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DIMENSION ON BOUNDARIES OF ε-SPHERES

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The purpose of this paper is to make somewhat more accessible the topological dimension-theoretic properties of metric spaces. We shall show that any metric for a space can be replaced by a topologically equivalent metric which has the following property: the boundary of any ε-sphere meets each of a specified countable collection of closed, finite-dimensional subsets in a set of lower dimension. An additional property of the new metric is that for any fixed ε, the collection of all ε-spheres is closure-preserving.

In the case of a separable metric space, the result can be sharpened to produce a totally bounded metric with the above properties, and in this case we obtain for each fixed ε at most finitely many distinct ε-spheres.

Our dimension function, denoted by *dim*, will be the covering dimension of Lebesgue. All spaces throughout this paper will be metric, and under this condition the Lebesgue dimension coincides with the inductive dimension in the large of Menger and Urysohn [6]. Now let us consider a space together with countably many closed, finite-dimensional subsets $X_1, X_2, \cdots$; these subsets need not cover X (X itself may not even be finite-dimensional). It is immediate from the Menger-Urysohn definition of dimension that for any point $p \in X$ and any fixed positive integer k, there exists a fundamental system of neighborhoods of p for which the boundaries meet X_k in a set of dimension lower than that of X_k. There is no reason why any of these neighborhoods should be ε-spheres under the given metric; J. Nagata, however, has shown [8] that if any $X_k = X$, then an equivalent metric can be defined on X such that the boundary of any ε-sphere (about any point) has dimension lower than that of X. We shall prove that this condition can be dispensed with and that an equivalent metric can be introduced on X in such a way that the boundary of any ε-sphere meets each of the X_k in a set of lower dimension:

THEOREM 1. *For each* $k = 1, 2, \cdots$ *let* X_k *be a nonvoid closed subset of* X *such that* $\dim X_k = n_k < \infty$. *Then there exists an equivalent metric* ρ *for* X *such that for any* $\varepsilon > 0$, *any* $x \in X$, *and any positive integer* k,

$$\dim [X_k \cap \operatorname{Bdry} S(x, \varepsilon)] \leqq n_k - 1 .$$

We shall define this metric by constructing a uniformity with certain desirable properties. To construct the uniformity we shall need several lemmas, and we make the following definitions to clarify the terminology.

DEFINITION. Let $\mathfrak{G}$ be a collection of subsets of X, and $p \in X$.

(i) local order$_p$ $\mathfrak{G} \leqq n$ if and only if there exists a neighborhood of p which meets at most n members of $\mathfrak{G}$.

(ii) local order $\mathfrak{G} \leqq n$ if and only if for all $x \in X$, local order$_x$ $\mathfrak{G} \leqq n$.

(iii) $\bar{\mathfrak{G}} = \{\bar{G}: G \in \mathfrak{G}\}$.

LEMMA 1.1. *Let F be a closed subset of (X, d), $\dim F \leqq n$, and $\mathfrak{U} = \{U_\alpha: \alpha \in A\}$ a collection of open subsets of X which covers F. Then there exists a collection $\mathfrak{V} = \{V_\alpha: \alpha \in A\}$ of open subsets of X which covers F, refines $\mathfrak{U}$ one-to-one, and such that local order $\mathfrak{V} \leqq n + 1$.*

Proof. Let $\mathfrak{G} = \{G_\alpha: \alpha \in A\}$ be a locally finite open cover of F such that $G_\alpha \subset U_\alpha$ for all $\alpha \in A$. Since $\dim F \leqq n$, there exists a locally finite closed cover $\mathfrak{F} = \{F_\alpha: \alpha \in A\}$ of F such that order $\mathfrak{F} \leqq n + 1$ and $F_\alpha \subset G_\alpha$ for all $\alpha \in A$ ([1], [5]). Then, by [5, Lemma in § 3 and Theorem 1.3] there exists a locally finite collection $\mathfrak{V} = \{V_\alpha: \alpha \in A\}$ of open subsets of X such that order $\bar{\mathfrak{V}} =$ order $\mathfrak{F}$ and for each $\alpha \in A$, $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$. For each $x \in X$ the set

$$X - \bigcup \{\bar{V}_\alpha: x \notin \bar{V}_\alpha\}$$

is a neighborhood of x which meets at most $n + 1$ members of $\mathfrak{V}$, so $\mathfrak{V}$ is the desired collection and the Lemma is proved[1].

LEMMA 1.2. *Let $X_1, \cdots, X_k$ be closed subsets of X such that $\dim X_i = n_i < \infty$ for all $i = 1, \cdots, k$, and let $\mathfrak{U}$ be an open cover of X. Then there exists a locally finite open cover $\mathfrak{V}$ of X satisfying*

(i) $\mathfrak{V} < \mathfrak{U}$

(ii) *for all $i = 1, \cdots, k$ and all $x \in X_i$, local order$_x$ $\mathfrak{V} \leqq n_i + 1$.*

Proof. We shall first prove the lemma for the case $n_1 \leqq \cdots \leqq n_k$. By Lemma 1.1 we obtain an open refinement $\mathfrak{V}_1$ of $\mathfrak{U}$ covering X_1 such that local order $\mathfrak{V}_1 \leqq n_1 + 1$; then the normality of X gives us an open set W_1 such that $X_1 \subset W_1 \subset \bar{W}_1 \subset \bigcup \mathfrak{V}_1$. If we define $\mathfrak{U}_2 = \mathfrak{U} \wedge \{X - \bar{W}_1\}$, we see that $\mathfrak{V}_1 \cup \mathfrak{U}_2$ is an open cover of $X_1 \cup X_2$, and we can hence apply Lemma 1.1 again to obtain an open one-to-one refinement $\mathfrak{V}_2$ which covers $X_1 \cup X_2$ and such that local order $\mathfrak{V}_2 \leqq$

[1] The author is indebted to the referee for shortening the proof of this lemma.

$n_2 + 1$. Continuing in this manner, we finally define the open set W_k such that $\bigcup_{i=1}^k X_i \subset W_k \subset \bar{W}_k \subset \bigcup \mathfrak{B}_k$, and the open collection $\mathfrak{U}_{k+1} = \mathfrak{U} \wedge \{X - \bar{W}_k\}$.

Now let $\mathfrak{B}$ be a locally finite open one-to-one refinement of $\mathfrak{B}_k \cup \mathfrak{U}_{k+1}$; we assert that $\mathfrak{B}$ is the desired cover. The cover $\mathfrak{B}$ satisfies (i), as

$$\mathfrak{B} < \mathfrak{B}_k \cup \mathfrak{U}_{k+1} < (\mathfrak{B}_{k-1} \cup \mathfrak{U}_k) \cup \mathfrak{U}_{k+1}$$
$$< \mathfrak{B}_{k-2} \cup \left(\bigcup_{i=k-1}^{k+1} \mathfrak{U}_i\right) < \cdots < \mathfrak{B}_1 \cup \left(\bigcup_{i=2}^{k+1} \mathfrak{U}_i\right) < \mathfrak{U} .$$

For condition (ii), let $x \in X_i$ and let $N(x, i)$ be an open neighborhood of x meeting at most $n_i + 1$ sets of $\mathfrak{B}_i$. Then we define the open neighborhood $N(x) = N(x, i) \cap (\bigcap_{j=i}^k W_j)$; we need only show that $N(x)$ meets at most $n_i + 1$ sets of $\mathfrak{B}$.

First we show that $N(x)$ meets at most $n_i + 1$ sets of $\mathfrak{B}_{i+1}$. Suppose $V_\alpha \cap N(x) \neq \varnothing$ for some $V_\alpha \in \mathfrak{B}_{i+1}$; then V_α is a subset of the corresponding $U_\alpha \in \mathfrak{B}_i \cup \mathfrak{U}_{i+1}$. As $N(x) \subset W_i$ and $(\bigcup \mathfrak{U}_{i+1}) \cap W_i = \varnothing$, we know $U_\alpha \in \mathfrak{B}_i$. But $N(x) \cap U_\alpha \neq \varnothing$ for at most $n_i + 1$ sets $U_\alpha \in \mathfrak{B}_i$ (as $N(x) \subset N(x, i)$), and hence $N(x) \cap V_\alpha \neq \varnothing$ for at most the corresponding $n_i + 1$ sets $V_\alpha \in \mathfrak{B}_{i+1}$ (as $\mathfrak{B}_{i+1}$ refines $\mathfrak{B}_i \cup \mathfrak{U}_{i+1}$ one-to-one). This argument can be repeated to show that $N(x)$ meets at most $n_i + 1$ sets of $\mathfrak{B}_{i+2}, \cdots, \mathfrak{B}_k$, and finally $\mathfrak{B}$, and we see that $\mathfrak{B}$ satisfies condition (ii).

To prove the lemma in its general form, we rearrange the X_i in such a way as to make their dimensions increase monotonely. Then the above proof will show that the conditions are satisfied, and the lemma is proved (as neither condition is concerned with the order in which the X_i are arranged).

LEMMA 1.3. *For each $i = 1, \cdots, k$, let X_i be a closed subset of X, and let $\mathfrak{U}$ be a locally finite open cover of X such that for all $i \leqq k$ and each $x \in X_i$, local order$_x$ $\mathfrak{U} \leqq m_i$. Then there exists an open cover $\mathfrak{B}$ of X such that*

(i) $\mathfrak{B}^{**} < \mathfrak{U}$,

(ii) *for each $i = 1, \cdots, k$ and every $x \in X_i$, $S^6(x, \mathfrak{B})$ meets at most m_i sets of $\mathfrak{U}$, and*

(iii) *for every $x \in X$, $S^6(x, \mathfrak{B})$ meets only a finite number of sets of $\mathfrak{U}$.*

Proof. For each $i = 1, \cdots, k$ and every $x \in X_i$ we define $N(x, i)$ to be an open neighborhood of x meeting at most m_i sets of $\mathfrak{U}$. We then define $\mathfrak{G}_i = \{X - X_i\} \cup \{N(x, i): x \in X_i\}$, and note that $\mathfrak{G}_i$ is an open cover of X, as X_i is closed. Now for any $x \in X$ we define another set $N(x)$ to be a neighborhood of x meeting only a finite number of members of $\mathfrak{U}$, and define $\mathfrak{G}_{k+1} = \{N(x): x \in X\}$. Now we define $\mathfrak{G} = (\bigwedge_{i=1}^{k+1} \mathfrak{G}_i) \wedge \mathfrak{U}$. The collection $\mathfrak{G}$ is an open cover of X, so we can define $\mathfrak{V}$ to be an open cover of X such that $\mathfrak{V}^{***} < \mathfrak{G}$ ($\mathfrak{V}$ exists by the full normality of X). We shall now prove that $\mathfrak{V}$ is the desired cover.

Condition (i) is clear, as $\mathfrak{V}^{***} < \mathfrak{G} < \mathfrak{U}$. For (ii) we let $i \leqq k$ and let $x \in X_i$; then there exists some $V \in \mathfrak{V}$ which contains x, as $\mathfrak{V}$ covers, X, and there exists a set $V^{***} \in \mathfrak{V}^{***}$ such that

$$\begin{aligned} B^{***} &= S(S(S(V, \mathfrak{V}), \mathfrak{V}^*), \mathfrak{V}^{**}) \supset S(S(S(x, \mathfrak{V}), \mathfrak{V}^*), \mathfrak{V}^{**}) \\ &= S(S^3(S(x, \mathfrak{V}), \mathfrak{V}), \mathfrak{V}^{**}) \supset S(S^3(x, \mathfrak{V}), \mathfrak{V}^{**}) \supset S^6(x, \mathfrak{V}) . \end{aligned}$$

Since $\mathfrak{V}^{***} < \mathfrak{G}$ we know there exists a set $G \in \mathfrak{G}$ such that $V^{***} \subset G$. As $\mathfrak{G} < \mathfrak{G}_i$ and $G \cap X_i \neq \emptyset$, there exists a point $y \in X_i$ such that $G \subset N(y, i)$. Thus $S^6(x, \mathfrak{V}) \subset V^{***} \subset G \subset N(y, i)$, so $S^6(x, \mathfrak{V})$ meets at most m_i elements of $\mathfrak{U}$.

In the same manner we see that, for any $x \in X$, there exist $W^{***} \in \mathfrak{V}^{***}$, $H \in \mathfrak{G}$, and $N(z)$ such that $S^6(x, \mathfrak{V}) \subset W^{***} \subset H \subset N(z)$, so $S^6(x, \mathfrak{V})$ meets only finitely many elements of $\mathfrak{U}$.

Proof of Theorem 1. We shall use Lemmas 1.2 and 1.3 to develop a sequence $\{\mathfrak{U}_i: i = 0, 1, 2, \cdots\}$ of open covers of X with special properties. First we define $\mathfrak{U}_0 = \{X\}$. Now set $\mathfrak{G}_1 = \{S(x, 1/4): x \in X\}$; by Lemma 1.2 there exists a locally finite open cover $\mathfrak{U}_1$ of X satisfying

(1′) $\mathfrak{U}_1 < \mathfrak{G}_1$

(2′) for all $x \in X_1$, local order$_x$ $\mathfrak{U}_1 \leqq n_1 + 1$

(3′) mesh $\mathfrak{U}_1 \leqq 1/2$, and

(4′) $\mathfrak{U}_1^{**} < \mathfrak{U}_0$.

Therefore an application of Lemma 1.3 yields an open cover $\mathfrak{V}_2$ of X such that

(i) $\mathfrak{V}_2^{**} < \mathfrak{U}_1$,

(ii) for all $x \in X_1$, $S^6(x, \mathfrak{V}_2)$ meets at most $n_1 + 1$ members of $\mathfrak{U}_1$, and

(iii) for all $x \in X$, $S^6(x, \mathfrak{V}_2)$ meets only finitely many members of $\mathfrak{U}_1$.

Now set $\mathfrak{G}_2 = \mathfrak{V}_2 \wedge \{S(x, 1/8): x \in X\}$, and apply Lemma 1.2 to obtain a locally finite open cover $\mathfrak{U}_2$ of X satisfying

(1′) $\mathfrak{U}_2 < \mathfrak{G}_2$, and

(2′) for $j = 1, 2$, all $x \in X_j$, local order$_x$ $\mathfrak{U}_2 \leqq n_j + 1$. We note also that $\mathfrak{U}_2$ satisfies

(3′) Mesh $\mathfrak{U}_2 \leqq 1/4$,

(4′) $\mathfrak{U}_2^{**} < \mathfrak{U}_1$,

(5′) for all $x \in X_1$, $S^6(x, \mathfrak{U}_2)$ meets at most $n_1 + 1$ members of $\mathfrak{U}_1$, and

(6′) for all $x \in X$, $S^6(x, \mathfrak{U}_2)$ meets only a finite number of members of $\mathfrak{U}_1$.

By alternating applications of Lemma 1.2 and Lemma 1.3, the same method can be used to construct $\mathfrak{U}_3, \mathfrak{U}_4, \cdots$ with properties analogous to (1′)-(6′) above. This procedure will yield a sequence $\{\mathfrak{U}_i: i = 0, 1, 2, \cdots\}$ of open covers of X satisfying the following four conditions:

(1) $\{X\} = \mathfrak{U}_0 > \mathfrak{U}_1^{**} > \mathfrak{U}_1 > \mathfrak{U}_2^{**} > \cdots$,

(2) for $i = 1, 2, \cdots$, mesh $\mathfrak{U}_i \leqq 1/2^i$,

(3) for $j = 1, 2, \cdots$, all $x \in X_j$, all $i \geqq j$, $S^6(x, \mathfrak{U}_{i+1})$ meets at most $n_j + 1$ members of $\mathfrak{U}_i$, and

(4) for $i = 1, 2, \cdots$, all $x \in X$, $S^6(x, \mathfrak{U}_{i+1})$ meets only finitely many members of $\mathfrak{U}_i$.

We shall use these covers to define certain open subsets of X, and the new metric will be defined in terms of collections of these subsets. For any finite set $0 \leqq m_1 < \cdots < m_p$ of integers and any open subset U of X we define inductively an open set $S_{m_2 \cdots m_p}(U)$ by

$$S_{m_2 \cdots m_p}(U) = U \qquad \text{for } p = 1, \text{ and}$$
$$S_{m_2 \cdots m_p}(U) = S^2(S_{m_2 \cdots m_{p-1}}(U), \mathfrak{U}_{m_p}) \qquad \text{for } p > 1 .$$

Next we define open covers of X by

$$\mathfrak{S}_{m_1} = \mathfrak{U}_{m_1}, \text{ and}$$
$$\mathfrak{S}_{m_1 \cdots m_p} = \{S_{m_2 \cdots m_p}(U): U \in \mathfrak{U}_{m_1}\} \qquad \text{for } p > 1 .$$

Now we define a nonnegatively valued function ρ on $X \times X$ by

$$\rho(x, y) = \inf \{1/2^{m_1} + \cdots + 1/2^{m_p}: y \in S(x, \mathfrak{S}_{m_1 \cdots m_p})\} .$$

J. Nagata has shown [7] that if a function γ is defined in this way from a sequence of covers which possesses properties (1) and (2), then γ is a metric on X equivalent to the given metric.

Now for any sequence $0 \leqq m_1 < m_2 < \cdots$ of integers and any open subset U of X, we define the open set

$$S_{m_2 m_3 \cdots}(U) = \bigcup_{i=1}^{\infty} S_{m_2 \cdots m_i}(U)$$

and the open cover

$$\mathfrak{S}_{m_1 m_2 \cdots} = \{S_{m_2 m_3 \cdots}(U): U \in \mathfrak{U}_{m_1}\} .$$

Now let ε be such that $0 < \varepsilon \leqq 1$. Then ε has a unique nonterminating expansion

(1.1) $\qquad \varepsilon = 1/2^{m_1} + 1/2^{m_2} + \cdots, \quad \text{where} \quad 1 \leqq m_1 < m_2 < \cdots ,$

and we can prove that for any $x \in X$,

$$S(x, \varepsilon) = S(x, \mathfrak{S}_{m_1 m_2 \cdots}) , \tag{1.2}$$

in the same manner used by Nagata [8] (where the ε-sphere is assumed to be with respect to ρ).

We shall use (1.2) to prove that for any $x \in X$ and any

$$k = 1, 2, \cdots, \dim [X_k \cap \text{Bdry}\, S(x, \varepsilon)] \leqq n_k - 1 .$$

Before doing this, we note that for any finite set $1 \leqq m_1 < \cdots < m_p$ of integers and any open subset U of X,

$$S^3(S_{m_2 \cdots m_{p-1}}(U), \mathfrak{U}_{m_p}) \subset S^3(U, \mathfrak{U}_{m_1+1}) ;$$

this relation is due to Nagata [8]. Hence it is immediate that for any sequence of integers $1 \leqq m_1 < m_2 < \cdots$, we have

$$S_{m_2 m_3 \cdots}(U) \subset S^3(U, \mathfrak{U}_{m_1+1}) ,$$

and therefore

$$\overline{S_{m_2 m_3 \cdots}(U)} \subset S^4(U, \mathfrak{U}_{m_1+1}) . \tag{1.3}$$

We are now in a position to prove that

$$\bar{\mathfrak{S}}_{m_1 m_2 \cdots} \text{ is locally finite, for } 1 \leqq m_1 < m_2 < \cdots . \tag{1.4}$$

We can show that if $x \in X$ and $x \in U' \in \mathfrak{U}_{m_1+1}$, then U' is the desired neighborhood of x. For if U' meets $\overline{S_{m_2 m_3 \cdots}(U)}$ for $U \in \mathfrak{U}_{m_1}$, then U' meets $S^4(U, \mathfrak{U}_{m_1+1})$ by (1.3); hence U meets $S^6(x, \mathfrak{U}_{m_1+1})$ (as $U' \in \mathfrak{U}_{m_1+1}$). By condition (4) this can occur for at most finitely many $U \in \mathfrak{U}_{m_1}$, so U' meets at most finitely many members of $\bar{\mathfrak{S}}_{m_1 m_2 \cdots}$.

By using condition (3) instead of (4) in the above proof, we see also that

$$\text{order}_y \bar{\mathfrak{S}}_{m_1 m_2 \cdots} \leqq n_k + 1 \quad \text{for } k \leqq m_1 < m_2 < \cdots \text{ and } y \in X_k . \tag{1.5}$$

We shall use this fact to prove that for $x \in X$, ε as in (1.1), $i \geqq k$, and $y \in [X_k \cap \text{Bdry}\, S(x, \varepsilon)]$, then $\text{order}_y \mathfrak{U}_{m_i} \leqq n_k$. For suppose there existed distinct members $U_1, \cdots, U_{n_k+1}$ of $\mathfrak{U}_{m_i}$ containing y. Now

$$\begin{aligned} y \in \overline{S(x, \varepsilon)} &= \overline{S(x, \mathfrak{S}_{m_1 m_2 \cdots})} \\ &= \overline{\bigcup \{S_{m_2 m_3 \cdots}(U) \colon U \in \mathfrak{U}_{m_1}, x \in S_{m_2 m_3 \cdots}(U)\}} \\ &= \bigcup \{\overline{S_{m_2 m_3 \cdots}(U)} \colon U \in \mathfrak{U}_{m_1}, x \in S_{m_2 m_3 \cdots}(U)\} \end{aligned}$$

by (1.4), so there exists a set $U' \in \mathfrak{U}_{m_1}$ such that

$$
\begin{aligned}
y \in \overline{S_{m_2m_3\cdots}(U')} &= \overline{S_{m_{i+1}m_{i+2}\cdots}(S_{m_2\cdots m_i}(U'))} \\
&= \overline{S_{m_{i+1}m_{i+2}\cdots}(\bigcup\{U \in \mathfrak{U}_{m_i}: U \subset S_{m_2\cdots m_i}(U')\})} \\
&= \overline{\bigcup\{S_{m_{i+1}m_{i+2}\cdots}(U): U \in \mathfrak{U}_{m_i},\ U \subset S_{m_2\cdots m_i}(U')\}} \\
&= \bigcup\{\overline{S_{m_{i+1}m_{i+2}\cdots}(U)}: U \in \mathfrak{U}_{m_i},\ U \subset S_{m_2\cdots m_i}(U')\}\ ,
\end{aligned}
$$

again by (1.4). Hence there exists a set $U \in \mathfrak{U}_{m_i}$ such that $y \in \overline{S_{m_{i+1}m_{i+2}\cdots}(U)}$ and such that $U \subset S_{m_2\cdots m_i}(U') \subset S_{m_2m_3\cdots}(U') \subset S(x, \varepsilon)$; this set U cannot be the same as any one of $U_1, \cdots, U_{n_k+1}$, as each of the latter contains the boundary point y of $S(x, \varepsilon)$. But

$$
y \in S_{m_{i+1}m_{i+2}\cdots}(U_j) \qquad \text{for all } j = 1, \cdots, n_k + 1\ ,
$$

and by the above we know that $y \in \overline{S_{m_{i+1}m_{i+2}\cdots}(U)}$, so

$$
\text{order}_y\, \bar{\mathfrak{S}}_{m_im_{i+1}\cdots} \geqq n_k + 2\ ,
$$

which contradicts (1.5).

To complete the proof of the theorem, we first note that the condition is trivially satisfied by $\varepsilon > 1$, so we may assume $0 < \varepsilon \leqq 1$ and hence ε may be written as in (1.1). For any $x \in X$ and $k = 1, 2, \cdots$, we define $\mathfrak{U}'_{m_i} = \mathfrak{U}_{m_i} \wedge \{X_k \cap \text{Bdry}\, S(x, \varepsilon)\}$ for all $i \geqq k$. Then $\{\mathfrak{U}'_{m_i}: i = k, k\, 1, \cdots\}$ is a sequence of relatively open covers of

$$
X_k \cap \text{Bdry}\, S(x, \varepsilon)
$$

of order $\leqq n_k$ with mesh $\to 0$ as $i \to \infty$ and such that for all $i \geqq k$, $\bar{\mathfrak{U}}'_{m_{i+1}} < \mathfrak{U}'_{m_i}$. Thus by a Theorem of C. H. Dowker and W. Hurewicz [2] we conclude that $\dim\,[X_k \cap \text{Bdry}\, S(x, \varepsilon)] \leqq n_k - 1$.

COROLLARY 2. *If ρ is the metric for X constructed in the proof of Theorem 1, then for each $\varepsilon > 0$ the collection $\{S(x, \varepsilon): x \in X\}$ is closure-preserving.*

Proof. As in Theorem 1 we note that the case $\varepsilon > 1$ is trivial, as $S(x, \varepsilon) = X$ for every $x \in X$. For $0 < \varepsilon \leqq 1$ we can write ε as in (1.1), so for any subset A of X we have

$$
\begin{aligned}
&\overline{\bigcup\{S(x, \varepsilon): x \in A\}} \\
&= \overline{\bigcup\{\bigcup\{S_{m_2m_3\cdots}(U): U \in \mathfrak{U}_{m_1},\ x \in S_{m_2m_3\cdots}(U)\}: x \in A\}} \\
&= \overline{\bigcup\{S_{m_2m_3\cdots}(U): U \in \mathfrak{U}_{m_1},\ A \cap S_{m_2m_3\cdots}(U) \neq \emptyset\}} \\
&= \bigcup\{\overline{S_{m_2m_3\cdots}(U)}: U \in \mathfrak{U}_{m_1},\ A \cap S_{m_2m_3\cdots}(U) \neq \emptyset\} \\
&= \bigcup\{\bigcup\{\overline{S_{m_2m_3\cdots}(U)}: U \in \mathfrak{U}_{m_1},\ x \in S_{m_2m_3\cdots}(U)\}: x \in A\} \\
&= \bigcup\{\overline{S(x, \varepsilon)}: x \in A\}
\end{aligned}
$$

by (1.4), which completes the proof of the corollary.

When we restrict our attention to the separable metric case, we would hope that the new metric would be totally bounded in addition to satisfying the conditions of Theorem 1 and Corollary 2. This is indeed the case if we use finite covers throughout.

THEOREM 3. *Let* (X, d) *be separable and for each* $k = 1, 2, \cdots$ *let* X_k *be a nonvoid closed subset of* X *such that* $\dim X_k = n_k < \infty$. *Then there exists a totally bounded metric* ρ *for* X, *equivalent to* d, *such that for any* $\varepsilon > 0$, *any* $x \in X$, *and any positive integer* k, $\dim [X_k \cap \text{Bdry}\, S(x \in \varepsilon)] \leqq n_k - 1$.

We shall need to restate Lemmas 1.1–1.3 in terms of finite covers before we can prove Theorem 3. In each case only a minor adjustment is required.

LEMMA 3.1. *Let* F *be a closed subset of* (X, d), $\dim F \leqq n$, *and* $\mathfrak{U}$ *a finite open cover of* F. *Then there exists a finite open one-to-one refinement* $\mathfrak{V}$ *of* $\mathfrak{U}$, *covering* F, *and such that local order* $\mathfrak{V} \leqq n + 1$.

Proof. The proof of Lemma 1.1 can be used without modification, as a one-to-one refinement of a finite cover must itself be finite.

LEMMA 3.2. *Let* $X_1, \cdots, X_k$ *be closed subsets of* X *such that* $\dim X_i = n_i < \infty$ *for all* $i = 1, \cdots, k$, *and let* $\mathfrak{U}$ *be a finite open cover of* x. *Then there exists a finite open cover* $\mathfrak{V}$ *of* X *satisfying*

(i) $\mathfrak{V} < \mathfrak{U}$, *and*

(ii) *for all* $i = 1, \cdots, k$ *and all* $x \in X_i$, *local order*$_x$ $\mathfrak{V} \leqq n_i + 1$.

Proof. We use the proof of Lemma 1.2, substituting Lemma 3.1 for Lemma 1.1. Then $\mathfrak{V}_1$ and $\mathfrak{U}_2$ are finite, so $\mathfrak{V}_1 \cup \mathfrak{U}_2$ and hence $\mathfrak{V}_2$ are finite. Similarly, $\mathfrak{V}_k \cap \mathfrak{U}_{k+1}$ is finite, so the one-to-one refinement $\mathfrak{V}$ is itself finite.

LEMMA 3.3. *For each* $i = 1, \cdots, k$, *let* X_i *be a closed subset of* X, *and let* $\mathfrak{U}$ *be a finite open cover of* X *such that for all* $i \leqq k$ *and each* $x \in X_i$, *local order*$_x$ $\mathfrak{U} \leqq m_i$. *Then there exists a finite open cover* $\mathfrak{V}$ *of* X *such that*

(i) $\mathfrak{V}^{**} < \mathfrak{U}$, *and*

(ii) *for each* $i = 1, \cdots, k$ *and every* $x \in X_i$, $S^6(x, \mathfrak{V})$ *meets at most* m_i *sets of* $\mathfrak{U}$.

