A CHARACTERIZATION OF ANALYTICALLY UNRAMIFIED SEMI-LOCAL RINGS AND APPLICATIONS

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It is proved that a semi-local ring R is analytically unramified if and only if R is a subspace of a ring which is isomorphic to a finite direct sum of semi-local Dedekind domains. Applying this, it is proved that a local domain R is analytically irreducible if and only if R is a subspace of a local Dedekind domain, and this is true if and only if R is a subspace of every local domain which dominates R and which satisfies the altitude formula relative to R. A final application proves that an analytically unramified local domain is unmixed if and only if it satisfies the altitude formula.

All rings in this article are commutative rings with a unit, and the terminology is in general the same as that in [2]. In particular a ring R^* *dominates* a ring R in case R is a subring of R^*, each maximal ideal in R^* contracts to a maximal ideal in R, and each proper ideal in R is contained in a maximal ideal in R^*. A semi-local (Noetherian) ring R is a *subspace* of a semi-local ring R^* in case R^* dominates R and R is a subspace of R^* for the natural (Jacobson radical) topologies. A *Dedekind domain* is an integrally closed Noetherian domain of altitude one.

In § 2 it is proved that a semi-local ring R is analytically unramified if and only if R is a subspace of a ring which is isomorphic to a finite direct sum of semi-local Dedekind domains (Theorem 2.1). When R is analytically unramified, Theorem 2.2 associates with each ideal B contained in the Jacobson radical J of R a ring W such that (1) W is a quotient ring of a finitely generated ring over R, and (2) W is isomorphic to a finite direct sum of semi-local Dedekind domains. Further, if Rad $B = J$, then (3) R is a subspace of W, and (4) if N is a maximal ideal in W, then trd $(W/N)/(R/N \cap R)$ is equal to the depth of one (uniquely determined by N) of the prime divisors of zero in the completion of R (Proposition 2.8). To prove Theorem 2.2 a number of preliminary lemmas are needed and among these results Lemmas 2.3 and 2.4 are of some interest in themselves although they follow quite readily from known results. Essentially the method of proof of Theorem 2.2 is a combination of the methods used by Rees in [5, 6, 7, 8] to prove the Valuation Theorem.

Applications of Theorem 2.2 are given in § 3, 4 and 5. Theorem 3.5 and Theorem 3.7 characterize the intersection of W and the total

quotient ring of R under the additional assumption that B is not contained in any of the prime divisors of zero in R. This ring also has the properties (1), (2), (3), and a variant of (4) listed above.

In § 4 it is proved that a local domain (R, M) is analytically irreducible if and only if R is a subspace of a local Dedekind domain (Theorem 4.1). When R is analytically irreducible, Theorem 4.2 associates with each M-primary ideal B a finite number of local Dedekind domains which have properties (1), (3), and (4) above. Theorem 4.5 generalizes Theorem 4.1 and states that R is analytically irreducible if and only if R is a subspace of every local domain (S, N) which dominates R and is such that altitude $S + \operatorname{trd}(S/N)/(R/M) =$ altitude $R + \operatorname{trd} S/R < \infty$. Finally in § 5, a special case of Theorem 5.2 proves that an analytically unramified local domain satisfies the altitude formula if and only if it is unmixed.

2. A characterization of analytically unramified semi-local rings. *The following notation is fixed throughout this section*:

R is a semi-local ring with maximal ideals $M_1, \cdots, M_h$, $J = \bigcap_1^h M_i$ is the Jacobson radical of R, and $q_1, \cdots, q_g$ are the prime divisors of zero in R. (The numbers h and g are fixed.)

Also, *the following two notational conventions will be used throughout this paper*:

(1) If $B = (b_1, \cdots, b_m)A$ is an ideal in a ring A, then $\mathscr{R}(A, B)$ will denote the Rees ring $A[tb_1, \cdots, tb_m, u]$ (t an indeterminant, $u = 1/t$) of A with respect to B. ($\mathscr{R}(A, (0)) = A[u]$, and $\mathscr{R}(A, A) = A[t, u]$); and

(2) If A is a ring, then A' will denote the integral closure of A in its total quotient ring.

THEOREM 2.1. *R is analytically unramified if and only if R is a subspace of a ring which is isomorphic to a direct sum of a finite number of semi-local Dedekind domains.*

Proof. It is well known that a ring which is isomorphic to a direct sum of a finite number of semi-local Dedekind domains is analytically unramified. Hence, if R is a subspace of such a ring, then R is analytically unramified. The converse follows from the following.

THEOREM 2.2. *Let R be analytically unramified, let $B \subsetneqq J$ be an ideal in R, let $\mathscr{R} = \mathscr{R}(R, B)$, and let S be the complement in $\mathscr{R}'$ of the union of the prime divisors of $u\mathscr{R}'$. Then $W = \mathscr{R}'_S$ is isomorphic to a direct sum of g semi-local Dedekind domains. If J is the radical of B, then R is a subspace of W.*

To prove Theorem 2.2 some preliminary results are needed, so the proof of this theorem will be given following Lemma 2.6 below.

LEMMA 2.3. R *is analytically unramified if and only if* $\bigcap_1^g q_i = (0)$ *and* R/q_i *is analytically unramified* ($i = 1, \cdots, g$).

Proof. Let R^* be the completion of R. If R is analytically unramified, then clearly $(0) = \bigcap_1^g q_i$. To prove that each R/q_i is analytically unramified is equivalent to proving that each $q_i R^*$ is an intersection of prime ideals, since $R^*/q_i R^*$ is isomorphic to the completion of R/q_i. Therefore, fix i and let P be a prime divisor of $q_i R^*$. Since q_i is a prime divisor of zero in R and since P is a prime divisor of $q_i R^*$, it follows from [2, (18.11)] that P is a prime divisor of zero in R^*. Hence, since R is analytically unramified, P is the P-primary component of $q_i R^*$. Therefore each R/q_i is analytically unramified. Conversely, if each $q_i R^*$ is an intersection of prime ideals and $(0) = \bigcap_1^g q_i$, then $(0)R^* = (\bigcap_1^g q_i)R^* = \bigcap_1^g (q_i R^*)$ is an intersection of prime ideals. Hence, from the uniqueness theorems on the primary decomposition of an ideal, R is analytically unramified.

It is known [1, p. 365] that if an integral domain A is a finite separable extension of an analytically unramified semi-local domain, then A' is a finite A-module and A_P is analytically unramified for every prime ideal P in A. This result is used in the proof of the following lemma.

