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ON THE VARIATION OF THE BERNSTEIN POLYNOMIALS OF A FUNCTION OF UNBOUNDED VARIATION

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The behavior of the ordinary Bernstein polynomials, $B_n f$, for discontinuous functions f can be quite erratic. The purpose of this note is to give an example of a function f which is quite irregular on the rationals but such that the total variation, $VB_n f$ of $B_n f$ tends to zero with n.

It is known that if f is of bounded variation, then $VB_n f$ tends to the variation of f taken over its points of continuity, [2 p. 25]. In [3] we consider arbitrary f, and give sufficient conditions for $VB_n f$ to tend to zero in terms of the sums $\sum_{r=0}^{n} |f(r/n)|$. It is shown in [2 p. 28] that $B_n f$, for unbounded f, can behave unusually in terms of pointwise convergence to f. Here we construct a function, unbounded on the rationals in every subinterval of $[0, 1]$, and which has the property that $B_n f$ converges in variation (and uniformly) to zero.

2. Preliminaries. The n-th Bernstein polynomial of the real function f on $[0, 1]$ is

$$B_n f \equiv \sum_{r=0}^{n} f\left(\frac{r}{n}\right) p_{nr}(x) , \tag{2.1}$$

where

$$p_{nr}(x) \equiv \binom{n}{r} x^r (1-x)^{n-r} , \qquad x \in [0, 1] .$$

Since $B_n f$ depends only on rational values of f, we restrict ourselves to "skeletons," i.e., functions defined only on the rationals in $[0, 1]$, in the manner of [1]. We need the following facts:

(A) If $r = 1, \cdots, n-1$, then for all n,

$$P(n, r) \equiv \operatorname*{Max}_{[0,1]} p_{nr}(x) < Cn^{\frac{1}{2}}[r(n-r)]^{-\frac{1}{2}} \tag{2.2}$$

where C is an absolute constant [1].

(B) If a is a positive integer, then

$$P(an, ar) < 2a^{-\frac{1}{2}} P(n, r) \tag{2.3}$$

for each $n \geqq 2$ and $r = 1, \cdots, n-1$. ((A) and (B) are applications of Stirling's formula.)

(C) For all n and f

$$VB_n f \leqq 2 \sum_{r=0}^{n} \left| f\left(\frac{r}{n}\right) \right| P(n, r) . \tag{2.4}$$

(D) If $\sum_{i=1}^{\infty} f_i$ is a pointwise convergent series of functions (skeletons) on [0, 1] then,

$$VB_n\left(\sum_{i=1}^{\infty} f_i\right) \leqq \sum_{i=1}^{\infty} VB_n f_i$$

where the right side may be $+\infty$.

3. Construction. We define a sequence of skeletons f_i such that each skeleton tends to $+\infty$ on a set of rationals tending to a limit rational r_i. The r_i will be dense in [0, 1]. It is shown that the skeleton $f \equiv \sum_{i=1}^{\infty} f_i$ has the following properties:

(1) f is unbounded on the rationals in every subinterval of [0, 1];

(2) $VB_n f \to 0$ as $n \to \infty$.

(Since f will satisfy $f(0) = f(1) = 0$, and since $B_n f(0) = f(0)$ and $B_n f(1) = f(1)$ for all f and n, (2) implies $B_n f \to 0$ uniformly on [0, 1].)

For all $i = 1, 2, \cdots$, pick $r_i \equiv p_i/q_i$ such that q_i is prime, $0 < p_i < q_i$, $q_i < q_{i+1}$, and $r_i \in I_i$, where $I_1 = [0, 1/2]$, $I_2 = [1/2, 1]$, $I_3 = [0, 1/4]$, $\cdots$ $I_6 = [3/4, 1]$, $I_7 = [0, 1/8]$, $\cdots$. Thus the r_i are dense in [0, 1]. Define

$$f_i\left(\frac{p_i}{q_i} + \frac{1}{q_i^{\alpha(i,l)}}\right) \equiv l \tag{3.1}$$

where for each i, $\alpha(i, l)$ is a strictly increasing sequence of positive integers to be determined later. For all other rationals in [0, 1], put $f_i \equiv 0$, and then set $f \equiv \sum_{i=1}^{\infty} f_i$. Since the supports of the f_i are disjoint, f is well defined at all rationals, and satisfies (1) by construction. We have

$$VB_n f \leqq \sum_{i=1}^{\infty} VB_n f_i \leqq \sum_{i=1}^{\infty} H(i, n) \tag{3.2}$$

by 2 (C) and (D), where we have put

$$H(i, n) \equiv 2 \sum_{r=0}^{n} \left| f_i\left(\frac{r}{n}\right) \right| P(n, r) . \tag{3.3}$$

LEMMA (3.1). *For fixed i, it is possible to choose $\alpha(i, l)$, $l = 1, 2, \cdots$ such that*

$$H(i, q_i^{\alpha(i,l)}) < \frac{1}{q_i^2 l} . \tag{3.4}$$

Proof. To simplify matters, let $p_i \equiv p$, $q_i \equiv q$ and $\alpha(i, l) \equiv \alpha_l$. When $n = q_i^{\alpha(i,k)} \equiv q^{\alpha_k}$, there are only k, nonzero terms on the right in (3.3), and these correspond to the points

$$\frac{r}{n} = \frac{p}{q} + \frac{1}{q^{\alpha_j}} = \frac{pq^{\alpha_k - 1} + q^{\alpha_k - \alpha_j}}{q^{\alpha_k}} \qquad (j = 1 \cdots k) .$$

