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IDEAL EXTENSIONS OF SEMIGROUPS

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An ideal extension (here called an extension) of a semigroup S by a semigroup with zero Q is a semigroup V such that S is an ideal of V and the Rees quotient semigroup V/S is isomorphic to Q. To study the structure of these extensions, special kinds of extensions are introduced, called strict and pure extensions. It is proved that any extension of S is a pure extension of a strict extension of S; also, if Q has no proper nonzero ideals, any extension of S by Q is either strict or pure. Dense extensions, closely related to Ljapin's "densely embedded ideals", are special cases of pure extensions. When S is weakly reductive, constructions of strict, pure, and arbitrary extensions of S are given, including descriptions of the ramification function.

Extensions were first systematically studied by Clifford [1] who gave the first general structure theorem in the case when S is weakly reductive (Theorem 4.21 of [2]) (later extended to arbitrary S by Yoshida [7]). In this theorem the multiplication in the extension V of S by Q is described in terms of the action of V on S and a ramification function. Our structure theorems are a refinement of this in that the ramification function is not used explicitly, or, equivalently, is described in terms of other functions. Our methods are not essentially new; except in § 3, we use exclusively the action of the extension V on S, this gives rise also to the notions of strict and pure extensions: the extension V of S is strict if every element of $V-S$ has same action on S as some element of S, pure if no element of $V-S$ has this property.

In the introductory § 1 we establish some preliminary results concerning extensions of an arbitrary semigroup S using the translational hull of S and introduce the notion of the type of an extension. This material is used in § 2 where we introduce strict and pure extensions, study their main properties and construct them in the case when S is weakly reductive. In § 3 we interpret some results of § 2 by means of congruences on an extension V of S whose restriction to S is the equality relation on S. This is particularly suitable for the study of dense extensions and leads in particular to another proof of a theorem of Gluskin concerning dense embeddings. Section 4 contains our main results; we construct all extensions of a weakly reductive semigroup S by an arbitrary semigroup with zero Q and establish when two such extensions are equivalent in the sense of Clifford.

Some of our results appeared previously in [4], [5]; we refer to these for proofs or give a different (and simpler) proof.

Definitions and notation. Throughout, S will denote an arbitrary semigroup unless stated otherwise. An *oversemigroup* of S is a semigroup containing S as a subsemigroup. If V is an extension of S, a *subextension* of V is a subsemigroup of V containing S; an *overextension* of V is an oversemigroup of V in which S is an ideal. If V and V' are extensions of S, an *S-homomorphism* of V into V' is a homomorphism of V into V' which leaves every element of S fixed; if for instance V is a subextension of V', then the canonical injection ι of V into V' (defined by: $\iota(a) = a$ for all $a \in V$) is an S-homomorphism.

We denote by $\Omega(S)$ the translational hull of S, i.e. the set of all pairs $\omega = (\lambda, \rho)$ of mappings (linked translations) of S into S such that

$$(\lambda x)y = \lambda(xy),\ x(y\rho) = (xy)\rho,\ x(\lambda y) = (x\rho)y$$

for all $x, y \in S$. The multiplication $(\lambda, \rho)\cdot(\lambda', \rho') = (\lambda\lambda', \rho\rho')$, where $(\lambda\lambda')x = \lambda(\lambda' x)$ and $x(\rho\rho') = (x\rho)\rho'$ for all $x \in S$, makes $\Omega(S)$ a semigroup. There is a canonical homomorphism π of S into $\Omega(S)$, which associates to each $a \in S$ the pair $\pi_a = (\lambda_a, \rho_a)$, where $\lambda_a x = ax$ and $x\rho_a = xa$ for all $x \in S$. The image $\Pi(S) = \{\pi_a;\ a \in S\}$ of π is an ideal of $\Omega(S)$ in view of the formulae

$$(\lambda, \rho)\cdot\pi_x = \pi_{\lambda x},\ \pi_x\cdot(\lambda, \rho) = \pi_{x\rho}$$

which hold for all $x \in S$ and $(\lambda, \rho) \in \Omega(S)$, as is readily verified. We shall also consider π as a homomorphism of S onto $\Pi(S)$. If π is one-to-one, S is *weakly reductive*. We call S *globally idempotent* if $S^2 = S$.

If $\mathscr{C}$ is a congruence on a semigroup V and if $A \subseteqq V$, we denote by $\mathscr{C}(A)$ the union of all classes of $\mathscr{C}$ which intersect A. We say that A is *saturated* for $\mathscr{C}$ if $\mathscr{C}(A) = A$, i.e. if A is a union of classes of $\mathscr{C}$. We write $x \mathscr{C} y$ if x and y are in the same class of $\mathscr{C}$.

The reader is referred to [2] for all concepts not defined in the paper.

1. Type of an extension.

DEFINITION 1.1. If V is an extension of S, define, for each $a \in V$, λ^a, ρ^a, τ^a by

(1) $$\tau^a = (\lambda^a, \rho^a),\ \lambda^a x = ax,\ x\rho^a = xa \qquad \text{for all } x \in S\,.$$

THEOREM 1.2. *If V is an extension of S, the mapping $\tau : a \rightsquigarrow \tau^a$*

defined by (1) *is a canonical homomorphism of V into $\Omega(S)$ (denoted by $\tau(V: S)$ if there is danger of confusion). Furthermore τ extends the canonical homomorphism π of S into $\Omega(S)$.*

The reader is referred to [4] for the proof, which is in fact quite simple.

