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ON A THEOREM OF MERGELYAN

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The following is a theorem of S. N. Mergelyan.

THEOREM. If X is a compact plane set with finitely many complementary components, then every continuous function on X which is analytic on the interior is a uniform limit on X of rational functions.

Here two short proofs of the theorem are given. Both proofs are obtained by considering the measures on X orthogonal to the algebra of rational functions.

Let X be a compact plane set and suppose its complement $C\backslash X$ has finitely many components. In 1951 Mergelyan [9] proved:

MERGELYAN'S THEOREM. *Every function continuous on X and analytic on the interior is uniformly approximable by rational functions.*

By examining the measures on X orthogonal to the rational functions, Glicksberg and Wermer [6] have found an elegant proof of Mergelyan's Theorem in the case $C\backslash X$ connected. Their argument has recently been extended by Ahern and Sarason [1] and by Glicksberg [5] to give functional analytic proofs of the general Mergelyan result. However those proofs are quite long. Here we give two shorter proofs of the theorem. The first proof begins with the known fact that the question of approximation is a local one, and the theorem is thereby reduced to the simply connected case. The second proof follows the reasoning of [6]. Here we use some results from [1] and [5], but avoid the more lengthy arguments in those papers. Though longer than the first, this second proof is included because the ideas therein yield related results not accessible via our first proof.

Denote by $A(X)$ the Banach algebra of functions continuous on X and analytic on X^0, the interior. Let $R(X)$ be the closed subalgebra of $A(X)$ spanned by the rational functions analytic on X. By a measure on X we mean a finite complex Borel measure on X. Write $\mu \perp R(X)$ is μ is a measure on X orthogonal to $R(X)$, and $\mu \perp A(X)$ if μ annihilates $A(X)$ as well. As have several previous authors, we prove the Mergelyan Theorem by showing $\mu \perp R(X)$ implies $\mu \perp A(X)$.

2. **The first proof.** We quote the following lemma of Bishop [3, p. 40]. This lemma was stated polynomials, but Bishop's proof is valid for rational functions.

LEMMA 2.1. (*Bishop*). *Let X be a compact subset of C. For x a real number, set*

$$L_x = X \cap \{z\colon \text{Real } z \leqq x\},\ R_x = X \cap \{z\colon \text{Real } z \geqq x\}.$$

Let $\mu \perp R(X)$. Then for almost all x there are measures ν_1 on L_x and ν_2 on R_x such that $\nu_1 \perp R(L_x)$, $\nu_2 \perp R(R_x)$ and $\mu = \nu_1 + \nu_2$.

This lemma gives the known result that questions concerning $R(X)$ are local ones.

COROLLARY 2.2. *Let f be a continuous function on the compact set X. Assume each point in X has a closed neighborhood K such that the restricted function $f \mid K$ is in $R(K)$. Then f is in $R(X)$.*

Proof. Let μ be a measure orthogonal to $R(X)$ and let $\{U_1, U_2, \cdots, U_n\}$ be any finite open cover of X. It follows from repeated applications of 2.1 that there are finitely many compact rectangles $\{T_1, \cdots, T_m\}$ and measures ν_j on T_j such that

(i) $T_j \subset U_i$ for some i

(ii) $\nu_j \perp R(T_j)$

(iii) $\mu = \sum_{j=1}^m \nu_j$.

If f satisfies the hypotheses, then using such a decomposition we see that $\int f d\mu = 0$. This implies $f \in R(X)$.

Actually we can localize measures as in the proof of 2.2 without appealing to Bishop's lemma. Instead we use an observation shown us by A. Browder, who credits it to K. Hoffman. For a measure μ on X define its *Cauchy transform*

$$\hat{\mu}(\alpha) = \int (z - \alpha)^{-1} d\mu(z) .$$

Then $\hat{\mu}$ is locally integrable with respect to plane Lebesgue measure. Also $\hat{\mu} = 0$ almost everywhere if and only if $\mu = 0$. See [11]. The observation, whose proof we only sketch, is this:

(a) $\mu \perp R(X)$ if and only if $\hat{\mu} = 0$ on $C \backslash X$. This is seen by taking partial fractions and by "pushing poles together."

