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A CO-TOPOLOGICAL APPLICATION TO MINIMAL SPACES

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A space (X, τ) which satisfies a topological property P is said to be minimal-P if $T = \{\tau' \mid \tau'$ is a P-topology on X; $\tau' \lneqq \tau\} = \emptyset$. For example, a Hausdorff space (X, τ) is minimal Hausdorff if there exists no Hausdorff topology on X which is strictly weaker than τ. The purpose of this paper is to show that for certain properties one need only consider a subset of T "induced" by τ to determine if (X, τ) is minimal-P.

Notation. Let β be an open base for the space (X, τ). τ_β will denote the topology on X generated by the subbase $\{X \backslash Cl_\tau B \mid B \in \beta\}$.

REMARK. J. de Groot in his investigation for a general classification of Baire spaces considered the above topologies (cf. [1], [4]). These topologies have come to be known as co-topologies.

DEFINITIONS. A filter base is regular if it is open and equivalent to a closed filter base.

A filter base $\mathscr{U}$ is Urysohn if for each nonadherent point a, there exists a neighborhood V and $G \in \mathscr{U}$ such that $Cl_\tau V \cap Cl_\tau G = \emptyset$.

REMARK. In this paper, the Bourbaki convention for the topological separation properties will be observed; specifically, all spaces are assumed to be Hausdorff.

The proof of the following lemmas are left to the reader. A proof for the regular case of Lemma 1 is similar to the proof of Theorem 2 in [3].

LEMMA 1. *Let (X, τ) be a Hausdorff (Urysohn; regular) space; let $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ be a nonconvergent open (Urysohn; regular) filter base with unique adherent point x_0; let $\beta = \mathscr{N} \cup \mathscr{M}$ where*

$$\mathscr{N} = \{N \mid N \in \tau \text{ and } x_0 \in \mathrm{Cl}_\tau N\}$$

and

$$\mathscr{M} = \{M \mid M \in \tau \text{ and } M \subset X \backslash \mathrm{Cl}_\tau U_\alpha \text{ for some } \alpha \in A\} .$$

Then (i) *β is a base for τ; and*

(ii) *τ_β is a Hausdorff (Urysohn; regular) topology strictly* *weaker than τ.*

LEMMA 2. *Let (X, τ) be a normal (completely normal) space; let $\mathscr{U}$ be a nonconvergent regular filter base with unique adherent point x_0; let β be defined as in Lemma 1. Then τ_β is a normal (completely normal) topology strictly weaker than τ.*

In the following theorem P denotes any of the following properties: (i) Hausdorff, (ii) Urysohn, (iii) regular, (iv) completely regular, (v) normal, (vi) completely normal, (vii) locally compact. In [2], [3], [5] it is shown that there exist minimal Hausdorff, minimal Urysohn, and minimal regular spaces which are not compact, while for properties (iv) through (vii) mininal-P is equivalent to compactness.

THEOREM. *A P-space (X, τ) is minimal-P if and only if $\{\tau_\beta \mid \tau_\beta$ is P; $\tau_\beta \neq \tau\} = \varnothing$.*[1]

Proof. Necessity, in each case, follows from the fact that $\tau_\beta \leqq \tau$ for every open base β.

Sufficiency for property (i) through (iii): Suppose (X, τ) is not minimal Hausdorff (Urysohn; regular). Then there exists an open (Urysohn; regular) filter base $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ with uniques adherent point x_0, which does not converge (see [5], [2]). By Lemma 1, there exists a base β for τ such that $\tau_\beta \lneqq \tau$ and τ_β is Hausdorff (Urysohn; regular).

Sufficiency for completely regular[2]: Suppose (X, τ) is not compact.