PROPOSITION 1.3. If S is weakly reductive, or globally idempotent, then $\tau(V : S)$ is the unique homomorphism of V into $\Omega(S)$ extending π.

Proof. Let ω be another homomorphism of V into $\Omega(S)$ such that $\omega(a) = \omega^a = \pi_a$ for all $a \in S$. Then

$$\omega^a \cdot \pi_x = \omega^a \cdot \omega^x = \omega^{ax} = \pi_{ax} = \tau^\alpha \cdot \pi_x$$

and dually $\pi_x \cdot \omega^a = \pi_x \cdot \tau^a$ for all $a \in V$, $x \in S$. The result now follows from the following:

LEMMA 1.4. *If S is weakly reductive, or globally idempotent, and if $\omega, \omega' \in \Omega(S)$ are such that $\omega \cdot \pi_x = \omega' \cdot \pi_x$ and $\pi_x \cdot \omega = \pi_x \cdot \omega'$ for all $x \in S$, then $\omega = \omega'$.*

Proof. Set $\omega = (\lambda, \rho)$, $\omega' = (\lambda', \rho')$; then

$$\lambda(xy) = \lambda\lambda_x y = \lambda'\lambda_x y = \lambda'(xy) ,$$
$$y(\lambda x) = (y\rho)x = y\rho\rho_x = y\rho'\rho_x = y(\lambda' x) ,$$

and dually $(yx)\rho' = (yx)\rho'$, $(x\rho)y = (x\rho')y$, for all $x, y \in S$. If $S = S^2$, the first and third equations imply $\lambda = \lambda'$, $\rho = \rho'$; if S is weakly reductive, all four equations imply $\lambda x = \lambda' x$, $x\rho = x\rho'$ for all $x \in S$, and thus $\lambda = \lambda'$, $\rho = \rho'$; in either case $\omega = \omega'$.

It is easily seen that 1.4 holds if and only if S satisfies: for any $\omega, \omega' \in \Omega(S)$, $\lambda(xy) = \lambda'(xy)$, $(xy)\rho = (xy)\rho'$ and $x(\lambda y) = x(\lambda' y)$ for all $x, y \in S$ imply $\omega = \omega'$. This condition does not hold in an arbitrary semigroup, the same is true of 1.3; a counterexample can be found in [4].

PROPOSITION 1.5. If V and V' are extensions of S and if φ is an S-homomorphism of V into V', then $\tau(V : S) = \tau(V' : S) \circ \varphi$.

Proof. For all $a \in V$, $x \in S$:

$$\varphi(a)x = \varphi(a)\varphi(x) = \varphi(ax) = ax$$

and dually $x\varphi(a) = xa$. The conclusion follows from (1).

DEFINITION 1.6. The image $T(V:S) = \{\tau^a; a \in V\}$ of $\tau(V:S) = \tau$ is called the *type* of the extension V of S.

The following results show how the types of extensions of S are located in $\Omega(S)$.

THEOREM 1.7. *A subset T of $\Omega(S)$ is the type of some extension of S if and only if*

(a) *T is a subsemigroup of $\Omega(S)$ and $\Pi(S) \subseteqq T$;*

(b) *for any $(\lambda, \rho), (\lambda', \rho') \in T$, λ and ρ' commute.*

Proof. Let V be an extension of S and $T = T(V:S)$. Then T satisfies (a) by 1.2; furthermore $(\lambda^a x)\rho^b = (ax)b = a(xb) = \lambda^a(x\rho^b)$ for all $a, b \in V$, $x \in S$, so that T verifies (b).

If conversely T is a subset of $\Omega(S)$ with properties (a) and (b), let V be the groupoid on the set $S \cup T$ with multiplication $*$ defined by:

$$\begin{aligned} x*y &= xy && \text{if } x, y \in S\,, \\ \omega*\omega' &= \omega\cdot\omega' && \text{if } \omega, \omega' \in T\,, \\ x*(\lambda, \rho) &= x\rho && \text{if } x \in S, (\lambda, \rho) \in T\,, \\ (\lambda, \rho)*x &= \lambda x && \text{if } x \in S, (\lambda, \rho) \in T\,. \end{aligned}$$

It is a routine calculation to verify that this multiplication is associative, using (b); so that V is an extension of S. To find the type of this extension, let $\tau(V:S) = \tau$; if $a \in S$, then $\tau^a = \pi_a$ by 1.2; if $\omega = (\lambda, \rho) \in T$, then $\lambda^\omega x = \omega * x = \lambda x$, $x\rho^\omega = x*\omega = x\rho$ for all $x \in S$ by (1), so that $\tau^\omega = \omega$. Therefore $T(V:S) = \Pi(S) \cup T = T$ by (a).

PROPOSITION 1.8. If S is weakly reductive, or globally idempotent, then a subset T of $\Omega(S)$ is the type of some extension of S if and only if it satisfies (a).

