\omega, \omega_1)$.*

LEMMA 1. *Let $0 < \alpha < \omega_1$. If X is a sum of n α-order indecomposables and $Y \in \mathscr{F}(X)$, then $| Y | < \omega^\alpha(n + 1)$.*

Proof. The proof is an induction on α. Assume inductively that for $\beta < \alpha$ if Z is a sum of m β-order indecomposables and $T \in \mathscr{F}(Z)$ then $| T | < \omega^\beta(m + 1)$. (Note that the remainder of the proof applies for all $\alpha \geqq 1$.) In particular, if Z is α-small and $T \in \mathscr{F}(Z)$ then $| T | < \omega^\alpha$. Suppose $X = X_1 + \cdots + X_n$ where $n \geqq 1$ and each $X_i \in P_\alpha$. Suppose $Y \in \mathscr{F}(X)$ and $| Y | \geqq \omega^\alpha(n + 1)$. Since Y is full, $Y = Y_1 + \cdots + Y_{n+1}$ where $| Y_i | = \omega^\alpha$ for $1 \leqq i \leqq n$ and $| Y_{n+1} | \geqq \omega^\alpha$. Each Y_i is full since initial segments and tails of full co-ordinals are full. $X = Z_1 + \cdots + Z_{n+1}$ where each Z_i is the field of Y_i. By the inductive hypotheses each Z_i is not α-small since $Y_i \in \mathscr{F}(Z_i)$ and $| Y_i | \geqq \omega^\alpha$. By the refinement property (Theorem 15 (l) of Dekker-Myhill [3]) there exist RETs $X_{i,j}$ satisfying the following system of equations.

$$\begin{array}{ccccccc}
X_{1,1} & + & \cdots & + & X_{1,n+1} & = & X_1 \\
+ & & & & + & & \\
X_{2,1} & + & \cdots & + & X_{2,n+1} & = & X_2 \\
+ & & & & + & & \\
\cdot & & \cdots & & \cdot & & \cdot \quad \cdot \\
\cdot & & \cdots & & \cdot & & \cdot \quad \cdot \\
\cdot & & \cdots & & \cdot & & \cdot \quad \cdot \\
+ & & & & + & & \\
X_{n,1} & + & \cdots & + & X_{n,n+1} & = & X_n \\
\| & & \cdots & & \| & & \\
Z_1 & & \cdots & & Z_{n+1} \,. & &
\end{array}$$

Since each Z_j is not α-small, for each j there is at least one i such that $X_{i,j}$ is not α-small. Since there are $n+1$ columns but only n rows there must be a row, say row i, in which there are at least two terms, $X_{i,j}$ and $X_{i,k}$, neither of which is α-small. Thus $X_i \notin P_\alpha$. Contradiction.

Lemma 1 shows that if X is a sum of n α-order indecomposables, then $\| \mathscr{F}(X) \| \subseteqq [\omega, \omega^\alpha(n+1))$. The next sequence of lemmas lead to the converse inclusion.

LEMMA 2. *If X is not α-small and β is any ordinal less than α, then there exist X_1 and X_2 such that $X = X_1 + X_2$, X_1 is not β-small, and X_2 is not α-small.*

Proof. Suppose X is not α-small. Since, in particular, $X \notin P_\beta$, there exist X_1 and X_2 such that $X = X_1 + X_2$ and neither X_1 nor X_2 is β-small. Not both X_1 and X_2 can be α-small.

LEMMA 3. *If X is not α-small, then there is a $Y \in \mathscr{F}(X)$ of order type ω^α.*

Proof. The proof is an induction on α. The base step, $\alpha = 1$, is easy since every infinite RET is the field of full co-ordinals of type ω. Let $1 < \alpha < \omega_1$. Let $\{\beta_i\}_{i<\omega}$ be a sequence of ordinals such that $1 \leqq \beta_0 \leqq \beta_1 \leqq \beta_2 \leqq \cdots < \alpha$ and such that for every $\beta < \alpha$ there is an i such that $\beta \leqq \beta_i$. (If $\alpha = \gamma + 1$, let each $\beta_i = \gamma$.)