LEMMA 2.4.   *Let* $\mathscr{R}$ *be a finitely generated extension ring of* R. *Assume:* (1) *the zero ideal in* $\mathscr{R}$ *is an intersection of prime ideals* q_i^*; (2) *each* $q_i^* \cap R$ *is a prime divisor of zero in* R; (3) *each* $\mathscr{R}/q_i^*$ *is separable over* $R/q_i^* \cap R$; *and* (4) R *is analytically unramified.* *Then the integral closure* $\mathscr{R}'$ *of* $\mathscr{R}$ *is a finite* $\mathscr{R}$-*module and* $\mathscr{R}_S$ *is analytically unramified, where* S *is the complement in* $\mathscr{R}$ *of the union of a finite number of prime ideals in* $\mathscr{R}$.

Proof. Let $\mathscr{T}$ be the total quotient ring of $\mathscr{R}$. Then by (1) $\mathscr{T}$ can be considered to be the (finite, since $\mathscr{R}$ is Noetherian) direct sum of the fields $\mathscr{T}/q_i^*\mathscr{T}$ and then $D = \bigoplus \mathscr{R}/q_i^*$ is a finite $\mathscr{R}$-module such that $\mathscr{R} \subseteq D \subseteq \mathscr{T}$. Therefore $\mathscr{R} \subseteq D \subseteq \mathscr{R}'$ and $\mathscr{R}' = \bigoplus (\mathscr{R}/q_i^*)'$. By (2), (4), and Lemma 2.3, $R/q_i^* \cap R$ is an analytically unramified semi-local domain, and by (3) $\mathscr{R}/q_i^*$ is a finite separable extension of $R/q_i^* \cap R$, hence $(\mathscr{R}/q_i^*)'$ is a finite $\mathscr{R}/q_i^*$-module [1]. Therefore $\mathscr{R}'$ is a finite D-module, and hence $\mathscr{R}'$ is a finite $\mathscr{R}$-module. Let S be the complement in $\mathscr{R}$ of the union of a finite number of prime ideals in $\mathscr{R}$. To prove $\mathscr{R}_S$ is analytically unramified it is sufficient to prove that $\mathscr{R}_P$ is analytically unramified, for every

prime ideal P in $\mathscr{R}$ such that $P\mathscr{R}_S$ is maximal [2, (36.7)]. Therefore, let K be the kernel of the natural homomorphism from $\mathscr{R}$ into $\mathscr{R}_P$, say $K = \bigcap_i^s q_i^*$. For each $i = 1, \cdots, s$, $\mathscr{R}_P/q_i^*\mathscr{R}_P$ is a quotient ring of $\mathscr{R}/q_i^*$ and $\mathscr{R}/q_i^*$ is separably generated over $R/q_i^* \cap R$. Therefore $\mathscr{R}_P/q_i^*\mathscr{R}_P$ is analytically unramified [1], hence $\mathscr{R}_P$ is analytically unramified (Lemma 2.3).

REMARK 2.5. Let R and $\mathscr{R}$ be as in Lemma 2.4, let $p_1, \cdots, p_n$ be the prime divisors of zero in $\mathscr{R}'$, and let b be a nonunit in $\mathscr{R}'$. Let S be the complement in $\mathscr{R}'$ of the union of the prime divisors of $b\mathscr{R}'$, and assume that $b \in p_i$ if and only if $i \leq k$, and $p_i + b\mathscr{R}' \neq \mathscr{R}'$ if and only if $i \leq k + r$. Then $\mathscr{R}_S'$ is isomorphic to a direct sum of k fields and r semi-local Dedekind domains.

Proof. $\mathscr{R}'$ is Noetherian, since $\mathscr{R}'$ is a finite $\mathscr{R}$-module. Therefore a prime ideal P is a prime divisor of an ideal B in $\mathscr{R}'$ if and only if $P\mathscr{R}_P'$ is a prime divisor of $B\mathscr{R}_P'$. Since $\mathscr{R}'$ is integrally closed, each prime ideal P in $\mathscr{R}'$ contains exactly one prime divisor p_i of zero, so $\mathscr{R}_P'$ is a quotient ring of the integrally closed Noetherian domain $\mathscr{R}'/p_i$. The proof is completed by using the primary decomposition of a principal ideal in an integrally closed Noetherian domain and the fact that $\mathscr{R}' \cong \bigoplus \mathscr{R}'/p_i$.

LEMMA 2.6. *Let A be a Noetherian ring, and let b be a nonunit nonzero-divisor in A. If A' is a finite A-module, then there is a positive integer n (depending on b) such that $b^{n+i}A' \cap A \subseteq b^i A$, for all $i \geq 1$.*

Proof. If A' is a finite A-module, then, since A is Noetherian, $D = A' \cap A[1/b]$ is a finite A-module. Since every element in D may be written in the form r/b^k, where $r \in b^k A' \cap A$, it follows that $b^n D \subseteq A$, for some $n \geq 1$. Therefore, if $c \in b^{n+i}A' \cap A$, then $c/b^{n+i} \in D$, hence $c \in b^i A$.

From the definition of the Rees ring $\mathscr{R} = \mathscr{R}(R, B)$ of R with respect to B it follows that $\mathscr{R}$ is a graded Noetherian ring (the homogeneous elements of degree k being the elements $t^k b$, where $b \in B^k$, if $k > 0$, and $b \in R$, if $k \leq 0$), u is not a zero divisor in $\mathscr{R}$, and $u^i\mathscr{R} \cap R = B^i$, for all $i \geq 1$. Also, it is known [9, Lemma 1.1] that if C is an ideal in R and $C^* = CR[t, u] \cap \mathscr{R}$, then

$$\mathscr{R}/C^* \cong \mathscr{R}(R/C, B + C/C) .$$

This implies that if q is a prime divisor of zero in R and $\mathscr{T}$ is the total quotient ring of $\mathscr{R}$, then $\mathscr{R}/q\mathscr{T} \cap \mathscr{R} \cong \mathscr{R}(R/q, B + q/q)$.

These facts are used in the following.

Proof of Theorem 2.2. Let $\mathscr{T}$ be the total quotient ring of $\mathscr{R}$. Since $\mathscr{T}$ is the total quotient ring of $R[t]$, and since $\mathscr{R}/q_i \mathscr{T} \cap \mathscr{R}$ is isomorphic to $\mathscr{R}_i = \mathscr{R}(R/q_i, B + q_i/q_i)$ $(i = 1, \cdots, g)$ [9], $\mathscr{R}'$ is a finite $\mathscr{R}$-module (Lemma 2.4). Further, $\mathscr{R}' \cong \bigoplus_1^g (\mathscr{R}_i)'$, since $\mathscr{R}_i \cong \mathscr{R}/q_i \mathscr{T} \cap \mathscr{R}$. Hence $u\mathscr{R}' + (q_i \mathscr{T} \cap \mathscr{R}') \neq \mathscr{R}'$ $(i = 1, \cdots, g)$, since $B \subseteq J$ implies u is a nonunit in $\mathscr{R}_i'$. Therefore $W = \mathscr{R}_S'$ is isomorphic to a direct sum of g semi-local Dedekind domains (Remark 2.5), and $W \subset \mathscr{T}$. Further, since $\mathscr{R}'$ is a finite $\mathscr{R}$-module, there is a positive integer n such that, for all $i \geq 1$, $u^{n+i}\mathscr{R}' \cap \mathscr{R} \subseteq u^i\mathscr{R}$ (Lemma 2.6). Therefore

$$u^{n+i} W \cap R = ((u^{n+i} W \cap \mathscr{R}') \cap \mathscr{R}) \cap R \subseteq u^i\mathscr{R} \cap R = B^i \subseteq u^i W \cap R \,,$$

for all $i \geq 1$. Now the Jacobson radical of W is the radical of uW, and the contraction of the radical of an ideal is the radical of the contraction of the ideal. Hence, if J is the radical of B, then R is a subspace of W.