Proof. In this case (b) is automatically verified in view of the following two lemmas, due to Clifford [1]:

LEMMA 1.9. *If S is weakly reductive and if $(\lambda, \rho), (\lambda', \rho') \in \Omega(S)$, then λ and ρ' commute.*

LEMMA 1.10. *If S is globally idempotent, then every left translation commutes with every right translation.*

THEOREM 1.11. *If S is any semigroup, the union $\Psi(S)$ of all types of extensions of S is the set of all $(\lambda, \rho) \in \Omega(S)$ such that λ and ρ commute.*

Proof. If $(\lambda, \rho) \in \Omega(S)$ belongs to some type of extension, then λ and ρ commute by 1.7. If conversely λ and ρ commute, where $(\lambda, \rho) \in \Omega(S)$, then the union T of $\Pi(S)$ and the subsemigroup of $\Omega(S)$ generated by (λ, ρ) satisfies condition (a) of 1.7 since $\Pi(S)$ is an ideal of $\Omega(S)$, and also condition (b) since λ commutes with all powers of ρ and with all inner right translations, and similarly for ρ. Therefore T is the type of some extension of S, and $(\lambda, \rho) \in \Psi(S)$.

If S is globally idempotent or weakly reductive, then $\Psi(S) = \Omega(S)$ by 1.8. In an arbitrary semigroup, $\Psi(S)$ need not be a type of extension nor even a subsemigroup of $\Omega(S)$; a counterexample can be found in [4].

2. Strict and pure extensions.

DEFINITION 2.1. A *strict* extension of S is an extension of S whose type is $\Pi(S)$.

Equivalently, an extension V of S is strict if and only if, for every $a \in V$, there exists $c \in S$ such that $ax = cx$, $xa = xc$ for all $x \in S$.

Strict extensions are closely related to extensions determined by partial homomorphisms.

LEMMA 2.2 (*Petrich*). *An extension V of S is determined by a partial homomorphism f if and only if there exists an S-homomorphism g of V onto S.*

In this case, f is the restriction of g to $V-S$. For the proof, see [6].

PROPOSITION 2.3. Every extension determined by a partial homomorphism is strict.

Proof. Let V be an extension of S determined by a partial homomorphism and g be an S-homomorphism of V onto S. Then $\tau(V:S) = \tau(S:S) \circ g = \pi \circ g$ by 1.5, so that $T(V:S) = \Pi(S)$.

PROPOSITION 2.4. Every extension of S is strict if and only if S has an identity.

Proof. If S has an identity, then every extension of S is determined by a partial homomorphism ([2], Th. 4.19) and is strict by 2.3. If conversely every extension of S is strict, so is the extension obtained by the adjuction of an identity e to S; then $\tau^e = \pi_c$ for some $c \in S$ and $cx = ex = x = xe = xc$ for all $x \in S$, whence S has an identity.

The following theorem may thus be considered as an extension of Theorem 4.19 of [2]:

THEOREM 2.5. *Let S be weakly reductive. Then every strict extension of S is determined by a partial homomorphism, and conversely.*

Proof. Let V be a strict extension of S. Then $\tau(V:S)=\tau$ is a homomorphism of V onto $\Pi(S)$ such that $\tau^a=\pi_a$ for all $a\in S$. Since π is an isomorphism of S onto $\Pi(S), \pi^{-1}\circ\tau$ is now an S-homomorphism of V onto S and the extension is determined by a partial homomorphism by 2.2. The converse follows from 2.3.

COROLLARY 2.6. *Let S be a weakly reductive semigroup and Q be a semigroup with zero. Then there exists a strict extension of S by Q if and only if there exists a partial homomorphism of $Q^*=Q-\{0\}$ into S.*

The assumption that S is weakly reductive cannot be omitted in 2.5 or 2.6, yet may be weakened in the following fashion. First recall that S is an *inflation* of a subsemigroup R of S if and only if there exists an idempotent homomorphism α of S onto R such that $xy=\alpha(x)\alpha(y)$ for all $x, y\in S$. We say that S is an *inflation of R over $R-R^2$* if $\alpha(x)\in R^2$ implies $x\in R$ (roughly speaking, if only $R-R^2$ is inflated to make S).

THEOREM 2.7. *Every strict extension of S is determined by a partial homomorphism if and only if S is an inflation over $R-R^2$ of a weakly reductive semigroup R. Moreover then $\Pi(S)\cong R$.*

Proof. Assume that every strict extension of S is determined by a partial homomorphism. Consider the extension of S by $\Pi(S)\cup\{0\}$ as defined in the proof of 1.7 (since $\Pi(S)$ is a type of extension); this extension has type $\Pi(S)$ and is therefore determined by a partial homomorphism f. Thus there exists a homomorphism f of $\Pi(S)$ into S such that $f(\pi_x)\,y=\pi_x*y=\lambda_x y=xy$ and dually $yf(\pi_x)=yx$ for all $x, y\in S$. This implies $\pi_{f(\pi_x)}=\pi_x$ for all $x\in S$; since π is onto, $\pi\circ f$ is the identity mapping of $\Pi(S)$; therefore f is one-to-one and $\alpha=f\circ\pi$ is an idempotent homomorphism of S into S. The image R of α is also the image of f and $R\cong\Pi(S)$ since f is one-to-one.

To prove that S is an inflation of R, observe that $\pi_{\alpha(x)}=\pi_x$ for all $x\in S$. Then, for all $x, y\in S$:

$$xy = \lambda_x y = \lambda_{\alpha(x)} y = \alpha(x)y = \alpha(x)\rho_y = \alpha(x)\rho_{\alpha(y)} = \alpha(x)\alpha(y) .$$

To prove that R is weakly reductive, we show that $\Pi(S)$ has this property. If $\pi_y, \pi_z \in \Pi(S)$ are such that $\pi_x \cdot \pi_y = \pi_x \cdot \pi_z$ and $\pi_y \cdot \pi_x = \pi_z \cdot \pi_x$ for all $x \in S$, then, since $\alpha = f \circ \pi$, $xy = \alpha(x)\alpha(y) = \alpha(x)\alpha(z) = xz$ and dually $yx = zx$ for all $x \in S$, so that $\pi_y = \pi_z$.

