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MAXIMAL NONNORMAL CHAINS IN FINITE GROUPS

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In a finite group G, knowledge of the distribution of the subnormal subgroups of G can be used, to some extent, to describe the structure of G. Here we show that if G is a finite nonnilpotent, solvable group such that every upper chain of length n in G contains a proper subnormal entry then:
(1) **the nilpotent length of G is less than or equal to n.**
(2) **$|G|$ has at most n distinct prime divisors, furthermore if $|G|$ has n distinct prime divisors, then G has abelian Sylow subgroups.**
(3) **if $|G|$ has at least $(n-1)$ distinct prime divisors, then G is a Sylow Tower Group, for some ordering of the primes.**
(4) **$r(G) \leq n$, where $r(G)$ denotes the minimal number of generators for G.**

Before proving these results it is necessary to have a few lemmas concerning upper chains and subnormal subgroups. All groups are assumed to be finite.

An *upper chain* of length r in G is a sequence of subgroups, $G = G_0 \supset G_1 \supset \cdots \supset G_r$ where for each i, G_i is maximal in G_{i-1}. Janko [4] has described the finite groups in which every upper chain of length four terminates in a normal subgroup. We define the function $h(G)$ as follows:

DEFINITION 1. $h(G) = n$ if every upper chain in G of length n contains a proper ($\neq G$) subnormal entry and there exists at least one upper chain of length $(n-1)$ which contains no proper subnormal entry.

Note that since a subnormal maximal subgroup is normal, $h(G) = 1$ if and only if G is nilpotent. From the definition it is clear that if $h(G) = n$ then there exists an upper chain of length n such that only the terminal entry is subnormal in G. Such a chain is called an *h-chain for G*. The following two lemmas are simple modifications of Lemmas 2, 3 [2].

LEMMA 1. *If H is a nonnormal maximal subgroup of G, then* $h(H) \leqq h(G) - 1$.

LEMMA 2. *If N is a normal subgroup of G, then $h(G/N) \leqq h(G)$.*

LEMMA 3. *If $G = H \times K$, where $h(H) \geqq 2$, then $h(G) \geqq h(H) + m$,* *where m is the length of the longest chain in K.*

Proof. Let $H = H_0 \supset H_1 \supset \cdots \supset H_r$ be an h-chain for H and $K = K_0 \supset K_1 \supset \cdots \supset K_m = \langle 1 \rangle$ be the longest chain in K. Then in $H \times K$ the upper chain:

$$H_0 \times K_0 \supset H_1 \times K_0 \supset H_1 \times K_1 \supset H_1 \times K_2 \supset \cdots \supset H_1 \times K_n = H_1 \supset H_2 \cdots \supset H_r ,$$

has $(r + m)$ entries. If one of these entries is subnormal in G, then its projection on H is subnormal in H. However these projections are simply $H_1, H_2, \cdots, H_r$, and of these, only H_r is subnormal in H. Thus $h(H \times K) \geqq r + m$.

For reference it is convenient to note here the notion of a *Saturated Formation* as defined by Gaschutz [3].

DEFINITION 2. A *Formation* $\mathscr{F}$ is a collection of finite solvable groups satisfying:

(1) $\langle 1 \rangle \in \mathscr{F}$.
(2) If $G \in \mathscr{F}$, and $N \lhd G$, then $G/N \in \mathscr{F}$.
(3) If $G/N_i \in \mathscr{F}$, $i = 1, 2$, then $G/(N_1 \cap N_2) \in \mathscr{F}$.

A formation $\mathscr{F}$ is called *saturated* if given a group G which does not belong to $\mathscr{F}$, if M is a minimal normal subgroup of G, such that $G/M \in \mathscr{F}$, then M has a complement in G, and all such complements are conjugate. Gaschütz showed later that conjugacy follows from existence and furthermore saturation can be characterized as follows:

A formation $\mathscr{F}$ is saturated if whenever $G/\phi(G)$ belongs to $\mathscr{F}$ then G also belongs to $\mathscr{F}$, where $\phi(G)$ denotes the Frattini subgroup of G. The collection of all finite solvable groups constitutes a formation, as does the collection of all finite nilpotent groups. This can be extended in a natural way to a theorem on all groups having a given bound on nilpotent length. By the *nilpotent length* (denoted by $l(G)$) of a solvable group we mean the length of the shortest normal chain with nilpotent factors. Example 4.5 [3] shows that the set, $\mathscr{F}_n$, of all solvable groups G such that the nilpotent length of G is less than or equal to n is a saturated formation for each n.

Theorem 1 shows the relation between $h(G)$ and $l(G)$.

THEOREM 1. *If G is a solvable group then $l(G) \leqq h(G)$.*

Proof. The proof is by induction on $h(G)$, the theorem being trivially true if $h(G) = 1$. So suppose the theorem is true for all groups K such that $h(K) \leqq (n-1)$ and is false for some group K where $h(K) = n$. Among such groups let G be one of minimal order. We show that such a group G cannot exist. Let M be a minimal normal subgroup of G. By Lemma 2, $h(G/M) \leqq h(G) = n$ so that by the minimality of G, $l(G/M) \leqq n$. If N is another minimal normal subgroup of G, then by the same argument $l(G/N) \leqq n$. By the saturated formation property $l(G/(M \cap N)) \leqq n$. Since $M \cap N = \langle 1 \rangle$, this is impossible, so M is the unique minimal normal subgroup of G. By the saturated formation property and minimality of G, M has a complement L in G. $G = ML$, $M \cap L = \langle 1 \rangle$. Since M is the unique minimal normal subgroup of G, L is a nonnormal, maximal subgroup. By Lemma 1 $h(L) \leqq (n-1)$. Hence by the induction hypothesis, $l(L) \leqq (n-1)$. Since $L \cong G/M$ and M is abelian $l(G) \leqq n$. This is a contradiction, therefore G does not exist.

By looking at the holomorph of a group of prime order p where $p = 2^n k + 1$ we see that no converse to Theorem 1 is possible, i.e., it is possible to have $l(G) = 2$ and $h(G)$ arbitrarily large.

For notation purposes let $\pi(G:K)$ denote the number of distinct prime divisors of $[G:K]$, with $\pi(G:\langle 1 \rangle)$ denoted simply by $\pi(G)$. Then there is a relationship between $h(G)$ and $\pi(G)$.

THEOREM 2. *If G is a solvable group such that $h(G) < \pi(G)$ then $h(G) = 1$, i.e., G is nilpotent.*

Proof. Suppose the theorem is false and let G be a counterexample. Let P be a nonnormal Sylow subgroup of G. Consider an upper chain from G through $N_G(P)$ to P. Since G is solvable this chain is at least $(\pi(G) - 1)$ entries long. Thus by hypothesis this chain must contain a subnormal entry. However $N_G(P)$ is not contained in a proper subnormal subgroup, and if $N_G(P)$ contains a subnormal subgroup containing P, P is subnormal. But a subnormal Sylow subgroup is normal. Thus we have a contradiction so G cannot exist.

S_3, the symmetric group on three symbols, has: $h(S_3) = \pi(S_3) = 2$, showing that the arithmetic condition of Theorem 2 cannot be relaxed. However this does suggest the question of what structure follows from the hypothesis that $h(G) - \pi(G)$ is small. G is called a *Sylow Tower Group* (STG) if G has a normal Sylow subgroup, and every homomorphic image of G has a normal Sylow subgroup.

THEOREM 3. *If G is solvable and $h(G) - \pi(G) \leqq 1$, then G is a Sylow Tower Group for some ordering of the prime divisors of G.*

Proof. The proof is by induction on $h(G)$, the theorem being trivially true if $h(G) = 1$. Suppose the theorem is true for all groups K for which $h(K) < n$, and is false for some group K for which $h(K) = n$. Among such groups let G be one of minimal order. We will show that G cannot exist thereby proving the theorem. G must satisfy the following:

(1) Every nonnormal maximal subgroup of G is STG.