COROLLARY 2.7. *With the same notation as Theorem 2.2, let U be the complement in R of the union of some of the minimal prime divisors of B. Then R_U is a subspace of W_U.*

Proof. Since R is analytically unramified, R_U is analytically unramified (Lemma 2.4). Let K be the kernel of the natural homomorphism from R into R_U. Then $K^* = K\mathscr{T} \cap \mathscr{R}$ is the kernel of the natural homomorphism from $\mathscr{R}$ into $\mathscr{R}_U$. Since $\mathscr{R}/K^*$ is isomorphic to $\mathscr{R}(R/K, B + K/K)$, it is easily seen that

$$\mathscr{R}(R, B)_U \cong R^* = \mathscr{R}(R_U, BR_U) \,,$$

and from this it follows that $W_U \cong (R^{*'})_{S'}$, where S' is the complement in $R^{*'}$ of the union of the prime divisors of $uR^{*'}$. Therefore R_U is a subspace of W_U (Theorem 2.2).

REMARK. With the same notation as Theorem 2.2, assume that $B \nsubseteq J$, $B \subseteq q_i$ if and only if $i \leq k$, and $B + q_i \neq R$ if and only if $i \leq k + r$. Then

$$\mathscr{R}/q_i \mathscr{T} \cap \mathscr{R} \cong \mathscr{R}_i = \mathscr{R}(R/q_i, B + q_i/q_i)$$

implies $\mathscr{R}_i = R/q_i[t, u]$ $(i = k + r + 1, \cdots, g)$, and $\mathscr{R}_i = R/q_i[u]$ $(i = 1, \cdots, k)$. Therefore $W = \mathscr{R}_S'$ is isomorphic to a direct sum of k local Dedekind domains and r semi-local Dedekind domains.

An integral domain A satisfies the *altitude formula* in case it is

true that for every finitely generated integral domain D over A, if P is a prime ideal in D, then

$$\text{height } P + \text{trd}\,(D/P)/(A/P \cap A) = \text{height } P \cap A + \text{trd}\, D/A$$

(where $\text{trd}\,(E/F)$ denotes the transcendence degree of the integral domain E over its subdomain F). It is known [3, Corollary 2.9] that an analytically irreducible local domain satisfies the altitude formula.

PROPOSITION 2.8. *With the notation of Theorem 2.2 let* $\text{Rad}\, B = J$ *so that the completion* R^* *of* R *is a subring of the completion* W^* *of* W. *Let* Q *be a prime divisor of* $u\mathscr{R}'$, *let* p^* *be the prime divisor of zero in* W^* *which is contained in* QW^*, *and let* $p = p^* \cap R^*$. *Then* p *is a prime divisor of zero in* R^* *and*

$$\text{depth } Q = \text{trd}\,(\mathscr{R}'/Q)/(R/Q \cap R) = \text{depth } p\,.$$

Proof. Let $\mathscr{R}^* = \mathscr{R}(R^*, BR^*)$. Then $\mathscr{R}'$ is a subring of $\mathscr{R}^{*\prime}$. Since W^* is isomorphic to a direct sum of local Dedekind domains and R^* is a subring of W^* (Theorem 2.2), it follows that $\mathscr{R}^{*\prime}$ is a subring of W^*. Since QW is a maximal ideal in W, QW^* contains only one prime divisor p^* of zero in W^* and $p' = p^* \cap \mathscr{R}^{*\prime} \subset Q^* = QW^* \cap \mathscr{R}^{*\prime}$ (since $u \notin p^*$). To prove that $p = p^* \cap R^*$ is a prime divisor of zero in R^* it suffices to prove that p' is a prime divisor of zero in $\mathscr{R}^{*\prime}$. Now Q^* contains a (height one) prime divisor N of $u\mathscr{R}^{*\prime}$, and then $N \cap \mathscr{R}' = Q^* \cap \mathscr{R}' = Q$. Also $\mathscr{R}^{*\prime}/N$ is integrally dependent on $\mathscr{R}^*/N \cap \mathscr{R}^*$, and $\mathscr{R}^*/N \cap \mathscr{R}^* = \mathscr{R}/N \cap \mathscr{R}$ (since $u \in N$ and $\text{Rad } B = J$ imply $R^*/N \cap R^* = R/N \cap R$). Hence, since $\mathscr{R}^{*\prime}/N \supseteq \mathscr{R}'/N \cap \mathscr{R}' \supseteq \mathscr{R}/N \cap \mathscr{R}$ and $(Q^*/N) \cap (\mathscr{R}'/N \cap \mathscr{R}') = (0)$, it follows that $Q^* = N$, so p' is a prime divisor of zero in $\mathscr{R}^{*\prime}$. Since $\mathscr{R}^{*\prime}/Q^*$ is integrally dependent on $\mathscr{R}'/Q$ and $\mathscr{R}'/Q$ is a finitely generated integral domain over the field $R/Q \cap R = R^*/Q^* \cap R^*$, it follows that

$$\text{trd}\,(\mathscr{R}^{*\prime}/Q^*)/(R^*/Q^* \cap R^*) = \text{trd}\,(\mathscr{R}'/Q)/(R/Q \cap R) = \text{depth } Q\,.$$

Finally, R^*/p satisfies the altitude formula [3] and

$$\mathscr{R}^{*\prime}/p' \cong \mathscr{R}(R^*/p, BR^* + p/p)'\,.$$

Hence $\text{trd}\,(\mathscr{R}^{*\prime}/Q^*)/(R^*/Q^* \cap R^*) = \text{altitude } R^*/p = \text{depth } p$, since height $Q^*/p' = \text{trd}\,(\mathscr{R}^{*\prime}/p')/(R^*/p) = 1$.

REMARK 2.9. With the notation of Proposition 2.8 $QW^* \cap \mathscr{R}^{*\prime}$ is a (height one) prime divisor of $u\mathscr{R}^{*\prime}$. Also, if a prime divisor p of zero in R^* is given, then there is a prime divisor Q of $u\mathscr{R}'$ such

that the prime divisor p^* of zero in W^* which is contained in QW^* is such that $p^* \cap R^* = p$ (since the radical of an ideal contracts to the radical of the contraction of the ideal).

3. **The ring $W \cap T$.** Let $R, B \subseteqq J$, and W be as in Theorem 2.2, and let T be the total quotient ring of R. It is known [2, (33.7)] that if V is a local Dedekind domain and F is a subfield of the quotient field of V, then $V \cap F$ is either a local Dedekind domain or F. This implies that if B is contained in exactly k of the ideals q_i, then $W \cap T$ is isomorphic to a direct sum of k fields and $g - k$ semi-local Dedekind domains. Under the assumption that $B \not\subseteqq q_i$ $(i=1, \cdots, g)$, Theorems 3.5 and 3.7 below describe $W \cap T$ as the quotient ring of a finitely generated ring over R and Theorem 3.10 gives a related result. To prove these theorems *the following notation will be fixed throughout this section*:

R is an analytically unramified semi-local ring with maximal ideals $M_1, \cdots, M_h$, J is the Jacobson radical of R, and $q_1, \cdots, q_g$ are the prime divisors of zero in R. $B = (b_1, \cdots, b_m)R$ is contained in J and $B \not\subseteqq q_i$ $(i = 1, \cdots, g)$ (so height $M_i \geqq 1$ $(i = 1, \cdots, h)$). Let $\mathscr{R} = \mathscr{R}(R, B)$, let $Q_1, \cdots, Q_s$ be the prime divisors of $u\mathscr{R}'$, and let $P_i = Q_i \cap \mathscr{R}$ and $P_i = P_i \cap R$ $(i = 1, \cdots, s)$. (There may be less than s distinct p_i or p_i.) Let $T(\mathscr{T})$ be the total quotient ring of $R(\mathscr{R})$, let $S = \mathscr{R}' \sim \bigcup_1^s Q_i$, and let $W = \mathscr{R}_S'$.

REMARK 3.1. Since $B \subseteqq J, W$ is isomorphic to a direct sum of g semi-local Dedekind domains (Theorem 2.2). Therefore, each $q_i \mathscr{T} \cap \mathscr{R}'$ is contained in at least one Q_j, so $q_i \mathscr{T} \cap \mathscr{R} \subset P_j$. Hence each q_i is contained in at least one p_j and the containment is proper, since $B \not\subseteqq q_i$ $(i = 1, \cdots, g)$ and $B \subseteqq p_j$ $(j = 1, \cdots, s)$.

LEMMA 3.2. (1) $\operatorname{trd} (\mathscr{R}'/Q_i)/(R/p_i) \geqq 1$ $(i = 1, \cdots, s)$.

(2) *There exists a positive integer k (depending on B) and an element $b \in B^k$ such that $t^k b \in p_i$ $(i = 1, \cdots, s)$.*

Proof. (1) Fix i and let $Q=Q_i$, $P=P_i$, and $p=p_i$. Let $q\mathscr{T} \cap \mathscr{R}'$ be the minimal prime ideal contained in Q. Since $\mathscr{R}'/Q$ is isomorphic to a factor ring of $\mathscr{R}(R/q, B + q/q)'$ [9], to prove (1) it may be assumed that R is an integral domain. Let $V^* = \mathscr{R}_Q'$, let $N^* = QV^*$, and let F be the quotient field of R. Since V^* is a local Dedekind domain and $N = N^* \cap F \supseteqq p \neq (0)$ (Remark 3.1), $V = V^* \cap F$ is a local Dedekind domain [2, (33.7)]. Therefore $b_j V = BV$, for some $j = 1, \cdots, m$, so $R[b_1/b_j, \cdots, b_m/b_j] \subseteqq V$. Thus $D = V[tb_j, u] \supseteqq \mathscr{R}$, so $\mathscr{R}_Q' = V^* = D'_{N^* \cap D'}$. Now D' is a finite D-module [1] and V satisfies

the altitude formula [3], hence $\operatorname{trd}(D'/N^* \cap D')/(V/N) = 1$. Since $D'/N^* \cap D'$ and $\mathscr{R}'/Q$ have the same quotient field, and since $V/N \cong R/p$, (1) is proved. It now follows with the above notation that:

(3.1)   In the general case (when R is not assumed to be an integral domain) $tb_j \notin Q$, and when R is an integral domain $V^* = V[tb]_{NV[tb]}$, for each $b \in B$ such that $bV = BV$.

Since $\mathscr{R}'$ is isomorphic to a direct sum of integrally closed Noetherian domains and each Q_i is a prime divisor of $u\mathscr{R}'$, it follows from [2, (33.11)] that each P_i is a prime divisor of $u\mathscr{R}$. Hence, since $\mathscr{R}$ is a graded ring, each P_i is a homogeneous ideal which contains all elements of negative degree. Also $B^* = (tb_1, \cdots, tb_m)\mathscr{R}$ is a homogeneous ideal in $\mathscr{R}$. Therefore to prove (2) it is sufficient to prove that $B^* \nsubseteq P_i$ $(i = 1, \cdots, s)$, and this follows from (3.1).

REMARK 3.3.   Since each $q_j \mathscr{T} \cap \mathscr{R}$ is contained in some P_i, if $b \in B^k$ is such that $t^k b \notin P_i$ $(i = 1, \cdots, s)$, then b is not a zero-divisor in R. Also, if R/M_j is an infinite field $(j = 1, \cdots, h)$, then it can be shown that there is an element $b \in B$ such that $tb \notin P_i$ $(i = 1, \cdots, s)$.

If B is an ideal in a ring A, then the set B_a of elements x in A such that x satisfies an equation of the form

$$x^n + b_1 x^{n-1} + \cdots + b_n = 0 ,$$

where $b_i \in B^i$ $(i = 1, \cdots, n)$, is the *integral closure* of B in A. Clearly $B \subseteq B_a$.

