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A REMARK ON INTEGRAL FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Let R_ν, $\nu = \mathrm{I, II, III, IV}$, be the 4 types of the classical Cartan domains and let $\mathscr{E}(R_\nu)$ denote the class of solutions u of the Laplace's equation $\Delta u = 0$ corresponding to the Bergman metric of R_ν which satisfy certain regularity conditions specified below.

In this note we give a distortion theorem for functions which are holomorphic in $\bar{R}_\nu$ and omit the value 0 there, and an application which leads to an interesting property of integral functions omitting the value 0. The tools used here are the generalized Harnack inequality for functions in the class $\mathscr{E}(R_\nu)$ and the classical theorem of Liouville for integral functions.

Let D be a bounded domain in the space C^p of p complex variables $z = (z^1, \cdots, z^p)$. The Laplace-Beltrami operator corresponding to the Bergman metric of D is

$$(1) \qquad \Delta_D = T^{\alpha\bar{\beta}}\partial^2/\partial z^\alpha \partial \bar{z}^\beta \; ;$$

here $T^{\alpha\bar{\beta}}$ are the contravariant components of the metric tensor $T_{\alpha\bar{\beta}} = \partial^2 \log K_D/\partial z^\alpha \partial \bar{z}^\beta$ and $K_D = K_D(z, \bar{z})$ is the Bergman kernel function of D[1]. Let $\mathscr{E}(D)$ be the class of real functions u satisfying: (a) u is continuous in $\bar{D}$. (b) In $\bar{D} - \boldsymbol{b}(D)$, u is of C^2 and satisfies $\Delta_D u = 0$, where $\boldsymbol{b}(D)$ is the Bergman-Šilov boundary of D. It is well-known that the class $\mathscr{E}(D)$ solves the Dirichlet problems for certain types of bounded symmetric domains D ([3], [4]). These are the classical Cartan domains. Let z be a matrix of complex entries, z' its transpose, z^* its conjugate transpose and I the identity matrix. By $H > 0$ we mean that a hermitian matrix H is positive definite. The first 3 types are defined by $R_\nu = [z: I - zz^* > 0]$, $\nu = \mathrm{I, II, III}$, where z is an $m \times n$ matrix ($m \leqq n$) for R_{I}, an $n \times n$ symmetric matrix for R_{II} and an $n \times n$ skew symmetric matrix for R_{III}. The fourth type R_{IV} is the set of all $1 \times n$ matrices satisfying the conditions:

$$1 + |zz'|^2 - 2zz^* > 0, \, |zz'| < 1 \; ,$$

or

$$1 > \bar{z}z' + [(\bar{z}z')^2 - |zz'|^2]^{1/2} \; .$$

By $\|z\|_\nu$ we denote the norm of the matrix $z \in R_\nu$, i.e., $\|z\|_\nu = \sup_{|x|=1} |zx|$, where x is an n-dimensional vector and $|x|$ the length of x. It can be shown that $\|z\|_\nu$ is the largest among the positive square roots of the characteristic roots of the hermitian matrix zz^*, and $R_\nu = [z: \|z\|_\nu < 1]$ ([2]). For any $r > 0$ we write

$$R_\nu(r) = [z: r^2 I - zz^* > 0] = [z: \|z\|_\nu < r] .$$

2. Distortion theorems. A generalization of Harnack's inequality to functions of the class $\mathscr{E}$ for the classical Cartan domains has been obtained in [6] and it is contained in the following lemma.

LEMMA 1. *If $u \in \mathscr{E}(R_\nu(r))$ is nonnegative on $\boldsymbol{b}(R_\nu(r))$ then on $R_\nu(r)$*

$$(2) \quad u(0)Q_\nu(r, z) \leqq u(z) \leqq u(0)Q_\nu(r, z)^{-1},\ Q_\nu(r, z) = \prod_{k=1}^{n_\nu} \left(\frac{r - \lambda_k}{r + \lambda_k}\right)^{N_\nu} ,$$

where

$$n_{\mathrm{I}} = m,\ n_{\mathrm{II}} = n,\ n_{\mathrm{III}} = [n/2],\ n_{\mathrm{IV}} = 2\ ;$$
$$N_{\mathrm{I}} = n,\ N_{\mathrm{II}} = (n + 1)/2,\ N_{\mathrm{III}} = n - 1$$

if n is even and $= n$ if n is odd, $N_{\mathrm{IV}} = n/2$; $\lambda_1, \lambda_2, \cdots, \lambda_{n_\nu}$ are the nonnegative square roots of the characteristic roots of the hermitian matrix zz^* for $z \in R_\nu(r)$, and $r > \lambda_1 \geqq \cdots \geqq \lambda_{n_\nu} \geqq 0$.