To prove that S is an inflation of R over $R - R^2$, take $a \in S$ such that $\alpha(a) \in R^2$, so that $\alpha(a) = bc$ for some $b, c \in R$. Construct a groupoid V by adjoining to S two elements B and C, with multiplication $*$ defined by

$$\begin{aligned} &x * y = xy , \\ &x * B = xb, B * x = bx, x * C = xc, C * x = cx , \\ &B * B = b^2, C * C = c^2, C * B = cb, B * C = a . \end{aligned}$$

Verification of associativity is tedious but straightforward, observing that $\pi_{bc} = \pi_{\alpha(a)} = \pi_a$ Then V is a strict extension of S. By the assumption it is determined by partial homomorphism f', so that $a = B * C = f'(B)f'(C) \in R$. This completes the proof of the direct part.

Conversely assume that S is an inflation over $R - R^2$ of a weakly reductive semigroup R, and let α be the idempotent inflation homomorphism of S onto R. Since α is idempotent, $\alpha(x)y = xy$, $y\alpha(x) = yx$ hold identically, so that $\pi_{\alpha(x)} = \pi_x$ for all $x \in S$. Moreover $\pi_x = \pi_y$ implies $x = y$ provided that $x, y \in R$, since R is weakly reductive.

If now V is a strict extension of S, then, for every $a \in V$, $\tau^a = \pi_c = \pi_{\alpha(c)}$ for some $c \in S$, so that $\tau^a = \pi_r$ for some $r \in R$; such r is, furthermore, unique. Set $r = h(a)$. From the uniqueness follows immediately that h is a homomorphism of V into R and that $h(a) = \alpha(a)$ if $a \in S$.

The restriction of h to $V - S$ is therefore a partial homomorphism, which in fact determines the extension. Indeed, if $a \in V - S$, $x \in S$, then $ax = h(a)x$ by definition of h, and dually $xa = xh(a)$. If $a, b \in V - S$ and $ab \in S$, then $h(a)h(b) = h(ab) = \alpha(ab)$; since the inflation is over $R - R^2$, $\alpha(ab) \in R^2$ implies $ab \in R$ and $ab = \alpha(ab) = h(a)h(b)$. This completes the proof.

COROLLARY 2.8. *If $\Pi(S)$ is globally idempotent and if every strict extension of S is determined by a partial homomorphism, then S is weakly reductive.*

Proof. Then $R \cong \Pi(S)$ is also globally idempotent so that $S = R$ is weakly reductive.

On the other hand, a semigroup R which is weakly reductive but not globally idempotent provides an example of semigroup S whose every strict extension is determined by a partial homomorphism, but which is not weakly reductive. (Such a semigroup R can be easily found in the tables of finite semigroups.)

Now we introduce pure extensions.

Let V be an extension of S of type T. The canonical homomorphism τ of V onto T sends S into $\Pi(S)$ by 1.2 and therefore induces a canonical homomorphism υ of V/S onto $T/\Pi(S)$.

DEFINITION 2.9. If Q and Q' are semigroups with zero, a homomorphism φ of Q into Q' is *pure* if $\varphi^{-1}(0) = \{0\}$. An extension V of S is *pure* if the canonical homomorphism υ of V/S onto $T/\Pi(S)$ is pure.

PROPOSITION 2.10. An extension V of S is pure if and only if, for any $a \in V$, $\tau^a \in \Pi(S)$ implies $a \in S$.

Proof. From the definition of υ it follows that $\upsilon^{-1}(0) = \tau^{-1}(\Pi(S))/S$, whence the result.

In particular, an extension V of S cannot be strict *and* pure unless $V = S$.

If S is weakly reductive and V is a pure extension of S, then υ determines the extension completely; more precisely, we have the following structure theorem, analogous to 2.5:

THEOREM 2.11. *Let S be weakly reductive and Q be a semigroup with zero, disjoint from S. Every pure homomorphism θ of Q onto a semigroup $T/\Pi(S)$, where T is a type of extension of S, determines a pure extension of S by Q, of type T, whose multiplication $*$ is given by the following formulae (where $Q^* = Q - \{0\}$ and $\theta(a) = \theta^a = (\lambda^a, \rho^a) \in T - \Pi(S)$ for $a \in Q^*$):*

$$a * b = \begin{cases} ab & \text{if } a, b \in S \\ \lambda^a b & \text{if } a \in Q^*, b \in S \\ a\rho^b & \text{if } a \in S, b \in Q^* \\ ab & \text{if } a, b \in Q^*, ab \neq 0 \\ c \in S \text{ such that } \theta^a \cdot \theta^b = \pi_c & \text{if } a, b \in Q^*, ab = 0\,. \end{cases}$$

Conversely, every pure extension of S can be constructed in this fashion.

Proof. The construction above gives a well-defined groupoid V on the set $S \cup Q^*$; indeed $\theta^a \in T - \Pi(S)$ for all $a \in Q^*$ since θ is pure, and the element c in the fifth case is unique since S is weakly reductive.

Associativity in V can be verified directly, but it is shorter to observe that θ induces a partial homomorphism of Q^* into T and therefore determines an extension V' of T by Q; the multiplication $*$ in V' is given by: $\omega * \omega' = \omega \cdot \omega'$, $\omega * q = \omega \cdot \theta^q$, $p * \omega = \theta^p \cdot \omega$, $p * q = pq$ if $pq \neq 0$, $p * q = \theta^p \cdot \theta^q$ if $pq = 0$, for all $p, q \in Q^*$, $\omega, \omega' \in T$. It is readily seen that the mapping φ of V into V' defined by $\varphi(x) = \pi_x$ if $x \in S$, $\varphi(q) = q$ if $q \in Q^*$, is a homomorphism; since S is weakly reductive, φ is one-to-one and associativity holds also in V.

Therefore V is an extension of S by Q; clearly $\tau^q = \theta^q$ for all $q \in Q^*$, so that the extension is pure, of type T.

Conversely, let V be any pure extension of S of type T, so that the canonical homomorphism υ of V/S onto $T/\Pi(S)$ is pure. Then the multiplication in V coincides with the multiplication $*$ in the statement of the theorem if we take $Q = V/S$, $\theta = \upsilon$; in the fifth case, for instance, $\theta^p \cdot \theta^q = \tau^p \cdot \tau^q = \tau^{pq} = \pi_{pq}$, where $pq \in S$. This completes the proof.

COROLLARY 2.12. *Let S be a weakly reductive semigroup and Q be a semigroup with zero. There exists a pure extension of S by Q if and only if there exists a pure homomorphism of Q into $\Omega(S)/\Pi(S)$.*

Proof. This follows from 2.11 and 1.8.

Given an arbitrary weakly reductive semigroup S, a semigroup with zero Q can be constructed such that there is no pure extension of S by Q [5]. Also, the construction in 2.11 can be extended to provide an extension of S by Q_1, given an extension of S by Q and a pure homomorphism of Q_1 into Q [5].

Finally we show that strict and pure extensions are naturally present in any extension.

THEOREM 2.13. *Let S be an arbitrary semigroup. Then every extension V of S is a pure extension of a strict extension of S. Precisely, the inverse image K of $\Pi(S)$ under $\tau(V: S)$ is the largest subextension of V which is a strict exension of S, and V is a pure extension of K.*

Proof. For any subextension V' of V, $\tau(V': S)$ is the restriction of $\tau(V: S)$ to V' by 1.5 (or directly), so that K is the largest strict subextension of V. Furthermore V is an extension of K, since $\Pi(S)$ is an ideal of $\Omega(S)$. To show that V is a pure extension of K, let $a \in V$ be such that $\tau(V: K)(a) \in \Pi(K)$. Then some $k \in K$ is such that $ax = kx$ and $xa = xk$ for all $x \in K$. Since this holds in particular for all $x \in S$, $\tau(V: S)(a) = \tau(V: S)(k) = \tau(K: S)(k)$, $\tau(V: S)(a) \in \Pi(S)$ and $a \in K$. This completes the proof.

THEOREM 2.14. *Any extension of S by a semigroup Q with zero having no proper nonzero ideal is either strict or pure.*

Proof. Let V be an extension of S by Q and K be the largest strict subextension of V. Then K/S is an ideal of $V/S \cong Q$. Therefore, either $K/S = V/S$ and $V = K$ is a strict extension of S, or $K/S = 0$ and V is a pure extension of $K = S$.

This result gives a particular interest to strict and pure extensions since any finite semigroup can be constructed from a (completely) simple semigroup (possibly 0) by finitely many extensions by semigroups having no nonzero proper ideals (i.e. (completely) 0-simple, or null with two elements).

3. Extension of congruences and dense extensions.

If V is an extension of S, we call S-*congruence* a congruence on V whose restriction to S is the equality relation. (The S-congruences are precisely the congruences induced by S-homomorphisms.)

THEOREM 3.1. *Let V be an extension of S and $\mathscr{T}$ be the congruence on V induced by $\tau = \tau(V: S)$. Then every S-congruence on V is contained in $\mathscr{T}$; if furthermore S is weakly reductive, $\mathscr{T}$ is the largest S-congruence on V. Moreover, $\mathscr{T}(S)$[1] is the largest strict subextension of V.*

Proof. Let $\mathscr{C}$ be an S-congruence on V; then $a \mathscr{C} b$ implies $ax \mathscr{C} bx$ and $xa \mathscr{C} xb$, thus $ax = bx$ and $xa = xb$ for all $x \in S$, and finally $\tau^a = \tau^b$. Therefore $\mathscr{C} \subseteqq \mathscr{T}$. If furthermore S is weakly reductive, then $\mathscr{T}$ is itself an S-congruence, so the largest. Moreover, if $a \in \mathscr{T}(S)$, then $\tau^a = \tau^x$ for some $x \in S$, $\tau^a \in \Pi(S)$ and a belongs to the largest strict subextension, and conversely.

COROLLARY 3.2. *If S is weakly reductive, then an extension V of S is pure if and only if S is saturated for every S-congruence on V.*

For strict extensions, we have:

PROPOSITION 3.3. Let V be an extension of S. If every congruence $\mathscr{C}$ on S is the restriction of some congruence $\overline{\mathscr{C}}$ on V such that $\overline{\mathscr{C}}(S) = V$, then V is a strict extension of S. The converse holds if S is weakly reductive.