(b) An application of Green's theorem yields, for φ a C^∞ function with compact support,

$$\varphi\hat{\mu} = \widehat{\varphi\mu} + \hat{\sigma} \quad \text{where} \quad \sigma = \frac{1}{2\pi i}\frac{\partial\varphi}{\partial\bar{z}}\hat{\mu}(z) dz \wedge d\bar{z} .$$

Now if $\mu \perp R(X)$ and K is the closed support of φ, then by (a) and (b),

$$\varphi\mu + \sigma \perp R(X \cap K) .$$

Letting φ run through a finite partition of unity $\{\varphi_j\}_{j=1}^m$ and setting $\nu_j = \varphi_j\mu + 1/2\pi i\, \partial\varphi_j/\partial\bar{z}\, \hat{\mu}(z)dz \wedge d\bar{z}$ we obtain a decomposition of μ satisfying (i), (ii) and (iii) in the proof of 2.2.

Proof of theorem. Let $H_1, H_2, \cdots, H_n$ be the bounded components of $C\backslash X$ and let $f \in A(X)$. For each point in X choose a compact neighborhood K with diameter less than the diameter of each H_j. Then $C\backslash K$ is connected and $f \mid K$ is in $A(K)$. Thus by [6], $f \mid K$ is in $R(K)$. Hence f is in $R(X)$ by 2.2.

We note that this proof is still valid when $C\backslash X$ has infinitely many components but the diameters of these components are bounded below. Examples of such sets are easily constructed.

3. The second proof. By the maximum modulus principle $A(X)$ can be identified with the algebra of boundary values of functions in $A(X)$. We will view $A(X)$ as a closed algebra of continuous functions in ∂X, the boundary of X, and we consider now only measures supported on $\partial(X)$.

We begin with a theorem of Walsh [10] and Lebesgue [8]. A more recent proof is contained in the first three lemmas of [4]. $C^R(\partial X)$ is the space of real continuous functions on ∂X, and Real $R(X)$ is the space (restrictions to ∂X) of real parts of functions in $R(X)$.

WALSH-LEBESGUE THEOREM. *Let $H_1, H_2, \cdots, H_n$ be the bounded components of $C\backslash X$ and let $a_j \in H_j$. Then Real $R(X) \oplus$ Span $\{\log|z - a_j|: 1 \leqq j \leqq n\}$ is uniformly dense in $C^R(\partial X)$.*

This means the space of real measures on ∂X orthogonal to $R(X)$ has dimension at most n. It also means the Dirichlet problem is solvable on X; if $u \in C^R(\partial X)$ then there is a unique $\hat{u}$ in $C^R(X)$ harmonic on X^0 with $u = \hat{u}$ on ∂X. Hence for $z \in X$ there is a unique probability measure λ_z on ∂X such that $\hat{u}(z) = \int_{\partial X} u\,d\lambda_z$ for all $u \in C^R(\partial X)$ It follows directly that λ_z is a *representing measure* for $A(X)$, as λ_z is positive and

$$f(z) = \int_{\partial X} f\,d\lambda_z \quad \text{for all} \quad f \in A(X)\ .$$

It also follows that λ_z is an *Arens-Singer measure* for $A(X)$,

$$\log|f(z)| = \int_{\partial X} \log|f|\,d\lambda_z \quad \text{for} \quad f \in A(X) \quad \text{and} \quad \frac{1}{f} \in A(X)\ .$$

Moreover the Walsh-Lebesgue theorem implies that λ_z is the unique

Arens-Singer measure for $R(X)$. Consequently the assumptions of the paper [1] of Ahern and Sarason apply to $R(X)$. We now state a lemma of theirs and some immediate consequences. All except Corollary 3.4 are proven on pp. 126–128 of [1].