Let H be a nonnormal maximal subgroup of G. $\pi(G:H) = 1$ so $\pi(H) \geqq (n-2)$. By Lemma 1, $h(H) \leqq (n-1)$. Thus by the induction hypothesis H is STG.

(2) G does not possess a normal Sylow subgroup.

Suppose P is a normal Sylow subgroup of G. Let K be a subgroup maximal with respect to the properties: $K \supseteqq P$, $K \lhd G$, K is a Hall subgroup of G, K is STG. Then $\langle 1 \rangle \subset K \subset G$, and G/K does not possess a normal Sylow subgroup since K is maximal with respect to the property of being STG. K is a normal Hall subgroup so K has a complement L. $L \cong G/K$ so L is not STG. L is Hall so $N(L)$ is abnormal, so if $N(L) \neq G$, $N(L)$ is contained in an abnormal maximal subgroup whence by (1) is STG. This contradicts the fact that L is not STG, so $N(L) = G$, and $G = H \times L$. Suppose $\pi(K) = m$, then $\pi(L) = \pi(G) - m$ so $h(L) \geqq \pi(G) - m + 2$ by induction. Hence by Lemma 3, $h(G) \geqq (\pi(G) - m + 2) + m = \pi(G) + 2$ which is a contradiction, so P does not exist.

(3) G possesses a unique minimal normal subgroup M; furthermore G/M is supersolvable.

Let M be a minimal normal subgroup of G. By (2), M is not a Sylow subgroup. Thus $\pi(G/M) = \pi(G)$. $h(G/M) \leqq h(G)$ so by the minimality of the order of G, G/M is STG. Now the groups having a Sylow tower for a given ordering of the primes constitute a saturated formation [1]. Thus M has a complement L in G, and L is STG. Let $L = L_1 \rhd L_2 \rhd \cdots \rhd L_{n-1} \rhd L_n \rhd \cdots \rhd \langle 1 \rangle$ be a Sylow tower for L. We refine this chain and adjoin G to obtain an upper chain. If for any $i < n$, L_{i-1}/L_i is not simple, L_n is subnormal in G. However this will give rise to a normal Sylow subgroup in G, contradicting (2). Hence each $L_{(i-1)}/L_i$ is of prime order and L_n is cyclic. Hence L is supersolvable. We have shown that the factor group to a minimal normal subgroup is supersolvable. Therefore if G has two distinct minimal normal subgroups N_1 and N_2, then G/N_i is supersolvable $i = 1, 2$, so that $G/(N_1 \cap N_2)$ is supersolvable. Since $N_1 \cap N_2 = \langle 1 \rangle$ this implies that G is supersolvable. However supersolvable groups are STG, so M is unique.

Using the same notation as in (3), since L does not contain a nontrivial normal subgroup, L does not contain a nontrivial subnormal subgroup thus from the chain obtained above we see that $|L|$ is square free.

Since L is supersolvable we may assume that the Sylow subgroup for the largest prime is normal in L. Let $|M| = p^{\alpha}$, p prime. Suppose Q is a Sylow q-subgroup of G where q is the largest prime divisor of $|G|$. We may assume $p \neq q$, $Q < L$, in fact $N(Q) = L$.

(4) $|G| = 24$, $h(G) = 3$.

Let P be a Sylow p-subgroup of G. Then since $|L|$ is square free, $|P| = |M| \cdot p$.