Proof. Taking first $\mathscr{C}$ to be the equality relation on S, we obtain an S-congruence $\overline{\mathscr{C}}$ on V such that $\overline{\mathscr{C}}(S) = V$; by 3.1, $\mathscr{C}(S) = V$ and

[1] See "Definitions and notation".

the extension is strict. If conversely S is weakly reductive and if V is a strict extension of S, then there exists an S-homomorphism g of V onto S, by 2.2 and 2.5. Hence, if $\mathscr{C}$ is any congruence on S, the congruence $\bar{\mathscr{C}}$ defined for all $a,b \in V$ by $a \bar{\mathscr{C}} b$ if and only if $g(a) \mathscr{C} g(b)$, has the required properties.

Note that for such $\bar{\mathscr{C}}$ we have $V/\bar{\mathscr{C}} \cong S/\mathscr{C}$. In fact, the result above can be rephrased in terms of homomorphisms, as follows:

PROPOSITION 3.4. Let V be an extension of S. If any homomorphism of S into another semigroup can be extended to V, then V is a strict extension of S. The converse holds if S is weakly reductive.

Finally congruences can be extended in yet another fashion.

PROPOSITION 3.5. If V is a strict extension of S, then any congruence $\mathscr{C}$ on S can be extended to a congruence $\bar{\mathscr{C}}$ on V such that $\{a\}$ is a class of $\bar{\mathscr{C}}$ for all $a \in V - S$.

Proof. Define $\bar{\mathscr{C}}$ by: $a \bar{\mathscr{C}} b$ if and only if either $a = b$ or $a, b \in S$ and $a \mathscr{C} b$. Then $\mathscr{C}$ has all the required properties. For instance, $a \mathscr{C} b$ implies $ca \bar{\mathscr{C}} cb$ for any $c \in V$; this is clear if $a = b$; if $a, b \in S$, and if $z \in S$ is such $\tau^c = \pi_z$, then $ca = za \ \mathscr{C} \ zb = cb$.

Now we apply these results to dense extensions of weakly reductive semigroups.

DEFINITION 3.6. An extension V of S is *dense* if the equality relation is the only S-congruence on V.

THEOREM 3.7. *Let S be weakly reductive and V be an extension of S. Then V is a dense extension if and only if $\tau(V\colon S)$ is one-to-one. Consequently, any dense extension of S is pure.*

Proof. This follow from 3.1.

COROLLARY 3.8. *Let S be weakly reductive and V be an oversemigroup of S. Then V is a dense extension of S if and only if there exists an isomorphism φ of V onto a type of extension of S which extends π.*

Proof. Only the converse is not obvious. If φ is such an isomorphism, then S is an ideal of V since $\Pi(S)$ is an ideal of $\Omega(S)$; furthermore $\varphi = \tau(V\colon S)$ by 1.3, so that $\tau(V\colon S)$ is one-to-one.

COROLLARY 3.9. (*cf.* 1.3.2, [3]). *If S is weakly reductive and if*

V is a dense extension of S, then every subextension of V is a dense extension of S.

Proof. For any subextension V' of V, $\tau(V'\colon S)$ is the restriction of $\tau(V\colon S)$ to V' and is therefore one-to-one if V is a dense extension of S.

COROLLARY 3.10. *Let S be weakly reductive. Then an extension V of S is pure if and only if there exists an S-homomorphism φ of V into a dense extension D of S, such that $\varphi^{-1}(S) = S$.*

Proof. Let $T = T(V : S)$; by 3.8 we can construct a dense extension D of S of type T; then $\varphi = \tau(D : S)^{-1} \circ \tau(V : S)$ is an S-homomorphism of V onto D. If furthermore V is a pure extension of S, then $\varphi^{-1}(S) = \tau(V : S)^{-1}(\Pi(S)) = S$.

Conversely, let φ be an S-homomorphism of V into a dense extension D of S, such that $\varphi^{-1}(S) = S$. Then $\tau(V : S) = \tau(D : S) \circ \varphi$ by 1.5 and $\tau(V : S)^{-1}(\Pi(S)) = \varphi^{-1}(S) = S$ using 3.7, so that V is a pure extension of S.

Finally we give another proof of a theorem due to Gluskin [3]. First note that in our terminology a semigroup S is a densely embedded ideal [3] of a semigroup V if and only if V is a maximal dense extension of V (under inclusion).

LEMMA 3.11. *Let S be weakly reductive and V be an extension of S of type $\Omega(S)$. Then no proper overextension of V is a dense extension of S.*

Proof. If V' is an overextension of V and if φ is the canonical injection of V into V', then $\tau(V : S) = \tau(V' : S) \circ \varphi$ by 1.5. By the hypothesis on V, $\tau(V : S)$ maps V onto $\Omega(S)$; if furthermore V' is dense, then $\tau(V' : S)$ is one-to-one by 3.7, so that φ is onto and $V' = V$.

THEOREM 3.12 (*Gluskin*). *Let S be weakly reductive. Then S is a densely embedded ideal of V if and only if there exists an isomorphism of V onto $\Omega(S)$ which extends π.*

Proof. If this condition is satisfied, then, by 3.8, V is a dense extension of S, maximal by 3.11. If conversely V is a maximal dense extension of S, it is enough to show that $T(V : S) = \Omega(S)$, in view of 3.8. But if $T(V : S)$ were different from $\Omega(S)$, then $\tau(V : S)$ could be extended to an isomorphism of an oversemigroup V' of V onto $\Omega(S)$; V' would then be a dense extension of S by 3.8, and a proper overextension of V, and V would not be maximal. This completes the proof.