We remark that n_ν is the rank of the domain R_ν, and $p_\nu = n_\nu N_\nu$ gives the (complex) dimension of R_ν.

A simple application of the above lemma leads to the following distortion theorem for holomorphic functions.

THEOREM 1. *Let $f(z)$ be a holomorphic function in $\overline{R_\nu(r)}$ which omits there the value 0. Then on $R_\nu(r)$*

$$(3) \quad |f(0)|^{Q_\nu(r,z)} m_\nu(r, f)^{1-Q_\nu(r,z)} \leqq |f(z)| \leqq |f(0)|^{Q_\nu(r,z)} M_\nu(r, f)^{1-Q_\nu(r,z)}$$

where $m_\nu(r, f) = \min_{\|z\|_\nu = r} |f(z)|$, $M_\nu(r, f) = \max_{\|z\|_\nu = r} |f(z)|$ and $Q_\nu(r, z)$ is given in Lemma 1.

Proof. Since $f(z)$ is holomorphic and omits the value 0 in $\overline{R_\nu(r)}$ the maximum principle of a holomorphic function yields:

$$m_\nu(r, f) \leqq |f(z)| \leqq M_\nu(r, f),\ z \in \overline{R_\nu(r)} .$$

Let $g_1(z) = f(z)/m_\nu(r, f)$ and $g_2(z) = M_\nu(r, f)/f(z)$. Since $m_\nu(r, f) \neq 0$ $g_k(z)$ is holomorphic in $\overline{R_\nu(r)}$ and $|g_k(z)| \geqq 1$ in $\overline{R_\nu(r)}$. Therefore, $u_k(z) = \log |g_k(z)|$ belongs to $\mathscr{E}(R_\nu(r))$ and satisfies all the hypotheses of Lemma 1. Applying the first inequality of (2) to $u_1(z)$ and the second inequality to $u_2(z)$ we have inequalities (3).

Specializing Theorem 1 to the hypersphere $H(r) = [z: |z| < r]$, $|z|^2 = |z^1|^2 + \cdots + |z^n|^2$, which can be obtained from $R_1(r)$ by taking $m = 1$, we obtain

COROLLARY 1. *Let $f(z)$ be a function which is holomorphic in $H(r)$ and continuous in $\overline{H(r)}$. If $f(z)$ omits the value 0 on $H(r)$ then on $H(r)$.*

$$(4) \qquad |f(0)|^{Q(r,z)} m(r, f)^{1-Q(r,z)} \leqq |f(z)| \leqq |f(0)|^{Q(r,z)} M(r, f)^{1-Q(r,z)},$$

where $m(r, f) = \min_{|z|=r} |f(z)|$, $M(r, f) = \max_{|z|=r} |f(z)|$ and $Q(r, z) = (r - |z|)^n/(r + |z|)^n$.

A slight modification of the above theroem is the following.

THEOREM 2. *Let $f(z)$ be a holomorphic function in $R_\nu(r)$ which omits there the value 0. Then for any $\delta > 0$*

$$(5) \qquad [|f(0)|\, m_\nu(r, f)^\delta]^{1/(1+\delta)} \leqq |f(z)| \leqq [|f(0)|\, M_\nu(r, f)^\delta]^{1/(1+\delta)}$$

holds for all $z \in R_\nu(r_\nu)$, where

$$r_\nu = \frac{t_\nu - 1}{t_\nu + 1} r, \; t_\nu = (1 + \delta)^{p_\nu^{-1}} .$$

Proof. For any $\delta > 0$ $f(z)$ is holomorphic in $\overline{R_\nu(r_\nu)}$ and omits the value 0. By Theorem 1,

$$(6) \quad |f(0)|^{Q_\nu(r_\nu,z)} m_\nu(r_\nu, f)^{1-Q_\nu(r_\nu,z)} \leqq |f(z)| \leqq |f(0)|^{Q_\nu(r_\nu,z)} M_\nu(r_\nu, f)^{1-Q_\nu(r_\nu,z)}$$

for $z \in R_\nu(r_\nu)$. Let $\delta_0 > 0$ be fixed arbitrarily. Since $r_\nu(\delta) \rightarrow r$ as $\delta \rightarrow \infty$, we have

$$(7) \quad |f(0)|^{Q_\nu(r,z)} m_\nu(r, f)^{1-Q_\nu(r,z)} \leqq |f(z)| \leqq |f(0)|^{Q_\nu(r,z)} M_\nu(r, f)^{1-Q_\nu(r,z)}$$

for $z \in R_\nu(r_\nu^0)$, $r_\nu^0 = (t_\nu^0 - 1)r/(t_\nu^0 + 1)$, $t_\nu^0 = (1 + \delta_0)^{p_\nu^{-1}}$. On the other hand, if $z \in R_\nu(r_\nu^0)$ then $\|z\|_\nu < r_\nu^0$ or $\{(r - \|z\|_\nu)/(r + \|z\|_\nu)\}^{p_\nu} > 1/(1 + \delta_0)$. Since $\|z\|_\nu \geqq \lambda_k$, $k = 1, \cdots, n_\nu$, $Q_\nu(r, z) > 1/(1 + \delta_0)$. Combining this with (7) and the inequalities: $m_\nu(r, f) \leqq |f(z)| \leqq M_\nu(r, f)$, we obtain the theorem.