LEMMA 3.1. (*Ahern-Sarason*). *Let* $(v_n)_{n=1}^{\infty}$ *be a sequence of non-negative functions in* $C^R(\partial X)$ *with* $\int v_n d\lambda_z \to 0$. *Then there is a subsequence* $(u_n)_{n=1}^{\infty}$ *and a sequence* $(f_n)_{n=1}^{\infty}$ *of functions in* $R(X)$ *such that* $|f_n| \leqq e^{-u_n}$ *and* $f_n \to 1$ *almost everywhere* λ_z.

COROLLARY 3.2. (*F. and M. Riesz Theorem*). *If* $z \in X$ *and* $\mu \perp R(X)$, *let* $\mu = \mu_z + \tilde{\mu}_z$ *be the Lebesgue decomposition of* μ *with respect to* λ_z. *Then* $\mu_z \perp R(X)$ *and* $\tilde{\mu}_z \perp R(X)$.

COROLLARY 3.3. F*or* $z \in X$, *every representing measure on* ∂X *for* z *is absolutely continuous with respect to* λ_z. *In particular, when* $z \in \partial X$, *the only representing measure is the point mass at* z.

The next corollary is a consequence of the F. and M. Riesz theorem, and is Theorem 2.8 of Glicksberg's paper [5].

COROLLARY 3.4. *If* $\mu \perp R(X)$, *then there is a sequence* $(z_n)_{n=1}^{\infty}$ *in* X *such that*

$$\mu = \mu_s + \sum_{n=1}^{\infty} \mu_{z_n} \tag{1}$$

where the series converges in norm, each term is orthogonal to $R(X)$, μ_{z_n} *is absolutely continuous with respect to* λ_{z_n} *and* μ_s *is singular with respect to all representing measures.*

Proof. First take the "simultaneous Lebesgue decomposition of μ with respect to all λ_z." That is, let $c = \sup\{\|\mu_z\| : z \in X\}$ and choose z_1 so that $\|\mu_{z_1}\| > c/2$. Take the Lebesgue decomposition $\mu = \tilde{\mu}_{z_1} + \mu_{z_1}$. Repeating the argument with $\tilde{\mu}_{z_1}$ and continuing by induction yields a decomposition (1), in which μ_s is singular with respect to all λ_z.

By Corollary 3.2, $\mu_{z_1} \perp R(X)$, and so by induction every μ_{z_n} (and hence μ_s as well) is ortogonal to $R(X)$. Finally by Corollary 3.3, μ_s is singular with respect to every representing measure.

From Corollary 3.4 we see that to show any $\mu \in (R(X))^{\perp}$ annihilates $A(X)$ it suffices to consider the case when $\mu = \mu_z$ for some z and the case when μ is singular with respect to all representing measures. The latter case is settled by the following argument of Wilken [12], valid for any compact plane set.

LEMMA 3.5. (*Wilken*). *Let X be any compact plane set and let $\mu \perp R(X)$. If μ is singular with respect to all representing measures then $\mu = 0$.*

Proof. As in [12], there is a point $z \in X$ such that

$$\int \frac{d\,|\mu|\,(\zeta)}{|z-\zeta|} < \infty \quad \text{and} \quad \int \frac{d\mu(\zeta)}{z-\zeta} = c \neq 0 \,.$$

Then the measure $\nu = 1/c\,\mu(\zeta)/z - \zeta$ is a complex measure representing z; so that by [7] there is a positive representing measure m for z absolutely continuous with respect to $|\nu|$. Then m is absolutely continuous with respect to $|\mu|$ and the assertion obtains.