LEMMA 3.4.   *Let b be a nonzero-divisor in $(B^k)_a$, where $k \geq 1$, let $C = \mathscr{R}'[1/t^k b]$, and let $A = R[\beta_1/b, \cdots, \beta_r/b]$ where $B^k = (\beta_1, \cdots, \beta_r)R$. Then there is a one to one correspondence between the prime divisors Q of uC and the prime divisors $\mathscr{P}$ of bA' such that if $\mathscr{P}$ and Q correspond, then $\mathscr{P} = Q \cap A'$ and $Q = Q_i C$, for some $i = 1, \cdots, s$.*

Proof.   Let $A^* = A'[t^k b, 1/t^k b]$. Then A^* is an integrally closed Noetherian ring since A' is (Lemma 2.4). An equation of integral dependence of b on B^k shows that $t^k b \in \mathscr{R}'$, so C is a quotient ring of $\mathscr{R}'$, hence C is integrally closed. Therefore, since the elements $t^k \beta_i$ $(i = 1, \cdots, r)$ are in $\mathscr{R}, A^* \subseteq C$. Further, C is integrally dependent on A^*, since $\mathscr{R}'$ is integrally dependent on

$$R[t^k \beta_1, \cdots, t^k \beta_r, u^k] \subseteq A^* .$$

(If $k = 1$, then $C = A^*$.)   Since C is integrally dependent on A^* and $bA^* = u^k A^*$, it follows that $bA' = A'$ if and only if $uC = C$, and if

$\mathscr{P}$ is a prime divisor of bA', then there is a prime divisor Q_iC (for some $i = 1, \cdots, s$) of uC such that $Q_iC \cap A^* = \mathscr{P}A^*$. Now let Q be a prime divisor of uC, and let $q\mathscr{T} \cap C$ be the minimal prime ideal contained in Q. Then $q\mathscr{T} \cap A^*$ is the minimal prime ideal contained in $Q \cap A^*$. Hence, since $A^*/q\mathscr{T} \cap A^*$ is integrally closed, height $Q \cap A^* = 1$ [2, (10.14)], so $Q \cap A'$ is a prime divisor of bA'. Finally, let $V^* = C_Q = \mathscr{R}'_{Q \cap \mathscr{R}'}$, let $N^* = QV^*$, and let F be the quotient field of R/q. Then $V^* \supseteqq A'/qT \cap A'$ and, since height $Q \cap A' = 1$, height $(N^* \cap (A'/qT \cap A')) = 1$. Therefore (3.1) implies that distinct prime divisors of uC which contain $q\mathscr{T} \cap C$ must contract to distinct prime divisors of bA'. Hence distinct prime divisors of uC contract to distinct prime divisors of bA'.

THEOREM 3.5. *With the fixed notation of this section, let*

$$b \in B^k = (\beta_1, \cdots, \beta_r)R$$

be such that $t^k b \notin P_i$ $(i = 1, \cdots, s)$, *let* $A = R[\beta_1/b, \cdots, \beta_r/b]$, *and let* U *be the complement in* A' *of the union of the prime divisors of* bA'. *Then* $A'_U = W \cap T$. *If* J *is the radical of* B, *then* R *is a subspace of* A'_U.

Proof. The existence of $t^k b \notin P_i$ $(i = 1, \cdots, s)$ is given by Lemma 3.2, and then Remark 3.3 shows that Lemma 3.4 is applicable for such $b \in B^k$. Therefore by Lemma 3.4 the ideals $\mathscr{P}_i = Q_i\mathscr{R}'[1/t^k b] \cap A'$ are distinct and are the prime divisors of bA', so $A'_U \subseteqq \mathscr{R}'[1/t^k b]_U = W$, hence $A'_U \subseteqq W \cap T$. Therefore for each $i = 1, \cdots, g$,

$$D_i = A'_U/q_iT \cap A'_U \subseteqq E_i = (W \cap T)/q_iT \cap (W \cap T)$$

and E_i is contained in the quotient field T/q_iT of D_i. Thus, since $A'_U \cong \bigoplus_1^g D_i$ and D_i is a semi-local Dedekind domain (Remark 2.5), and since $W \cap T$ is integrally closed in T, to prove that $A'_U = W \cap T$ it is sufficient to prove that $D_i = E_i$ $(i = 1, \cdots, g)$. Since $D_i = \bigcap D_{iN}$, where N runs over the maximal ideals in D_i, and since D_{iN} is a maximal proper subring of its quotient field, $D_i = E_i$ if it is shown that for each maximal ideal N in D_i there is a prime ideal N' in E_i such that $N' \cap D_i = N$. This, for each $i = 1, \cdots, g$, follows immediately on considering the ideals $Q_jW \cap (W \cap T)$ $(j = 1, \cdots, s)$. Finally, if Rad $B = J$, then R is a subspace of W (Theorem 2.2), hence R is a subspace of A'_U.

COROLLARY 3.6. A'_U *depends only on* B. *That is,* A'_U *is independent of the choice of* k *and* $b \in B^k$ *such that* $t^k b \notin P_i$ $(i = 1, \cdots, s)$.

Proof. This follows from $A'_U = W \cap T$.

REMARK. With the notation of Theorem 3.5 let Rad $B = J$. Let $\mathscr{P}$ be a prime divisor of bA' and let Q be the prime divisor of $u\mathscr{R}'$ such that $QW \cap A' = \mathscr{P}$ (Lemma 3.4). Then it follows from (3.1) and Proposition 2.8 that

$$\text{depth } \mathscr{P} = \text{trd } (A'/\mathscr{P})/(R/\mathscr{P} \cap R) = \text{depth } Q - 1 .$$

Theorem 3.7 below shows in particular that $W \cap T$ is determined by any ideal between B^n and $(B^n)_a$, where $n \geq 1$.

THEOREM 3.7. *Let $R, B, t^k b, A, U,$ and T be as in Theorem 3.5. Let R^* be a ring such that $R \subseteq R^* \subseteq R'$, let $n \geq 1$, let B^* be an ideal in R^* such that $B^n R^* \subseteq B^* \subseteq B^n A' \cap R^*$, and let $\Sigma = \mathscr{R}(R^*, B^*)$. Then:*

(1) $A' = R^*[\gamma_1/b^n, \cdots, \gamma_d/b^n]'$, *where* $B^{*k} = (\gamma_1, \cdots, \gamma_d)R^*$;

(2) $u\Sigma'$ *has exactly s prime divisors;*

(3) $\Sigma'_{S'} \cap T = A'_U$, *where S' is the complement in Σ' of the union of the prime divisors of $u\Sigma'$; and*

(4) $u^i\Sigma' \cap R^* = (B^{*i})_a$ *and* $b^{ni}A' \cap R^* = (B^{*ki})_a$, *for all $i \geq 1$.*

Proof. Since

$$B^{kn} R^* \subseteq B^{*k} \subseteq B^{kn} A' \cap R^* = b^n A' \cap R^*$$

and $A' = R[c_1/b^n, \cdots, c_e/b^n]'$, where $B^{kn} = (c_1, \cdots, c_e)R$, (1) is straightforward. To prove (2) it will first be shown that $u\mathscr{R}_n'$ has exactly s prime divisors, where $\mathscr{R}_n = \mathscr{R}(R, B^n)$. Now $\mathscr{R}_n'[1/t^k b^n] \cong A'$ (by (1)), so Lemma 3.4 (applied to $b^n \in (B^{nk})_a$) and (1) show that there is a one to one correspondence between the prime divisors of $u\mathscr{R}_n'[1/t^k b^n]$ and $b^n A'$. Therefore, since $b^n A'$ and bA' have the same prime divisors and bA' has s prime divisors (by the proof of Theorem 3.5), $u\mathscr{R}_n'$ has at least s prime divisors. Also, if $B^n = (d_1, \cdots, d_f)R$, then $\mathscr{R}_n$ is isomorphic to $D = R[t^n d_1, \cdots, t^n d_f, u^n]$ and $\mathscr{R} = \mathscr{R}(R, B)$ is integrally dependent on D. Hence, since $t^k b$ is not in any prime divisor of $u\mathscr{R}'$, the integral dependence of $\mathscr{R}'$ on $D' \cong \mathscr{R}_n'$ implies that $t^k b^n$ is not in any prime divisor of $u\mathscr{R}_n'$. Therefore $u\mathscr{R}_n'$ has exactly s prime divisors. Now $\mathscr{R}_n \subseteq \Sigma$, so (2) holds if $\Sigma \subseteq \mathscr{R}_n'$. For this it suffices to prove that if $x \in B^*$, then $tx \in \mathscr{R}_n'$. Now $x \in B^*$ implies

$$x^k \in B^{nk} A' = b^n A' \subseteq b^n A'[t^k b^n, 1/t^k b^n] \subseteq u^k \mathscr{R}_n'[1/t^k b^n] .$$

Hence, since $t^k b^n$ is not in any prime divisor of $u\mathscr{R}_n'$ and $x \in R'$, $(tx)^k \in \mathscr{R}_n'$. Thus, since $\mathscr{R}_n'$ is integrally closed, $tx \in \mathscr{R}_n'$, hence (2) holds. R^* is analytically unramified since R is. Therefore, (3) holds by (1), (2), and the proof of Theorem 3.5, since

$$\Sigma' = \mathscr{R}_n', \quad \mathscr{R}_n'[1/t^k b^n] \cong A' \, ,$$

and $t^k b^n$ is not in any prime divisor of $u\mathscr{R}_n'$. For (4), if $x \in u^i \Sigma' \cap R^*$, then $t^i x \in \Sigma'$, so there are elements $f_1, \cdots, f_j$ in Σ such that

$$(t^i x)^j + f_1 (t^i x)^{j-1} + \cdots + f_j = 0 \, .$$

Since Σ is a graded subring of $R^*[t, u]$, and since each term on the left side of this equation is in $R^*[t, u]$, the definition of $\mathscr{R}(R^*, B^*)$ and the ji-th component show that $x \in (B^{*i})_a$. If $x \in b^{ni} A' \cap R^*$, then

$$x \in b^{ni} A'[t^k b^n, 1/t^k b^n] \cap R^* \subseteq u^{ki} \Sigma'[1/t^k b^n] \cap \Sigma' \cap R^* \, .$$

Therefore, since $t^k b^n$ is not in any prime divisor of $u\Sigma'$, $x \in u^{ki} \Sigma' \cap R^* = (B^{*ki})_a$. The opposite inclusions follow readily by dividing an equation of integral dependence (of degree j) by u^{ij} (by b^{nij}) and using the definition of Σ (using (1)).

Theorem 3.10 below gives a result which is related to Theorems 3.5, 3.7, and the "necessity" part of Theorem 4.1 below. Its proof is based on Lemma 3.8 below. The following facts will be used in the proof of Lemma 3.8.

Let b be a nonzero-divisor in a Noetherian ring A. Then it is known [4, Lemma 1] that $(bA)_a = bA' \cap A$, and an element $c \in A$ is in $(bA)_a$ if and only if $c/b \in A'$. Also [4, Lemmas 2 and 4] $bA = (bA)_a$ if and only if A_p is a local Dedekind domain, for each prime divisor p of bA. Finally, if A' is isomorphic to a (finite) direct sum of Krull domains, then it follows from [2, (33.11)] that each prime divisor of bA' contracts in A to a prime divisor of bA.

LEMMA 3.8. *Let $Q_1, \cdots, Q_v$ be specified prime divisors of $u\mathscr{R}'$ ($v \leq s$). Then there is an integer n (depending on B and $Q_1, \cdots, Q_v$) and a nonzero-divisor $c \in (B^n)_a$ such that $t^n c \in Q_i$ ($i = 1, \cdots, v$) and $t^n c \notin Q_i$ ($i = v + 1, \cdots, s$).*

Proof. Let b and A be as in Theorem 3.5 and let $C = \mathscr{R}'[1/t^k b]$ so that bA' has the s ideal $Q_i C \cap A'$ as its prime divisors (Lemma 3.4). Let $D = A' \cap A[1/b]$. Then D is Noetherian, since A' is a finite A-module (Lemma 2.4), and D is the set of elements x/b^m, where $x \in b^m A' \cap A$. By the definition of D and the remarks preceding this lemma it follows that $bD = (bD)_a$, and then that the prime divisors of bD are the s height one prime ideals $\mathscr{P}_i = (Q_i C \cap A') \cap D$. Hence, since there are no containment relations among the $\mathscr{P}_i$, there is an element x/b^m in D which is in $\mathscr{P}_i$ if and only if $i \leq v$, and it can be assumed that x/b^m is not a zero-divisor. Then, by the definition of A, there is an integer i and a nonzero-divisor $c \in B^{ki}$ such that $x/b^m = c/b^{m+i}$. Since $A' \subseteq C$ and $t^k b$ is a unit in C, c/b^{m+i} and ct^{mk+ik} generate

the same ideal in C. Let $n = k(m + i)$. Then $c \in u^n C \cap \mathscr{R}' = u^n \mathscr{R}'$, hence $t^n c \in \mathscr{R}'$ and $t^n c \in Q_i C \cap \mathscr{R}'$ if and only if $i \leq v$. Finally, $c \in u^n \mathscr{R}' \cap R = (B^n)_a$ (Theorem 3.7).

COROLLARY 3.9. *There is an integer* n *(depending on* B*) and a nonzero-divisor* $c \in (B^n)_a$ *such that* cA' *is a primary ideal, where* $B^n = (c_1, \cdots, c_e)R$ *and* $A = R[c_1/c, \cdots, c_e/c]$.

Proof. This follows immediately from Lemma 3.8 and 3.4.

