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A THEOREM OF ROLLE'S TYPE IN E^n FOR FUNCTIONS OF THE CLASS C^1

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In a note recently published W. Leighton presents the following theorem.

THEOREM 1. Let $f(x) = (x_1, \cdots, x_n)$ be of class C^2 in E^n. Suppose that $f(x)$ has an isolated relative minimum at $x = 0$, and that $f(0) = 0$. If there is a point $x = 0$ where $f(x) = 0$, then $f(x)$ must have at least one critical point, finite or infinite, in addition to that at the origin.

The proof employed by Prof. Leighton of Theorem (1) is based upon the theory of Morse and for this the condition that $f(x)$ belongs to class C^2 is essential.

In what follows we shall give a proof of Theorem (1) for functions of class C^1.

This proof will be based upon the so-called "extension theorem" proved in the book by N. P. Bhatia and G. P. Szegö [1, p. 362].

In the sequel, small latin letters will denote vectors (notable exception t = time), small greek letters scalars and capital letters sets.

With E^n we shall denote the n-dimensional euclidean space and with $E^n \cup \{\infty\}$ the n-dimensional euclidean space with the point at infinity adjoined; thus by saying that $y \in E^n \cup \{\infty\}$ is such that $g(y) = 0$ we mean that either exists $y \in E^n$ such that $g(y) = 0$ or that there exists a sequence $\{y^n\} \subset E^n$, $\|y^n\| \to \infty$, such that $g(y^n) \to 0$. With $S(\{y\}, \alpha)$ we shall denote the sphere of radius $\alpha > 0$ with center in the point $y \in E^n$.

Let us first state the extension theorem proved in [1].

(2) EXTENSION THEOREM. *Let $v = \phi(x)$ and $w = \Psi(x)$ be real-valued functions defined on E^n. Let $M \subset E^n$ be a compact set. Assume that*

(i) $v = \phi(x) \in C^1$

(ii) $\phi(x) = 0$ *for* $x \in M$

(iii) *for any sequence* $\{x^n\}$, $\Psi(x^n) \to 0$ *implies* $x^n \to M$.

(iv) $\Psi(x) = \langle \text{grad}\, \phi(x), f(x) \rangle$.

Then whatever the local stability properties of M for the system

$$\dot{x} = f(x) \tag{3}$$

may be, these properties are global.

The complete proof and the historical background of this theorem can be found in [1]. For the case in which $\phi(x) \in C^2$ or for the case in which the solutions of the differential system (3) define a dynamical system, the proof is due to Bhatia and Szegö [2]. The proof for the case $\phi(x) \in C^1$ makes use of a lemma proved by C. Olech [1, Lemma 3. 8.15].

We can then prove the following theorem

THEOREM 4. *Let $v = \phi(x)$ be a real-valued function defined on E^n. Assume that*

(i) $v = \phi(x) \in C^1$

(ii) $\phi(0) = 0$

(iii) $\phi(x) \neq 0, x \neq 0, x \in S(\{0\}, \eta), \eta > 0$.

(iv) $\phi(x^0) = 0, x^0 \neq 0$.

There then exists a point $x^c \in E^n \cup \{\infty\}, x^c \neq 0$, such that

$$\text{grad } \phi(x_c) = 0 . \tag{5}$$

Proof. Consider the differential equation

$$\dot{x} = -\text{grad } \phi(x) , \tag{6}$$

the real-valued functions $v = \phi(x)$ and

$$\Psi(x) = - \langle \text{grad } \phi(x), \text{grad } \phi(x) \rangle = - (\text{grad } \phi(x))^2 . \tag{7}$$

Suppose that grad $\phi(x^0) \neq 0$, if not, the theorem is trivially true. Assume that for any sequence $\{x^n\}$, $\Psi(x^n) \to 0$ implies $x^n \to 0$, i.e., that grad $\phi(x) \neq 0$, $x \in (E^n \cup \{\infty\})\backslash\{0\}$. Then from the extension theorem (3) it follows that $\{0\}$ is either globally asymptotically stable or globally completely unstable depending from the sign properties of $\phi(x)$ in $S(\{0\}, \eta)$. On the other hand, if the thesis of the theorem is not true, there must exist a set $M \subset E^n$, $M\backslash\{x^0 \cup \{0\}\} \neq \phi$, $x^0 \in M$, which is unbounded, such that $\phi(x) = 0$ for all $x \in M$. Assume that $\phi(x)$ is indefinite and changes its sign on M, if not, $\phi(x)$ would be semidefinite and the theorem proved. If $\phi(x)$ changes its sign on M, then there exist $y \in M$, whose stability properties are opposite to the local stability properties of $\{0\}$. For instance if $\{0\}$ is locally asymptotically stable y is unstable and such that $\lim_{t \to +\infty} x(y, t) \neq 0$, where $x(y, t)$ is a solution of equation (6) with $x(y, 0) = y$. This contradicts the fact that $\{0\}$ is either globally asymptotically stable or globally completely unstable.

In [5] it has been proved that Theorem (4) implies the extension Theorem (2). We have now proved that the Extension Theorem (2) implies, and it is therefore equivalent to, Theorem (4).

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