Since X is not α-small and $\beta_0 < \alpha$, by Lemma 2 there exist X_0 and Z_0 such that $X = X_0 + Z_0$, X_0 is not β-small, and Z_0 is not α-small. Inductively for each n, $X = X_0 + \cdots + X_n + Z_n$ where Z_n is not α-small. Since $\beta_{n+1} < \alpha$ by Lemma 2 there exist X_{n+1} and Z_{n+1} such that $Z_n = X_{n+1} + Z_{n+1}$, X_{n+1} is not β_{n+1}-small and Z_{n+1} is not α-small. Thus

$$X = X_0 + \cdots + X_n + X_{n+1} + Z_{n+1}$$

where each X_i is not β_i-small and Z_{n+1} is not α-small.

Since $\beta_n < \alpha$ the inductive hypothesis asserts the existence of a full co-ordinal Y_n of order type ω^{β_n} with field X_n. Unfortunately the ordinal sum of the Y_n is not well defined and even if it were it would not, in general, be a co-ordinal with field X. However, it would be a full co-ordinal of type ω^α. To remove these difficulties, it seems necessary to work with a representative of X.

Let $\langle\xi\rangle = X$. For each n let ξ_n be a representative of X such that $\bigcup_n \xi_n \subseteqq \xi$ and $\bigcup_{i=1}^n \xi_i$ is RE separable from $\xi - \bigcup_{i=1}^n \xi_i$. Define

$$\xi'_n = \xi_n \bigcup \left(\{n\} \cap \xi \cap \overline{\bigcup_{m<n} \xi_m}\right)$$

and $X'_n = \langle\xi'_n\rangle$. For any RET Z both Z and $Z + 1$ have the same order of indecomposability, if any. Thus X'_n is not β_n-small. Now apply the inductive hypothesis to obtain full co-ordinals Y'_n of order type ω^{β_n} with field X'_n. Let $Y'_n = \langle\xi'_n, <_n\rangle$ and define

$$Y = \left\langle \bigcup_{n<\omega} \xi'_n, \bigcup_{n<\omega} <_n \cup \bigcup_{m<n} (\xi'_m \times \xi'_n) \right\rangle .$$

Y is a full co-ordinal of type $\sum \omega^{\beta_n} = \omega^\alpha$ with field X.

LEMMA 4. *If* $Y \in \mathscr{F}(X), |Y| = \alpha$, *and* $\omega \leqq \beta < \alpha$, *then there is a* $Y' \in \mathscr{F}(X)$ *such that* $|Y'| = \beta$.

Proof. First consider the case in which $\alpha - \beta$ is finite. $Y = Y_1 + Y_2$ where $|Y_1| = \beta$ and $|Y_2| = \alpha - \beta < \omega$. $Y' = Y_2 + Y_1$ is then a full co-ordinal of type $(\alpha - \beta) + \beta = \beta$ with field X.

Next consider the case in which $\alpha - \beta \geqq \omega$ and $\omega^2 \leqq \beta$. $Y = Y_1 + Y_2$ where $|Y_1| = \beta$ and $|Y_2| = \alpha - \beta \geqq \omega$. Let Y'_2 be a (necessarily full) ordering of type ω of the field of Y_2. $Y' = Y'_2 + Y_1$ satisfies the conclution of the lemma.

Finally consider the case in which $\alpha - \beta \geqq \omega$ and $\omega^2 > \beta$. Observe that for every $k < \omega$, any co-ordinal of type $\omega + k$ is full. Let $\beta = \omega n + k$ where n and k are finite. If $n = 1$ the field of Y has full co-ordinals of type β by the observation just made. Otherwise let $n = m + 1$ where $m \geqq 1$. $Y = Y_1 + Y_2$ where $|Y_1| = \omega m$ and

$$|Y_2| = \alpha - \omega m \geqq \alpha - (\omega n + k) = \alpha - \beta \geqq \omega .$$

Let Y'_2 be a full co-ordinal of type $\omega + k$ with the same field as the field of Y_2. $Y = Y_1 + Y'_2$ is a full co-ordinal of type $\omega m + \omega + k = \beta$ with field X.

LEMMA 5. *If X is not α-small then* $[\omega, \omega^\alpha] \subseteqq ||\mathscr{F}(X)||$.

Proof. Apply Lemmas 3 and 4.

Proof of theorem. First suppose X is not α-small for any countable ordinal α. $\omega^\alpha \in ||\mathscr{F}(X)||$ for each countable α by Lemma 3. Thus $||\mathscr{F}(X)|| = [\omega, \omega_1)$ by Lemma 4.