Proof. For each $x \in X$ we define the set

$$N(x) = \bigcap \{U \in \mathfrak{U}: x \in U\} - \bigcup \{\bar{U} \in \bar{\mathfrak{U}}: x \notin \bar{U}\} ;$$

for all $i \leqq k$, $N(x)$ is a neighborhood of x which meets at most m_i members of $\mathfrak{U}$. Furthermore, there are only finitely many distinct $N(x)$, as there are only finitely many ways in which elements of the finite set $\mathfrak{U}$ can be combined, so for each $i = 1, \cdots, k$, the family

$$\mathfrak{G}_i = \{X - X_i\} \cup \{N(x): x \in X_i\}$$

is a finite open cover of X (assuming we count each distinct element only once). Hence $\mathfrak{G} = (\bigwedge_{i=1}^{k} \mathfrak{G}_i) \wedge \mathfrak{U}$ is a finite open cover of X.

To find a finite open cover $\mathfrak{V}$ such that $\mathfrak{V}^{***} < \mathfrak{G}$, it suffices to establish that any finite open cover has a finite Δ-refinement. This follows from a proof of K. Morita [5, Th. 1.2], in which a Δ-refinement is constructed from intersections of binary covers; if the original cover were finite, then the total number of possible intersections would be finite, so the resulting Δ-refinement would have to be a finite cover.

The proof that $\mathfrak{V}$ satisfies (i) and (ii) is exactly the same as in Lemma 1.3, and the lemma is proved.

Proof of Theorem 3. We shall closely follow the proof of Theorem 1, using Lemmas 3.2 and 3.3 (instead of 1.2 and 1.3) to construct a sequence $\{\mathfrak{U}_i: i = 0, 1, 2, \cdots\}$ of finite open covers which satisfies conditions (1)–(4). This can be done exactly as in Theorem 1 if we know that for each positive integer i, there exists a finite cover of X of $1/2^i$-spheres. Although this may not be possible with respect to the given metric d, the fact that X is separable guarantees the existence of a totally bounded metric d' for X which is equivalent to d [3, Th. V4]. Therefore we can proceed as in Theorem 1 to construct $\{\mathfrak{U}_i: i = 0, 1, 2, \cdots\}$ and then to define the metric ρ which is equivalent to d' and hence to d. The proof that the boundaries of the ε-spheres (with respect to ρ) meet the X_k in sets of lower dimension is exactly the same as in Theorem 1, so to prove Theorem 3 it suffices to show that ρ is totally bounded.

We need only consider ε such that $0 < \varepsilon \leqq 1$, which we can write as in (1.1). But $\{S(x, \varepsilon): x \in X\} = \{S(x, \mathfrak{S}_{m_1 m_2 \ldots}): x \in X\}$ by (1.2), and $\mathfrak{S}_{m_1 m_2 \ldots}$ is finite as $\mathfrak{U}_{m_1}$ is finite, so there are only finitely many distinct sets of the form $S(x, \mathfrak{S}_{m_1 m_2 \ldots}) = S(x, \varepsilon)$. This proves Theorem 3, and also proves the following analogue of Corollary 2:

COROLLARY 4. *If ρ is the metric for X constructed in the proof of Theorem 3, then for each $\varepsilon > 0$ the collection $\{S(x, \varepsilon): x \in X\}$ is finite.*

BIBLIOGRAPHY

1. C. H. Dowker, *Lebesgue dimension of a normal space*, Bull. Amer. Math Soc. **52** (1946), 243 (abstract).

2. C. H. Dowker and W. Hurewicz, *Dimension of metric spaces*, Fund. Math. **43** (1956), 83-88.

3. W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton, 1941.

4. S. Lefschetz, *Algebraic Topology*, Amer. Math. Soc. Colloquium, Vol. XXVII, 1942.

5. K. Morita, *On the dimension of normal spaces II*, J. Math. Soc. of Japan, **2** (1950), 16-33.

6. K. Morita, *Normal families and dimension theory for metric spaces*, Math. Ann. **128** (1954), 350-364.

7. J. Nagata, *Note on dimension theory for metric spaces*, Fund. Math. **45** (1958), 143-181.

8. ———, *Two theorems for the n-dimensionality of metric spaces*, Comp. Math. **15** (1963), 227-237.

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