For our analysis of the absolutely continuous orthogonal measures we need two more lemmas. For $z \in X$, write $\lambda = \lambda_z$, and let $H^2(R(X))$ be the closure of $R(X)$ in $L^2(\lambda)$. Set $H^\infty(R(X)) = H^2(R(X)) \cap L^\infty(\lambda)$. Define $H^2(A(X))$ and $H^\infty(A(X))$ analogously. The first lemma is from Ahern and Sarason [1] and follows directly from Lemma 3.1 by an argument of Hoffman and Wermer. (See also [5] and [11].)

LEMMA 3.6. *If h is in $H^\infty(R(X))$, then there is a sequence $(h_n)_{n=1}^\infty$ in $R(X)$ with $\|h_n\|_\infty \leqq \|h\|_\infty$ and $h_n \to h$ almost everywhere λ.*

The second lemma is a minor variation of Glicksberg's Lemma 3.16 of [5]. The proof is due to T. W. Gamelin and the author.

LEMMA 3.7. $H^2(R(X)) = H^2((A(X))$.

Proof. Let $\overline{H_0^2(R(X))}$ be the space of complex conjugates of functions in $H^2(R(X))$ having integral zero. Since λ is multiplicative on $R(X)$, $L^2(\lambda)$ has the following orthogonal decomposition:

$$L^2(\lambda) = H^2(R(X)) \oplus \overline{H_0^2(R(X))} \oplus E \,.$$

The space E is spanned by the real functions in L^2 annihilating $R(X)$ and thus by the Walsh-Lebesgue theorem E is finite dimesional. Also λ is multiplicative on $A(X)$, so an orthogonality argument gives

$$H^2(R(X)) \subset H^2(A(X)) \subset H^2(R(X)) \oplus E \,.$$

This means $H^2(A(X)) = H^2(R(X)) \oplus N$ where N is finite dimensional. Let $G_1, G_2, \cdots, G_p$ be an orthonormal basis of N and for an appropriately small ε choose $g_1, g_2, \cdots, g_p$ in $A(X)$ with $\|g_j - G_j\|_2 < \varepsilon$. Let J be the span of $\{g_1, \cdots, g_p\}$. Then

$$H^2(A(X)) = H^2(R(X)) + J$$

where the sum is direct but not necessarily orthogonal. Let P_J be the projection of $H^2(A(X))$ onto J which is zero on $H^2(R(X))$ For $f \in R(X)$ and $g \in J$ define $T_f(g) = P_J(fg)$. Then T is a representation of $R(X)$ on J. Since $R(X)$ is commutative and J is finite dimensional there is a common eigenvector $g \in J$. Therefore for $f \in R(X)$ we have

$$fg = \varphi(f) \cdot g + r \tag{2}$$

where $r \in H^2(R(X))$ and $\varphi(f) \in \boldsymbol{C}$.

Clearly φ is a multiplicative linear functional on $R(X)$, so that there is a point $z_1 \in X$ such that $\varphi(f) = f(z_1)$ for all $f \in R(X)$. Also because g is bounded φ extends continuously to $H^2(R(X))$ and so there is an $F \in L^2(\lambda)$ such that $f(z_1) = \int fF d\lambda$. Hence by (7) z_1 has a positive representing measure absolutely continuous with respect to λ. By Corollary 3.3 then $z_1 \notin \partial X$.

Taking $f = z$ in (2), we have $h = (z - z_1)g \in H^2(R(X))$ Hence there is a sequence $(h_n)_{n=1}^{\infty}$ in $R(X)$ converging to h in $L^2(\lambda)$. Let $c = \int hF d\lambda = \lim_{n\to\infty} \int h_n F d\lambda$. Then since $z_1 \in X^0$, $r_n = h_n - h_n(z_1)/z - z_1$ is in $R(X)$ and r_n converges to $g - c(z - z_1)^{-1}$ in $L^2(\lambda)$. Thus