THEOREM 3.10. *With the fixed notation of this section let* Rad $B = J$, *and let* $q_1^*, \cdots, q_d^*$ *be the prime divisors of zero in the completion* R^* *of* R. *Then there is an integer* n *(depending on* B*) and a nonzero-divisor* $c \in (B^n)_a$ *such that* cA' *has exactly* d *prime divisors* $\mathscr{P}_i$ *and* R *is a subspace of* A'_U, *where* $B^n = (c_1, \cdots, c_e)R$, $A = R[c_1/c, \cdots, c_e/c]$, *and* U *is the complement in* A' *of the union of the prime divisors of* cA'. *Further,*

$$\text{trd } (A'/\mathscr{P}_i)/(R/\mathscr{P}_i \cap R) = \text{depth } q_i^* - 1 .$$

Proof. Let $\mathscr{R}^* = \mathscr{R}(R^*, BR^*)$ and let $\mathscr{T}^*$ be the total quotient ring of $\mathscr{R}^*$. Then $\mathscr{R}^{*\prime}$ is a subring of the completion W^* of W (as in the proof of Proposition 2.8). Also, by renumbering if necessary, $Q_i^* = Q_i W^* \cap \mathscr{R}^{*\prime} \supset q_i^* \mathscr{T}^* \cap \mathscr{R}^{*\prime}$ and Q_i^* is a (height one) prime divisor of $u \mathscr{R}^{*\prime}$ $(i = 1, \cdots, d)$ (Remark 2.9). Let $U^* = \mathscr{R}^{*\prime} \sim \bigcup_1^d Q_i^*$ and let $U' = \mathscr{R}' \sim \bigcup_1^d Q_i$. Then $V^* = \mathscr{R}_{U^*}^{*\prime}$ is isomorphic to a direct sum of d local Dedekind domains, V^* dominates R^*, and $V^* \cap \mathscr{T} = \mathscr{R}_{U'}'$. Also, since R^* is complete, R^* is a subspace of V^* [2, (30.2)]. Therefore, since V^* dominates $\mathscr{R}_{U'}'$ and $\mathscr{R}_{U'}'$ dominates R, it follows that R is a subspace of $\mathscr{R}_{U'}'$. Let n be chosen such that there is an element $c \in (B^n)_a$ such that $t^n c \in Q_i$ if and only if $i > d$ (Lemma 3.8), let $B^n = (c_1, \cdots, c_e)R$, and let $A = R[c_1/c, \cdots, c_e/c]$. Then by Lemma 3.4 cA' has the ideals $\mathscr{P}_i = Q_i \mathscr{R}'[1/t^n c] \cap A'$ $(i = 1, \cdots, d)$ as its prime divisors. Let $U = A' \sim \bigcup_1^d \mathscr{P}_i$. Then $\mathscr{R}_{U'}'$ dominates A'_U and A'_U dominates R, hence R is a subspace of A'_U. The last statement follows from Proposition 2.8 and the Remark following Corollary 3.6.

4. A characterization of analytically irreducible local domains. Throughout this section R is a local domain with maximal ideal M.

THEOREM 4.1. *R is analytically irreducible if and only if* R *is a subspace of a local Dedekind domain.*

Proof. It is well known that a local Dedekind domain is analytically irreducible. Hence, if R is a subspace of such a ring, then R is analytically irreducible. The converse (for altitude $R \geq 1$) follows 3.10. With no restriction on altitude R the converse follows from from Theorem the following.

THEOREM 4.2. *Let R be analytically irreducible, let B be an M-primary ideal in R, let $\mathscr{R} = \mathscr{R}(R, B)$, and let Q be a prime divisor of $u\mathscr{R}'$. Then R is a subspace of $\mathscr{R}'_Q$, and*

$$\operatorname{trd} (\mathscr{R}'/Q)/(R/M) = \text{altitude } R.$$

Proof. Let S be the complement in $\mathscr{R}'$ of the union of the prime divisors of $u\mathscr{R}'$. Then $W = \mathscr{R}'_S$ is a semi-local Dedekind domain and R is a subspace of W (Theorem 2.2). Therefore $\mathscr{R}^{*\prime}$ is a subring of $W^* \cap \mathscr{T}^*$, where R^* (W^*) is the completion of R (W) and $\mathscr{T}^*$ is the quotient field of $\mathscr{R}^* = \mathscr{R}(\mathscr{R}^*, BR^*)$. Let Q be a prime divisor of $u\mathscr{R}'$ and let $Q^* = QW^* \cap \mathscr{R}^{*\prime}$. Then height $Q^* = 1$ (Remark 2.9), thus $V^* = \mathscr{R}_{Q^*}^{*\prime}$ is a local Dedekind domain which contains R^* and $N^* \cap R^* = MR^*$, where $N^* = Q^*V^*$. Hence, since R^* is complete, R^* is a subspace of V^* [2, (30.2)]. Let $\mathscr{T}$ be the quotient field of $\mathscr{R}$ and let $V = V^* \cap \mathscr{T}$. Then R is a subspace of V, since R is a subspace of V^*, and $V = \mathscr{R}'_Q$, since $N^* \cap \mathscr{R}' = Q$. The last statement follows since R satisfies the altitude formula [3].

The following Corollary is a special case of Corollary 4.6 below. It is included here because it will be used in the proof of Theorem 4.5 below.

COROLLARY 4.3. *Let R be analytically irreducible, let altitude $R \geq 1$, and let $B = (b_1, \cdots, b_s)R$ be an M-primary ideal. If $b \in B$ is such that bA' is a proper ideal, where $A = R[b_1/b, \cdots, b_s/b]$, then R is a subspace of A'_P, for every prime divisor P of bA'.*

Proof. Let $\mathscr{R} = \mathscr{R}(R, B)$ and let $C = \mathscr{R}'[1/tb]$. Then, if P is a prime divisor of bA', there is a prime divisor Q of $u\mathscr{R}'$ such that $QC \cap A' = P$ (Lemma 3.4). This implies that $\mathscr{R}'_Q$ dominates A'_P. Hence, since A'_P dominates R and R is a subspace of $\mathscr{R}'_Q$ (Theorem 4.2), R is a subspace of A'_P.

REMARK 4.4. Let R be analytically irreducible. (1) R' is a local domain. (2) If altitude $R \leq 1$ and if (S, N) is a local domain that dominates R, then R is a subspace of S.

Proof. (1) Since R is analytically irreducible, R' is a finite R-

module, and the completion R^* of R is a complete local domain.
Therefore, the completion R'^* of R' is $R' \bigotimes_R R^*$ [2, (17.8)], so $R'^* = R^*[R']$, hence $R^* \subseteq R'^* \subseteq R^{*'}$. Also, $R^{*'}$ is a local domain [2, (30.3)
and (30.5)], so it follows that R' is a local domain. (2) If altitude
$R = 0$, then R is a field, so R is a subspace of S. If altitude $R = 1$,
then R' is a local Dedekind domain (by (1)) which is a finite R-module.
Thus $A = S[R']$ is a finite S-module, hence A is a semi-local domain
that dominates R'. Therefore R' is a subspace of A, since the maximal
ideal in R' is a principal ideal. Since R is a subspace of R', it follows
that R is a subspace of S.

Theorem 4.5 below gives a generalization of Remark 4.4. The
following result [10, Lemma 1] is used in the proof of Theorem 4.5:
If $x_1, \cdots, x_n$ $(n > 1)$ is a system of parameters in a local domain
(S, N), then, with $D = S[x_2/x_1, \cdots, x_n/x_1]$, ND is a height one prime
ideal and the ND-residues of the x_i/x_1 are algebraically independent
over S/N.