3. Main theorem.

The following lemma is a simple application of Theorem 2.

LEMMA 2. *Let $\{f_k\}$ be a sequence of holomorphic functions in $R_\nu(r)$ such that f_k omits the value 0 there. Suppose that for some $\delta > 0$ there exists an $A > 0$ such that*

(8) $$|f_k(0)|\, M_\nu(r, f_k)^\delta \leqq A,\ k = 1, 2, \cdots .$$

Then for $\|z\|_\nu \leqq (t_\nu - 1)r/(t_\nu + 1)$, $t_\nu = (1+\delta)^{p_\nu^{-1}}$,

(9) $$|f_k(z)| \leqq A^{(1+\delta)^{-1}},\ k = 1, 2, \cdots .$$

We observe that the hypothesis that each f_k omits the value 0 is essential for the validity of Lemma 2, as is shown by the following example in C^2. Let

(10) $$f_k(z^1, z^2) = k(z^1 + z^2 + 1/k^2),\ k = 1, 2, \cdots$$

be a sequence of holomorphic functions in the unit hypersphere H. A formal computation shows that

$$M(1, f_k) = [(3k + 1/k)(k + 1/k)]^{1/2},\ f_k(0) = 1/k .$$

For $\delta = 1$ we find $A = 8^{1/2}$. But no (z^1, z^2) with $|z^1|^2 + |z^2|^2 < 1$ satisfies (9).

Using Lemma 2 and the classical theorem of Liouville on integral functions we prove:

THEOREM 3. *Let f be an integral function in the space C^{p_ν} omitting the value 0, where $p_\nu = mn, n(n+1)/2, n(n-1)/2, n$ if $\nu =$ I, II, III, IV, respectively. If there exists a $\delta > 0$ and a monotonically increasing $\{s_k\}$ of positive numbers without bound such that for*

$$\tau > 2(1+\delta)^{p_\nu^{-1}}/((1+\delta)^{p_\nu^{-1}} - 1)$$

(11) $$\lim_{k\to\infty} m_\nu(s_k, f) M_\nu(\tau s_k, f)^\delta < \infty ,$$

then f is constant.

Proof. Let z_k be a point on $\|z\|_\nu = s_k$ such that

(12) $$\zeta = \zeta_k(z) = (z - z_k)/(\tau - 1)s_k,\ k = 1, 2, \cdots .$$

Then (12) defines a biholomorphic mapping of C^{p_ν} for each $\delta > 0$. Hence, $g_k(\zeta) = f[\zeta_k^{-1}(\zeta)]$ is again an integral function in C^{p_ν} which omits the value 0. Further, the set $[z\colon \|z - z_k\|_\nu < s_k(\tau - 1)]$ is contained in $R_\nu(\tau s_k)$, and hence,

$$M_\nu(1, g_k) \leqq M_\nu(\tau s_k, f),\ k = 1, 2, \cdots ,$$

Since $|g_k(0)| = |f(z_k)| = m_\nu(s_k, f)$, from (11) we have

$$\lim_{k\to\infty} |g_k(0)|\, M_\nu(1, g_k)^\delta < \infty .$$

Hence there exists a number $A > 0$ such that

$$|g_k(0)|\, M_\nu(1, g_k)^\delta \leqq A,\ k = 1, 2, \cdots .$$

By Lemma 2,

$$|g_k(\zeta)| \leqq A^{(1+\delta)^{-1}},\ k = 1, 2, \cdots$$

for all

$$\zeta \in R_\nu\left(\frac{t_\nu - 1}{t_\nu + 1}\right),\ t_\nu = (1 + \delta)^{p_\nu^{-1}} .$$

This together with (12) implies that $f(z)$ is bounded by $A^{(1+\delta)^{-1}}$ in $R_\nu(s_k\sigma_\nu)$ for each k, where $\sigma_\nu(\delta) = (t_\nu - 1)(\tau - 1)/(t_\nu + 1) - 1$. Since $\sigma_\nu(\delta) > 0$ for $\tau > 2t_\nu/(t_\nu - 1)$, $\{R_\nu(s_k\sigma_\nu)\}$ covers the entire space C^{p_ν}. The theorem now follows from the theorem of Liouville.

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