Let (Y, τ') denote a compact extension of (X, τ). Take and fix $p \in Y \backslash X$. Let $\mathscr{S}$ be the filter base of open neighborhoods of p, and $\mathscr{S}^*$ denote the trace of $\mathscr{S}$ in X. Considered as a filter base in (Y, τ'), $\mathscr{S}^*$ has a unique adherent point, namely p. Thus $\mathscr{L}^*$ has no adherent point in (X, τ). Fix and element x_0 in X. Let $\beta = \mathscr{N} \cup \mathscr{M}$ where $\mathscr{N} = \{N \mid N \in \tau$ and $x_0 \in Cl_\tau N\}$ and $\mathscr{M} = \{M \mid M \in \tau$ and $M \subset X \backslash Cl_\tau S^*$ for some $S^* \in \mathscr{S}^*\}$. One can show β is an open base for τ. Similarly one can show that $\mathscr{K} = \{X/Cl_\tau H \mid H \in \mathscr{N} \cup \mathscr{M}\}$ is a base for τ_β.

We will now show that $\tau_\beta \neq \tau$ and (X, τ_β) is completely regular. Let us first note that since (X, τ) is regular and since $\mathscr{N} \subset \beta$, then $G \in \tau_\beta$ whenever $G \in \tau$ and $x_0 \notin G$. Hence if f is continuous on (X, τ) then f is continuous everywhere on (X, τ_β) except possibly at x_0. Now there exists $S^* \in \mathscr{S}^*$ such that $x_0 \notin Cl_\tau S^*$. Since τ is regular, then there exists $U \in \tau$ such that $x_0 \in U$ and $Cl_\tau U \cap Cl_\tau S^* = \varnothing$. Since any element of τ_β which contains x_0 must meet S^*, then $U \notin \tau_\beta$. Thus

[1] The result for p = Hausdorff was independently obtained by G. Strecker.

[2] The technique used by Berri in [2] to show that a space is compact if it is minimal completely regular is extensively used in this proof.

$\tau_\beta \neq \tau$.

We complete the proof by showing τ_β is completely regular. Take $b \in X$ and $X \backslash Cl_\tau H \in \mathscr{K}$ where $b \notin X \backslash Cl_\tau H$ and $H \in \mathscr{N} \cup \mathscr{M}$. We wish to show there exists a continuous, real-valued function f on (X, τ_β), such that $f(b) = 1$ and $f(x) = 0$ for all $x \in Cl_\tau H$. Suppose $H \in \mathscr{N}$. Then $x_0 \in Cl_\tau H$. Let $S^* \in \mathscr{S}^*$ be such that $b \notin Cl_\tau S^*$; Since (X, τ) is regular, then there exists $V \in \tau$ such that $b \in V$ and

$$Cl_\tau V \cap Cl(H \cup S^*) = \varnothing .$$

Since (X, τ) is completely regular, then there exists a continuous, real-valued function f such that $f(b) = 1$ and $f(x) = 0$ for all $x \in X \backslash V$. By a previous remark, f is continuous at every point of (X, τ_β) except possibly at $x = x_0$. We will now show f is continuous at $x = x_0$. Now for all $x \in X \backslash V$, $f(x) = 0$. Since $Cl_\tau V \cap Cl_\tau(H \cup S^*) = \varnothing$, then $f(x) = 0$ for all $x \in X \backslash Cl_\tau V$. Thus f is continuous at all $x \in X \backslash Cl_\tau V$, and hence at all $x \in Cl_\tau(H \cup S^*)$. Therefore f is continuous at x_0.

Similarly one can show that if $H \in \mathscr{M}$, then there exists a real-valued continuous function f on (X, τ_β) such that $f(b) = 1$ and $f(x) = 0$ for each $x \in Cl_\tau H$.

Sufficiency for properties (v) and (vi): Suppose the normal (completely normal) space (X, τ) is not compact. Then X is not minimal regular since a minimal regular normal (completely normal) space is minimal completely regular. Hence there exists a nonconvergent regular filter base $\mathscr{U}$ with a unique adherent point x_0. By Lemma 2, there exists a base β for τ such that $\tau_\beta \lneqq \tau$ and τ_β is normal (completely normal).

Sufficiency for locally compact: Suppose (X, τ) is not minimal locally compact (i.e., not compact). Let (Y, τ') denote the Alexandroff compactification of X with $Y = X \cup \{p\}$ where $p \notin X$. Fix an element x_0 in X and construct $\beta = \mathscr{N} \cup \mathscr{M}$ as in the proof of sufficiency for completely regular spaces. One can show $\tau_\beta \lneqq \tau$ and τ_β is locally compact, and in fact, compact.

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