4. The structure theorem. The next theorem extends 2.5 and 2.11 to the general case of an arbitrary extension of a weakly reductive semigroup.

THEOREM 4.1. *Let S be a weakly reductive semigroup and Q be a semigroup with zero, disjoint from S. For a given ideal I of Q, a partial homomorphism f of $I^* = I - \{0\}$ into S and a mapping ω of $Q - I$ into $\Omega(S) - \Pi(S)$ satisfying the following conditions*:

$$\begin{array}{lll} \text{(a)} & \varphi^u(f(a)) = f(ua) & \textit{if } u \in Q - I,\, a \in I^*,\, ua \neq 0\,, \\ \text{(b)} & (f(a))\psi^u = f(au) & \textit{if } u \in Q - I,\, a \in I^*,\, au \neq 0\,, \\ \text{(c)} & \omega^u \cdot \omega^v = \omega^{uv} & \textit{if } u, v \in Q - I,\, uv \notin I\,, \\ \text{(d)} & \omega^u \cdot \omega^v = \pi_{f(uv)} & \textit{if } u, v \in Q - I,\, uv \in I^*\,, \\ \text{(e)} & \omega^u \cdot \omega^v \in \Pi(S) & \textit{if } u, v \in Q - I,\, uv = 0\,, \end{array}$$

(*where* $\omega(u) = \omega^u = (\varphi^u, \psi^u)$ *for all* $u \in Q - I$), *there exists an extension* $V = S[I \subseteqq Q; f, \omega]$ *of S by Q whose multiplication $*$ is defined by*:

$$a * b = \begin{cases} ab & \textit{if } a, b \in S \\ af(b) & \textit{if } a \in S,\, b \in I^* \\ a\psi^b & \textit{if } a \in S,\, b \in Q - I \\ f(a)b & \textit{if } a \in I^*,\, b \in S \\ \varphi^a b & \textit{if } a \in Q - I,\, b \in S \\ ab & \textit{if } a, b \in Q^*,\, ab \neq 0 \\ f(a)f(b) & \textit{if } a, b \in I^*,\, ab = 0 \\ (f(a))\psi^b & \textit{if } a \in I^*,\, b \in Q - I,\, ab = 0 \\ \varphi^a(f(b)) & \textit{if } a \in Q - I,\, b \in I^*,\, ab = 0 \\ c & \textit{if } a, b \in Q - I,\, ab = 0,\, \omega^a \cdot \omega^b = \pi_c\,; \end{cases}$$

and every extension of S by Q can be constructed in this fashion.

Proof. Rather than proving the direct part by verifying the associativity in all 54 possible cases, we shall derive the proof from Theorem 4.21 of [2]. First observe that, if we set

$$\begin{array}{ll} \lambda^a = \varphi^a,\, \rho^a = \psi^a & \text{if } a \in Q - I\,, \\ \lambda^a = \lambda_{f(a)},\, \rho^a = \rho_{f(a)} & \text{if } a \in I^*, \text{ and} \end{array}$$

$$\Phi(a, b) = \begin{cases} f(a)f(b) & \text{if } a, b \in I^*, ab = 0 \\ (f(a))\psi^b & \text{if } a \in I^*, b \in Q - I, ab = 0 \\ \varphi^a(f(b)) & \text{if } a \in Q - I, b \in I^*, ab = 0 \\ c & \text{if } a, b \in Q - I, ab = 0, \omega^a \cdot \omega^b = \pi_c , \end{cases}$$

then the multiplication $*$ is given by λ^a, ρ^a and the ramification Φ by the formulae:

$$\begin{aligned} x * y &= xy && \text{if } x, y \in S , \\ a * x &= \lambda^a x && \text{if } a \in Q^*, x \in S , \\ x * a &= x\rho^a && \text{if } x \in S, a \in Q^* , \\ a * b &= ab && \text{if } a, b \in Q^*, ab \neq 0 , \\ a * b &= \Phi(a, b) && \text{if } a, b \in Q^*, ab = 0 , \end{aligned}$$

which are formulae (N 1) to (N 4) of Theorem 4.21 of [2]. The direct part follows then from the fact that λ, ρ, Φ satisfy the conditions (C 1), (C 2), (C 3) of the same theorem. This verification is straightforward using the assumptions on f and ω and is left to the reader; it is much shorter than direct verification of associativity, though not very different in nature. Then the construction in the statement of the theorem gives an extension of S by Q.

It is readily verified that the largest strict subextension of this extension is just $S \cup I^*$.

Conversely, let $(V, \circ)$ be any extension of S; V is an extension of S by $Q = V/S$ and $S \cap Q = \varnothing$. If K is the largest strict subextension of V, then $I = K/S$ is an ideal of Q and by 2.5 there exists a partial homomorphism f of $I^* = K - S$ into S such that

$$\begin{aligned} x \circ a &= xf(a) && \text{if } x \in S, a \in I^* , \\ a \circ x &= f(a)x && \text{if } a \in I^*, x \in S , \\ a \circ b &= f(a)f(b) && \text{if } a, b \in I^*, ab = 0 \text{ in } Q . \end{aligned}$$

If now $u \in Q - I = V - K$, then $\tau^u \in \Omega(S) - \Pi(S)$; let ω be the restriction of τ to $Q - I$.