To prove that if X is a sum of n strictly α-order indecomposables than $||\mathscr{F}(X)|| = [\omega, \omega^\alpha(n + 1))$, consider first the case $n = 1$. Since X is not α-small there is, by Lemma 3, a full co-ordinal Y with field X of order type ω^α. Let $\omega^\alpha \leqq \beta < \omega^\alpha \cdot 2$. For some $\gamma < \omega^\alpha$, $\beta = \omega^\alpha + \gamma$. Let $Y = Y_1 + Y_2$ where $|Y_1| = \gamma$ and $|Y_2| = \omega^\alpha - \gamma = \omega^\alpha$. $Y' = Y_2 + Y_1$ is a full co-ordinal with field X of type $|Y_2| + |Y_1| = \omega^\alpha + \gamma = \beta$. This result, Lemma 5, and Lemma 1 show $||\mathscr{F}(X)|| = [\omega, \omega^\alpha \cdot 2)$.

Finally consider the case in which $X = X_1 + \cdots + X_n$, $n \geqq 2$, and each X_i is strictly α-order indecomposable. Let

$$\beta \in [\omega^\alpha \cdot n, \omega^\alpha(n + 1)) .$$

For some $\gamma < \omega^\alpha$, $\beta = \omega^\alpha \cdot n + \gamma$. Let Y_i be a full co-ordinal with field X_i of order type ω^α for each $i < n$. Let Y_n be a full co-ordinal of order type $\omega^\alpha + \gamma$ with field X_n. $Y = Y_1 + \cdots + Y_n$ is a full co-ordinal of type β with field X. This result, Lemma 4, and Lemma 1 show $||\mathscr{F}(X)|| = [\omega, \omega^\alpha(n + 1))$.

It remains to show that if the RET X is not an isol then $||\mathscr{F}(X)|| = [\omega, \omega_1)$ and show the existence of c isols such that $||\mathscr{F}(X)|| = [\omega, \omega_1)$. For the former result it suffices to show that if X is not an isol then $X \notin \bigcup_{\alpha<\omega_1} I_\alpha$ and hence it suffices to show that $X \notin \bigcup_{\alpha<\omega_1} P_\alpha$. Let α be the least ordinal β such that some non-isol X is β-order indecomposable. $X = X_1 + X_2$ where neither X_1 nor X_2 is an isol but one of them, say X_1, is a finite sum of β-order indecomposables for some $\beta < \alpha$. Since $\beta < \alpha$ every β-order indecomposable is an isol. Since every finite sum of isols is an isol, X_1 must be an isol. This contradiction shows that every nonisol is not in $\bigcup_{\alpha<\omega_1} I_\alpha$.

Examples of isols X such that $||\mathscr{F}(X)|| = [\omega, \omega_1)$ are provided by first-order highly decomposable isols in the sense of Manaster [4]. It will be shown, as in the preceding paragraph, that if X is first-order highly decomposable then $X \notin \bigcup_{\alpha<\omega_1} P_\alpha$. Let α be the least ordinal β such that some first-order highly decomposable X is β-order indecomposable. $X = X_1 + X_2$ where both X_1 and X_2 are infinite but one, say X_1, is a finite sum of β-order indecomposables for some $\beta < \alpha$. Since X_1 is infinite, there is an infinite $X_3 \leqq X_1$ such that X_3 is β-order indecomposable. Since $X_3 \leqq X_1 \leqq X$, X_3 is highly decomposable contradicting the minimality of α. The existence of c first-order highly decomposable isols is shown in Dekker-Myhill [3, pp. 112–113] and Manaster [4]. Thus there are c isols X such that $\|\mathscr{F}(X)\| = [\omega, \omega_1)$.

REFERENCES

1. J. N. Crossley, *Constructive order types, I,* in Formal Systems and Recursive Functions, Proc. of the Eighth Logic Colloquium, Oxford 1963, Ed. by J. N. Crossley and M. A. E. Dummett, Amsterdam (1965), 189-264.

2. ———, *Constructive order types, II,* Journal of Symbolic Logic **31** (1966), 525-538.

3. J. C. E. Dekker and J. Myhill, *Recursive equivalence types,* University of California Publications in Mathematics (New Series) **3** (1960), 67-214.

4. A. B. Manaster, *Higher-order indecomposable isols,* Trans. Amer. Math. Soc. **125** (1966), 363-383.

Received May 23, 1967.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY