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A GENERALIZED COROLLARY OF THE BROWDER-KIRK FIXED POINT THEOREM

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This paper generalizes a corollary, due to W. A. Kirk, of the F. E. Browder-W. A. Kirk fixed point theorem for nonexpansive self-mappings of closed, bounded, convex sets in uniformly convex Banach spaces.

F. E. Browder [1] and W. A. Kirk [4] have independently proved that if F is a closed, bounded, convex subset of a uniformly convex Banach space, and if T is a nonexpansive mapping from F into F, then T has a fixed point in F. The following corollary was proved by Kirk [4] and also by Browder and Petryshyn [2]: If E is a uniformly convex Banach space, and $T: E \rightarrow E$ is a nonexpansive mapping, and if for some $x_1 \in E$ the sequence $\{T^n x_1\}$ of Picard iterates of T is bounded, then T has a fixed point in E. Browder and Petryshyn also observed that if the nonexpansive mapping T has a fixed point in E, then for any $x_1 \in E$ the sequence $\{T^n x_1\}$ will be bounded. Outlaw and Groetsch [6] have recently announced the following extension of this corollary: If E is a uniformly convex Banach space, and $T: E \rightarrow E$ is a nonexpansive mapping, and $S_\lambda = \lambda I + (1-\lambda)T$ for a given $\lambda, 0 < \lambda < 1$, then T has a fixed point in E if and only if the sequence $\{S_\lambda^n x_1\}$ of Picard iterates of S_λ is bounded for each $x_1 \in E$. The purpose of this note is to show that this corollary and its extension are both special cases of a considerably more general corollary of the Browder-Kirk theorem.

W. R. Mann [5] introduced the following general iterative process: Suppose $A = [a_{np}]$ is an infinite real matrix satisfying (1) $a_{np} \geqq 0$ for all n, p, and $a_{np} = 0$ for $p > n$; (2) $\sum_{p=1}^n a_{np} = 1$ for each n; (3) $\lim_n a_{np} = 0$ for each p. If F is a closed convex subset of a Banach space E, and $T: F \rightarrow F$ is a continuous mapping, and $x_1 \in F$, then the process $M(x_1, A, T)$ is defined by

$$v_n = \sum_{p=1}^{n} a_{np} x_p, \quad x_{n+1} = T v_n, \quad n = 1, 2, 3, \cdots .$$

Various choices of the matrix A yield many interesting iterative processes as special cases. With A the infinite identity matrix, one gets the Picard iterates of T: $v_{n+1} = x_{n+1} = Tv_n$, whence $v_{n+1} = T^n v_1 = T^n x_1$. With $0 < \lambda < 1$ and $A = [a_{np}]$ defined by $a_{np} = \lambda^{n-1}$ if $p = 1$, $a_{np} = \lambda^{n-p}(1-\lambda)$ if $1 < p \leqq n$, $a_{np} = 0$ if $p > n$, $n = 1, 2, 3, \cdots$, one gets $v_{n+1} = \lambda v_n + (1-\lambda) T v_n = S_\lambda v_n$, whence $v_{n+1} = S_\lambda^n v_1 = S_\lambda^n x_1$. If T is linear then an appropriate choice of A yields

$$v_{n+1} = (x_1 + Tx_1 + \cdots + T^n x_1)/(n+1) ,$$

thus providing a connection with mean ergodic theorems for linear operators. Another choice of A yields an iterative process recently investigated by Halpern [3], provided $x_1 = 0$. Many other choices are possible, of course. Our main theorem is as follows.

THEOREM 1. *If E is a uniformly convex Banach space, and if $T: E \to E$ is a nonexpansive mapping, and if there exist $x_1 \in E$ and a process $M(x_1, A, T)$ such that either of the sequences $\{x_n\}, \{v_n\}$ is bounded, then T has a fixed point in E.*

To prove this, we will make use of the following lemma which is a straightforward consequence of uniform convexity.

LEMMA 1. *Suppose E is a uniformly convex Banach space, and suppose $r > 0$. For each $\varepsilon > 0$ let $p_\varepsilon = \sup\{s: s = \| u - v \|$ where $u, v \in E, \| u \| = 2r, 2r < \| v \| \leqq 2r + \varepsilon$, and $\| (1-t)u + tv \| > 2r$ for all $t \in (0, 1)\}$. Given any $c > 0$, there exists $\varepsilon > 0$ such that $p_\varepsilon \leqq c$.*

Proof of Theorem 1. We first observe that if either of the sequences $\{x_n\}, \{v_n\}$ in the process $M(x_1, A, T)$ is bounded, then the other is also bounded. For if $\| x_n \| \leqq b$ for all n, then

$$\| v_n \| = \left\| \sum_{p=1}^{n} a_{np} x_p \right\| \leqq \sum_{p=1}^{n} a_{np} \| x_p \| \leqq b \sum_{p=1}^{n} a_{np} = b$$

for all n; and if $\| v_n \| \leqq b$ for all n, then

$$\| x_{n+1} - T(0) \| = \| T(v_n) - T(0) \| \leqq \| v_n - 0 \| \leqq b$$

for all n. So, given $x_1 \in E$ and a process $M(x_1, A, T)$ in which both of the sequences $\{x_n\}, \{v_n\}$ are bounded, we wish to show that T has a fixed point. This will be done by showing that T maps a certain bounded, closed, convex set into itself. We use the notation $D_r(p) = \{x: \| x - p \| \leqq r\}, r > 0, p \in E$. Let $r > 0$ be such that $x_n \in D_r(0)$ and $v_n \in D_r(0)$ for all n. For each $i = 1, 2, 3, \cdots$, define sets C_i and G_i by

$$C_i = \bigcap_{n=i}^{\infty} D_{2r}(x_n), \quad G_i = \bigcap_{n=i}^{\infty} \{D_{2r}(x_n) \cap D_{2r}(v_n)\} .$$

For each i, we have

$$D_r(0) \subset G_i \subset C_i \subset D_{2r}(x_i) \subset D_{3r}(0) .$$

Each C_i and each G_i is a nonempty bounded, closed, convex set, and it is clear that $C_i \subset C_{i+1}$ and $G_i \subset G_{i+1}$. We now show $T(G_i) \subset C_{i+1}$: $x \in G_i$ implies $\| x - v_n \| \leqq 2r$ for all $n \geqq i$, which gives $\| Tx - Tv_n \| \leqq$

$\| x - v_n \| \leqq 2r$ for all $n \geqq i$; but, since $x_{n+1} = Tv_n$, this can be written $\| Tx - x_{n+1} \| \leqq 2r$ for all $n \geqq i$, so that $Tx \in C_{i+1}$. Define sets C and G by

$$C = \bigcup_{i=1}^{\infty} C_i,\ G = \bigcup_{i=1}^{\infty} G_i \ .$$

Clearly, $D_r(0) \subset G \subset C \subset D_{3r}(0)$; and $\bar{G}, \bar{C}$ are bounded, closed, convex sets. Since $T(G_i) \subset C_{i+1}$ for each i, we have $T(G) \subset C$. Since T is continuous, $T(\bar{G}) \subset \overline{T(G)} \subset \bar{C}$. The proof will be completed by showing $C \subset \bar{G}$, so that $T(\bar{G}) \subset \bar{C} \subset \bar{G}$ (i.e., T maps the bounded, closed, convex set $\bar{G}$ into itself). Since $C = \bigcup_{i=1}^{\infty} C_i$, it suffices to show that for each i, $C_i \subset \bar{G}$. Suppose i is a given positive integer, and $x \in C_i$. We wish to show that $x \in \bar{G}$. The first step toward this end is set off as the following lemma.

LEMMA 2. *For each $\varepsilon > 0$ there exists a positive integer $j_\varepsilon \geqq i$ such that $x \in \bigcap_{n=j_\varepsilon}^{\infty} \{D_{2r}(x_n) \cap D_{2r+\varepsilon}(v_n)\} = F_{j_\varepsilon}$.*

Proof of Lemma 2. Since $x \in C_i$ we have $\| x - x_p \| \leqq 2r$ for all $p \geqq i$. For all $n \geqq i$ we have

$$\| x - v_n \| = \left\| \sum_{p=1}^{n} a_{np} x - \sum_{p=1}^{n} a_{np} x_p \right\| = \left\| \sum_{p=1}^{n} a_{np}(x - x_p) \right\|$$

so that

$$\| x - v_n \| \leqq \sum_{p=1}^{n} a_{np} \| x - x_p \| = \sum_{p=1}^{i-1} a_{np} \| x - x_p \| + \sum_{p=i}^{n} a_{np} \| x - x_p \| \ ,$$

whence, for all $n \geqq i$,

$$\| x - v_n \| \leqq \left(\sum_{p=1}^{i-1} a_{np} \right) \cdot \max_{1 \leqq p \leqq i-1} \| x - x_p \| \ + \ 2r \ .$$

Since i and x are fixed, and since $\lim_n a_{np} = 0$ for each $p = 1, 2, \cdots, i-1$, it is clear that for any $\varepsilon > 0$ there exists a positive integer $j_\varepsilon \geqq i$ such that $\| x - v_n \| \leqq 2r + \varepsilon$ for all $n \geqq j_\varepsilon$. But $n \geqq j_\varepsilon \geqq i$ also implies $\| x - x_n \| \leqq 2r$ since $x \in C_i$. Hence $n \geqq j_\varepsilon$ implies

$$x \in D_{2r}(x_n) \cap D_{2r+\varepsilon}(v_n) \ ,$$

and so $x \in \bigcap_{n=j_\varepsilon}^{\infty} \{D_{2r}(x_n) \cap D_{2r+\varepsilon}(v_n)\} = F_{j_\varepsilon}$.

Proof of Theorem 1 *continued*. We return now to the final problem of showing $x \in \bar{G}$ (see immediately before Lemma 2). Given any $c > 0$, choose $\varepsilon > 0$ such that $p_\varepsilon \leqq c$ (this can be done by Lemma 1, in which $r > 0$ is taken as the r we are using in this proof). For this ε, there exists a positive integer $j_\varepsilon \geqq i$ such that $x \in F_{j_\varepsilon}$ (by Lemma 2). We will show $G_{j_\varepsilon} \cap D_c(x) \neq \phi$. Since c is arbitrary, this will show $x \in \bar{G} = (\bigcup_{i=1}^\infty G_i)^-$. We suppose $G_{j_\varepsilon} \cap D_c(x) = \phi$ and obtain a contradiction. Since $0 \in D_r(0) \subset G_{j_\varepsilon}$, $0 \notin D_c(x)$, and so $0 < c/\|x\| < 1$. Let $t_1 = 1 - (c/\|x\|)$. Then $0 < t_1 < 1$ and $\|t_1x - x\| = (1 - t_1)\|x\| = c$. Since $t_1x \in D_c(x)$, we have $t_1x \notin G_{j_\varepsilon}$. Now $x \in F_{j_\varepsilon} \subset \bigcap_{n=j_\varepsilon}^\infty D_{2r}(x_n) = C_{j_\varepsilon}$, and since C_{j_ε} also contains 0 and is convex, $t_1x \in C_{j_\varepsilon}$. Since $t_1x \notin G_{j_\varepsilon}$ and $t_1x \in C_{j_\varepsilon}$, we have $t_1x \notin \bigcap_{n=j_\varepsilon}^\infty D_{2r}(v_n)$. Let n be a positive integer, $n \geqq j_\varepsilon$, such that $t_1x \notin D_{2r}(v_n)$. Let

$$t_2 = \sup\{t: 0 < t < 1 \quad \text{and} \quad tx \in D_{2r}(v_n)\} .$$

This set of t's is nonempty since $D_r(0) \subset D_{2r}(v_n)$. Since $D_{2r}(v_n)$ is closed, we have $t_2x \in D_{2r}(v_n)$; and it is easily seen from the definition of t_2 that we must have $\|t_2x - v_n\| = 2r$. If $t_2 \geqq t_1$, then, since 0 and t_2x are in the convex set $D_{2r}(v_n)$, we would have $t_1x \in D_{2r}(v_n)$ which is not true. Hence $t_2 < t_1$. Similarly we have $\|x - v_n\| > 2r$, since 0 is in the convex set $D_{2r}(v_n)$ and t_1x is not. Since $x \in F_{j_\varepsilon}$ and since $n \geqq j_\varepsilon$, $x \in D_{2r+\varepsilon}(v_n)$, so we have $2r < \|x - v_n\| \leqq 2r + \varepsilon$. Next we observe that if $t \in (0, 1)$

$$\|(1 - t)(t_2x - v_n) + t(x - v_n)\| = \|[(1 - t)t_2 + t\cdot 1]x - v_n\| > 2r$$

since $t_2 < (1 - t)t_2 + t\cdot 1 < 1$ so that $[(1 - t)t_2 + t\cdot 1]x \notin D_{2r}(v_n)$. With $u = t_2x - v_n$ and $v = x - v_n$ we now have $\|u\| = 2r, 2r < \|v\| \leqq 2r + \varepsilon$, and $\|(1 - t)u + tv\| > 2r$ for all $t \in (0, 1)$. Hence $\|u - v\| = \|t_2x - x\| \leqq p_\varepsilon$ (see Lemma 1). But ε was chosen so that $p_\varepsilon \leqq c$. So we have

$$\|t_2x - x\| = (1 - t_2)\|x\| \leqq p_\varepsilon \leqq c .$$

This gives $t_2 \geqq 1 - (c/\|x\|) = t_1$, which is a contradiction.

For completeness, we include the following theorem which is somewhat stronger than the converse of Theorem 1.

THEOREM 2. *If E is a normed linear space, and if $T: E \to E$ is a nonexpansive mapping, and if T has a fixed point $p \in E$, then for any $x_1 \in E$ and any process $M(x_1, A, T)$, the sequences $\{x_n\}$, $\{v_n\}$ are bounded.*

Proof. For each $n = 1, 2, 3, \cdots$, we have

$$\|x_{n+1} - p\| = \|Tv_n - Tp\| \leqq \|v_n - p\| = \left\|\sum_{j=1}^n a_{nj}(x_j - p)\right\|$$
$$\leqq \sum_{j=1}^n a_{nj}\|x_j - p\| \leqq \max_{j=1,\cdots,n} \|x_j - p\| .$$

Thus $\| x_2 - p \| \leqq \| x_1 - p \|$, $\| x_3 - p \| \leqq \max_{j=1,2} \| x_j - p \| = \| x_1 - p \|$, etc., so that we have $\| x_j - p \| \leqq \| x_1 - p \|$ for all $j = 1, 2, 3, \cdots$; and hence with $b = \| x_1 - p \| + \| p \|$ we get $\| x_j \| = \| (x_j - p) + p \| \leqq \| x_j - p \| + \| p \| \leqq b$ for all $j = 1, 2, 3, \cdots$. Finally,

$$\| v_n \| = \left\| \sum_{j=1}^{n} a_{nj} x_j \right\| \leqq \sum_{j=1}^{n} a_{nj} \| x_j \| \leqq b \cdot \sum_{j=1}^{n} a_{nj} = b$$

for all $n = 1, 2, 3, \cdots$.

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