We may assume that P contains a Sylow p-subgroup T of L. Then since T is not subnormal, P contains a maximal (in P) non-subnormal (in G) subgroup J. $P = MJ$, $[P : M \cap J] = p^2$. Now J is $(n-1)$-th maximal and not subnormal, and $h(G) = n$, thus each maximal subgroup of J is subnormal in G. Hence J has just one maximal subgroup, and so J is cyclic. However M is elementary abelian, therefore $|M \cap J| = 1$ or $|M \cap J| = p$. Thus $|M| = p$ or p^2. However $|M| = [G : L] \equiv 1 \pmod{q}$, by the Sylow theorems. Now $p < q$ so $|M| = p^2$. Since $q \mid (p^2 - 1)$, $q = p + 1$, so that $q = 3$, $p = 2$, and $|G| = 24$, $h(G) = 3$.

(5) The final contradiction.

Note that G is not S_4 since $h(S_4) = 4$. Now in G the subgroups of order 2 are subnormal. Thus the normalizer of the Sylow 3-subgroup is cyclic. By Burnside's theorem the 3-Sylow subgroup has a normal complement contrary to (2). Thus G does not exist.

Note that $h(S_4) = 4$, $\pi(S_4) = 2$ and S_4 is not STG.

In the special case where $h(G) = \pi(G)$, even more can be said.

THEOREM 4. *If G is solvable and $h(G) = \pi(G) \geqq 2$, then the Sylow subgroups of G are cyclic or elementary abelian. Furthermore if there exist at least two nonisomorphic nonnormal Sylow subgroups of G, then all nonnormal Sylow subgroups of G are of prime order.*

Proof. Let $\pi(G) = h(G) = n$. Let P be a nonnormal Sylow subgroup of G. As in Theorem 2, $\pi(G : P) = (n-1)$ so that P is at least $(n-1)$-th maximal in G.

Considering a chain through $N(P)$ to P, as in the proof of Theorem 2 we see that this chain can have at most $(n-1)$ entries, hence exactly $(n-1)$ entries. Therefore P is cyclic, since every maximal subgroup of P is subnormal in G, and P is not. In this chain we have $(n-1)$ distinct primes and $(n-1)$ entries. Therefore each entry is a Sylow complement in its predecessor. However this implies that the Sylow subgroup is elementary abelian. If there were two non-normal Sylow subgroups, then by this same argument P is elementary abelian. However P is cyclic so that P is of prime order.

Note that under the hypothesis of Theorem 4, if we let K denote the product of all the normal Sylow subgroups in G, then K is abelian and G/K has cyclic Sylow subgroups, so that $l(G) \leqq 3$. Also we should note that an extension of the Quaternion group of order 8 by an automorphism which permutes the subgroups of order 4 will yield a non-A-group G having $h(G) = 3$ and $\pi(G) = 2$.

To see how these theorems restrict the structure of a solvable group in a particular case, consider the groups G having $h(G) = 2$.

THEOREM 5. *Suppose* $h(G) = 2$. *Then* $G = PQ$; P *and* Q *are Sylow subgroups of* G; P *is a minimal normal subgroup*; Q *is cyclic*; Q_1, *the maximal subgroup of* Q, *is normal in* G, *in fact*, $Q_1 = \phi(G) = Z(G)$.

Note that a theorem due to Rose [5] shows that $h(G) = 2$ implies solvability for G. More generally, we can effectively duplicate the proofs of the theorems in [2] to prove:

THEOREM 6. *If* G *is a finite group, and* $h(G) \leqq 3$, *then* G *is solvable. Moreover if* $h(G) \leqq 4$ *and* $(|G|, 3) = 1$, *then* G *is solvable.*

Note that A_5, the simple group of order sixty, has $h(A_5) = 4$.

The groups described in Theorem 5 have the property that they can be generated by two elements. This can be extended to a more general theorem.

Let $r(G)$ denote the minimal number of generators for G.

THEOREM 7. *If* $h(G) \geqq 2$, *then* $r(G) \leqq h(G)$.

Proof. The condition $h(G) \geqq 2$ is certainly necessary since we can find abelian groups K with $r(K)$ large. To prove Theorem 7 we only need to note that the next to last entry in an h-chain for G is $(h(G) - 1)$-th maximal in G and is cyclic.

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