Investigation of the different cases shows that the multiplication $\circ$ coincides with the multiplication $*$ defined by the formulae in the statement of the theorem by means of f and ω. For instance, if $a \in Q - I, b \in I^*, ab = 0$ in Q, then $\pi_{a \circ b} = \tau^a \cdot \tau^b = \omega^a \cdot \pi_{f(b)} = \pi_{\varphi^a(f(b))}$, whence $a \circ b = \varphi^a(f(b))$ since S is weakly reductive; the other cases are similar or trivial. Similarly it is verified that f and ω satisfy conditions (a) through (e); if for instance $a \in Q - I, b \in I^*$ and $ab \neq 0$ in Q, then $\pi_{f(a \circ b)} = \tau^{a \circ b} = \tau^a \cdot \tau^b = \pi_{\varphi^a(f(b))}$ as above, so that $f(a \circ b) = \varphi^a(f(b))$ and (a) holds; the other cases are similar or dual. This completes the proof.

Note that 2.5 follows by taking $I = Q$, and 2.11 by taking $I = \{0\}$.

Also note that not every partial homomorphism f of I^* into S can be used in 4.1. For if $a, c \in I^*$ and $b \in Q - I$ are such that $ab \neq 0$, $bc \neq 0$, $abc = 0$, then it follows from (a) and (b) that

$$f(a)f(bc) = f(a)(\varphi^b(f(c))) = ((f(a))\psi^b)f(c) = f(ab)f(c) \, ,$$

In our last theorem we give necessary and sufficient conditions that $S[I \subseteqq Q; f, \omega] = V$ and $S[I' \subseteqq Q'; f', \omega'] = V'$ be equivalent in the sense of [2]; i.e., that there exists an isomorphism α of V onto V' such that $\alpha(S) = S$ (see p. 143 of [2]).

THEOREM 4.2. *It S is weakly reductive, then $V = S[I \subseteqq Q; f, \omega]$ and $V' = S[I' \subseteqq Q'; f', \omega']$ are equivalent if and only if there exists an automorphism $\beta : x \rightsquigarrow x'$ of S and an isomorphism $\gamma : a \rightsquigarrow a'$ of Q onto Q' such that:*

(i) $\gamma(I) = I'$,

(ii) $(f(a))' = f'(a')$ *for all* $a \in I^*$,

$$\left.\begin{array}{ll}\text{(iii)} & (\varphi^a x)' = \varphi'^{a'} x' \\ \text{(iv)} & (x\psi^a)' = x'\psi'^{a'}\end{array}\right\} \textit{ for all } x \in S \textit{ and } a \in Q - I \, .$$

Proof. If there exists an isomorphism α of V onto V' such that $\alpha(S) = S$, then α induces an automorphism β of S and an isomorphism γ of Q onto Q'. With $a' = \alpha(a)$ for all $a \in V$ and the obvious notation, we observe first that

$$\text{(v)} \quad (\lambda^a x)' = \lambda'^{a'} x' \, , \qquad (x\rho^a)' = x'\rho'^{a'}$$

for all $x \in S$, $a \in V$.

In particular, $\tau^a \in \Pi(S)$ implies $\tau'^{a'} \in \Pi(S)$; in other words, $\alpha(K) \subseteqq K'$. Since similarly $\alpha^{-1}(K') \subseteqq K$, we conclude that $\alpha(K) = K'$. (Consequently, if V is strict or pure, so is V'.) In particular (i) holds. Also, for any $a \in I^* = K - S$, $a' \in I'^* = K' - S$ so that $\pi_{f'(a')} = \tau'^{a'} = \pi_{(f(a))'}$ by (v); since S is weakly reductive, $f'(a') = (f(a))'$ which establishes (ii). Finally (iii) and (iv) are special cases of (v).

Conversely, let β and γ be as in the statement of the theorem. Define α by $\alpha(x) = x'$ if $x \in S$, $\alpha(a) = a'$ if $a \in Q^*$, so that α is a one-to-one mapping of V onto V'. Using the formulae in 4.1 and conditions (i) through (iv), it is easy to prove that α is an isomorphism; if for instance $a \in I^*$, $b \in Q - I$ and $ab = 0$ in Q, then

$$(ab)' = (f(a)\psi^b)' = f'(a')\psi'^{b'} = a'b'$$

if the multiplications $*$ are denoted by juxtaposition. This completes the proof.

References

1. A. H., Clifford, *Extensions of semigroups*, Trans. Amer. Math. Soc. **68** (1950), 165–173.
2. A. H., Clifford, and G. B., Preston, *The algebraic theory of semigroups*, Vol. I, Math. Surveys **7**, Amer. Math. Soc., Providence, 1961.
3. L. M., Gluskin, *Ideals of semigroups*, Mat. Sbornik **55** (1961), 421–448 (Russian).
4. P. A., Grillet, *Sur les extensions idéales d'un demi-groupe*, Sém. Dubreil-Pisot (Univ. de Paris) **18** (1964/65), No. 1, 1–27.
5. ———, *Extensions idéales strictes et pures d'un demi-groupe*, Sém. Dubreil-Pisot (Univ. de Paris) **18** (1964/65), No. 11, 1–21.
6. M., Petrich, *On extensions of semigroups determined by partial homomorphisms*, Nederl. Akad. Wetensch. Indag. **28** (1966), 49–51.
7. R. Yoshida, *Ideal extensions of semigroups and compound semigroups*, Mem. Res. Inst. Sci. Eng., Ritumeikan Univ. **13** (1965), 1–8.

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