THEOREM 4.5. *R is analytically irreducible if and only if R is
a subspace of every local domain* (S, N) *such that* (1) *S dominates R
and* (2) *altitude* S + trd $(S/N)/(R/M)$ = altitude R + trd $S/R < \infty$.

Proof. Let R be analytically irreducible and let (S, N) be a local
domain such that (1) and (2) hold. It may be assumed by Remark
4.4 that altitude $R \geqq 2$ (so by (1) altitude $S \geqq 1$). If altitude $S = 1$,
let $V^* = S'_P$ and let $Q^* = PV^*$, where P is a maximal ideal in S'.
If altitude $S = n > 1$, let $x_1, \cdots, x_n$ be a system of parameters in S
and let $D = S[x_2/x_1, \cdots, x_n/x_1]$. Then ND is a height one prime ideal
in D and trd $(D/ND)/(S/N) = n - 1$ [10]. Let $V^* = D'_P$ and let
$Q^* = PV^*$, where P is a (height one) prime ideal in D' such that
$P \cap D = ND$. Then in both cases (V^*, Q^*) is a local Dedekind domain
[2, (33.10)] which satisfies (1) and (2). Let T be the quotient field of
R, let $V = V^* \cap T$, and let $Q = Q^* \cap T$. Then (V, Q) is a local
Dedekind domain that dominates R. Also,

$$1 + \text{trd } (V/Q)/(R/M) \leqq \text{altitude } R$$

[11, Prop. 1, p. 330], and, since Q is a principal ideal,

$$\text{trd } (V^*/Q^*)/(V/Q) \leqq \text{trd } V^*/V = \text{trd } V^*/R < \infty .$$

Adding these inequalities and using the fact that (V^*, Q^*) satisfies
(2), it follows that both inequalities are equalities. Therefore (V, Q)
also satisfies (2), so there are elements $v_1, \cdots, v_k$ in V (altitude $R = k + 1$) such that V/Q is algebraic over $C/Q \cap C$, where $C = R[v_1, \cdots, v_k]$.
Let $b \in R$ be a common denominator of the v_i, and let $bV \cap R = B =$

$(b_1, \cdots, b_s)R$.   Then B is M-primary, and

$$C \subseteq R[b_1/b, \cdots, b_s/b] = A \subseteq A' \subseteq V \, .$$

Therefore trd $(A'/Q \cap A')/(R/M) = k$, hence, since R satisfies the altitude formula, height $Q \cap A' = 1$. Thus $V = A'_{Q \cap A'}$, hence R is a subspace of V (Corollary 4.3). Also V is a subspace of V^* (Remark 4.4). Hence, since V^* dominates S and S dominates R, R is a subspace of S. Conversely, if R is a subspace of every local domain (S, N) which satisfies (1) and (2), then let $x_1, \cdots, x_n$ be a system of parameters in R. It may be assumed that $n = $ altitude $R > 0$. Let

$$D = R[x_2/x_1, \cdots, x_n/x_1] \qquad (D = R, \text{ if } n = 1) \, .$$

Then $D^* = D_{MD}$ is a local domain of altitude one which satisfies (1) and (2) [10]. Therefore R is a subspace of the local Dedekind domain $D_P^{*\prime}$, where P is a maximal ideal in $D^{*\prime}$, hence R is analytically irreducible.

COROLLARY 4.6. *Let R be analytically irreducible, and let A be an integral domain which is finitely generated over R. Then R is a subspace of A_P, for every prime ideal P in A such that $P \cap R = M$.*

Proof. R satisfies the altitude formula [3].

5. **A further application.** A semi-local ring R is *unmixed* (*quasi-unmixed*) in case each prime divisor (each minimal prime divisor) p of zero in the completion of R satisfies depth $p = $ altitude R. It is known [3, Prop. 2.2 and Corollary 2.7] that a quasi-unmixed semi-local integral domain satisfies the altitude formula and the second chain condition for prime ideals. It is also known [2, exercise, p. 135] that a pseudo-geometric semi-local domain satisfies the second chain condition for prime ideals if and only if it is unmixed. Theorem 5.2 below contains a form of the converse of the first of these results (since "unmixed" and "quasi-unmixed" are equivalent conditions under the presence of analytical unramifiedness). Also, it is closely related to the second of these results since a pseudo-geometric semi-local domain is analytically unramified [2, (36.4)] and since there is a close relationship between the altitude formula and the second chain condition for prime ideals.

DEFINITION 5.1. A local ring R satisfies the *altitude formula* in case for each minimal prime ideal q in R it is true that depth $q = $ altitude R and R/q satisfies the altitude formula. A Noetherian ring A satisfies the *altitude formula* in case A_N satisfies the altitude for-

mula, for each maximal ideal N in A.

THEOREM 5.2.   *Let (A, N) be an analytically unramified local ring. A satisfies the altitude formula if and only if A is unmixed.*

Proof.   Let A satisfy the altitude formula and let q be a prime divisor of zero in A.   Then by Lemma 2.3 and hypothesis $R = A/q$ is an analytically unramified local domain which satisfies the altitude formula and altitude R = altitude A.   Let M be the maximal ideal in R, let B be an M-primary ideal, let $\mathscr{R} = \mathscr{R}(R, B)$, and let S be the complement in $\mathscr{R}'$ of the union of the prime divisors of $u\mathscr{R}'$.   Then R is a subspace of $W = \mathscr{R}'_S$ (Theorem 2.2).   Let W^* be the completion of W, and let p be a (minimal) prime divisor of zero in the completion R^* of R.   Then there is a prime divisor Q of $u\mathscr{R}'$ such that $p^* \cap R^* = p$, where p^* is the prime divisor of zero in W^* which is contained in QW^* (Remark 2.9), and then

$$\text{depth } p = \text{trd } (\mathscr{R}'/Q)/(R/M)$$

(Proposition 2.8).   Now $\mathscr{R}'$ is a finitely generated integral domain over R (Lemma 2.4) and trd $\mathscr{R}'/R$ = height Q = 1, so, since R satisfies the altitude formula, it follows that depth p = trd $(\mathscr{R}'/Q)/(R/M)$ = altitude R.   Therefore R is unmixed.   Hence A/q is unmixed and depth q = altitude A, for each minimal prime ideal q in A.   Therefore A is quasi-unmixed [3, Lemma 2.2] and analytically unramified, thus A is unmixed.   Conversely, let A be unmixed and let q be a minimal prime ideal in A.   Then depth q = altitude A and A/q satisfies the altitude formula [3, Lemma 2.2 and Prop. 2.2].   Therefore A satisfies the altitude formula.

COROLLARY 5.3.   *An analytically unramified semi-local ring R satisfies the altitude formula if and only if R_M is unmixed, for each maximal ideal M in R.*

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