$$g - c(z - z_1)^{-1} \in H^2(R(X)) . \tag{3}$$

If $c = 0$, then $g \in H^2(R(X))$ and we are done. On the other hand if $c \neq 0$, then $(z - z_1)^{-1} \in H^2(A(X))$. Being bounded on ∂X, $(z - z_1)^{-1}$ is therefore in $H^\infty(A(X))$. Now $H^\infty(A(X))$ is an algebra, and consequently $(z - z_1)^{-n} \in H^2(A(X))$ for all n. But $H^2(R(X))$ has finite codimension in $H^2(A(X))$; whence some polynomial in $(z - z_1)^{-1}$ is in $H^2(R(X))$. That is

$$\frac{a_1}{z - z_1} + \frac{a_2}{(z - z_1)^2} + \cdots + \frac{a_n}{(z - z_1)^n} \in H^2(R(X)) . \qquad (a_n \neq 0) .$$

Multiplying by $(z - z_1)^{n-1}$ we have $(z - z_1)^{-1} \in H^2(RX))$. By (3), then $g \in H^2(R(X))$. Therefore $J = \{0\}$ and

$$H^2(R(X)) = H^2(A(X)) .$$

To conclude the proof of Mergelyan's theorem, let $\mu \perp R(X)$. We can assume μ is absolutely continuous with respect to some λ_z. Let $h \in A(X)$. Then by 3.7 $h \in H^2(R(X))$ and so by 3.6 there is a sequence $(h_n)_{n=1}^{\infty}$ in $R(X)$ converging boundedly pointwise to h almost everywhere λ_z. As $\int h_n d\mu = 0$, dominated convergence implies $\int h d\mu = 0$ and μ is orthogonal to $A(X)$.

We should remark that the proof of Lemma 3.7 actually shows that $H^{\infty}(R(X))$ is maximal among the subalgebras of $L^{\infty}(\lambda)$ on which λ is multiplicative. This means that one can compute the defect of Real $R(X)$ in $C^{R}(\partial X)$ and obtain the other results that Ahern and Sarason do in [2] without using the deeper theorems of their earlier paper [1].

Added in proof. After a closer reading of the literature we have found that the proof of Mergelyan's theorem in § 2 was essentially given by Laura Kodama in "Boundary measures of analytic differentials" Pacific J. Math., 15 (1965), 1261-1277.

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REFERENCES

1. P. R. Ahern and Donald Sarason, *The H^p spaces of a class of function of algebras*, Acta. Math. **117** (1967), 123-163.
2. P. R. Ahern and Donald Sarason, *On some hypodichlet algebras of analytic functions*, Amer. J. Math. **89** (1967), 932-941.
3. E. Bishop, *Subalgebras of functions on a Riemann surface*, Pacific J. Math. **8** (1958), 29-50.
4. L. Carleson, *Mergelyan's theorem on uniform polynomial approximation*, Math. Scand. **15** (1964), 167-175.
5. I. Glicksberg, *Dominant representing measures and rational approximation* Trans. Amer. Math. Soc. **130** (1968), 425-462.
6. I. Glicksberg and J. Wermer, *Measures orthogonal to a Dirichlet algebra*, Duke Math. J. **30** (1963), 661-666.
7. K. Hoffman and H. Rossi, *On the extension of positive weak continuous functionals*, Duke Math. J. **34** (1967), 453-466.
8. H. Lebesgue, *Sur le problem de Dirichlet*, Rend. Palmero, **29** (1907), 371-402.
9. S. N. Mergelyan, *Uniform approximation to functions of a complex variable*, Amer. Math. Soc. Translation No. 101.
10. J. L. Walsh, *The approximation of harmonic functions by harmonic polynomials and harmonic rational functions*, Bull. Amer. Math. Soc. **35** (1929), 499-544.
11. J. Wermer, *Seminar über Funktionen-Algebren*, Springer, Berlin, 1964.
12. D. R. Wilken, *Lebesgue measure for parts of $R(X)$*, Proc. Amer. Math. Soc. **18** (1967), 508-512.

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