Since the value of f_i at the j-th point is j, (3.3) becomes

$$\sum_{j=1}^{k} 2jP(q^{\alpha_k}, pq^{\alpha_k - 1} + q^{\alpha_k - \alpha_j}) . \tag{3.5}$$

By applying (2.2), one gets each term in (3.5) less than

$$\begin{aligned} & 2jC\Big[\frac{q^{\alpha_k}}{[pq^{\alpha_k-1} + q^{\alpha_k - \alpha_j}][q^{\alpha_k} - pq^{\alpha_k - 1} - q^{\alpha_k - \alpha_j}]}\Big]^{\frac{1}{2}} \\ & = 2jC\Big[q^{\alpha_k}\Big(\frac{p}{q} - \frac{p^2}{q^2} - \frac{2p}{q^{\alpha_j + 1}} + \frac{1}{q^{\alpha_j}} - \frac{1}{q^{2\alpha_j}}\Big)\Big]^{-\frac{1}{2}} . \end{aligned} \tag{3.6}$$

Thus, for $k = j = 1$, α_1 may be chosen so large that (3.6), hence (3.5), is less than $1/q^2$. (We pick $\alpha_1 \geqq 2$ so that $p/q + 1/q^{\alpha_1} < 1$.) Now suppose α_k, $k = 1, \cdots, l-1$ have been chosen so that $\alpha_k > \alpha_{k-1}$, and so that (3.5) is less than $1/q^2k$. When $k = l$, (3.6) shows that α_l can be chosen so that each term, $j = 1, \cdots l$ is less than $1/q^2l^2$. Thus (3.5) is less than $l\cdot(ql)^{-2} = 1/q^2l$.

We can factor every integer n uniquely as:

$$n \equiv d\cdot \prod_{j=1}^{T} n_j, \quad n_j = q_{i_j}^{\alpha(i_j, L_j)} \quad q_{i_j} < q_{i_{j+1}} . \tag{3.7}$$

The q_{i_j} are those q_i which appear in n to a power greater than or equal $\alpha(i_j, 1)$, and L_j is the largest index l of the exponents $\alpha(i_j, l)$ such that $q_{i_j}^{\alpha(i_j, l)}$ divides n. For any n,

$$\sum_{i=1}^{\infty} H(i, n) = \sum_{j=1}^{T} H(i_j, n) \leqq \sum_{j=1}^{T} 2\left(\frac{n_j}{n}\right)^{\frac{1}{2}} H(i_j, n_j) \tag{3.8}$$

where the inequality follows from (2. B) with $a = n/n_j$. If we apply the lemma to each term, we get the last sum less than

$$\sum_{j=1}^{T} 2\left(\frac{n_j}{n}\right)^{\frac{1}{2}} \frac{1}{q_{i_j}^2 L_j} \leqq \frac{2}{n_T^{1/2}}\Big(\sum_{j=1}^{T-1} \frac{1}{q_{i_j}^2 L_j}\Big) + \frac{1}{q_{i_T}^2 L_T} \tag{3.9}$$

where the decomposition applies if $T > 1$. In this case, the sum on the right is dominated by $\sum 1/m^2$ and is thus bounded. (If $T = 1$, the assertion is that (3.9) holds if the sum is regarded as vacuous, and a similar remark holds for (3.11) below.) Therefore if the largest of the q_{i_j}, q_{i_T} is as large as, let us say, q_{i_*}, n_T will also be large, and (3.9) can be made less than ε.

Now suppose n is such that every $q_{i_j} < q_{i_*}$. As before

$$(3.10) \qquad \sum_{i=j}^{\infty} H(i, n) \leqq \sum_{j=1}^{T} 2\left(\frac{n_j}{n}\right)^{\frac{1}{2}} \frac{1}{q_{i_j}^2 L_j} \qquad (q_{i_j} < q_{i_*})$$

Let k be the first index where $\mathrm{Max}_{1 \leqq j \leqq T} L_j$ occurs. Then (3.10) becomes

$$(3.11) \qquad \begin{aligned} &\sum_{j \neq k} \left[2\left(\frac{n_j}{n}\right)^{\frac{1}{2}} \cdot \frac{1}{q_{i_j}^2 L_j}\right] + \frac{1}{q_{i_k}^2 L_k} \leqq \\ &\left[2(2)^{-\alpha(i_k, L_k)/2}\left(\sum_{j \neq k} \frac{1}{q_{i_j}^2 L_j}\right)\right] + \frac{1}{q_{i_k}^2 L_k}, \end{aligned}$$

since $q_{i_k} \geqq 2$ and appears in every n^j/n for $j \neq k$. As in (3.9), the sum is bounded. Thus if L_k is large enough, say $L_k \geqq L$, $\alpha(i_k, L_k)$ is also large, and (3.10) is less than ε.
Now suppose every q_i in n is less than q_{i_*} and all the indices L_j are less than L. There are only a finite number of such combinations $\prod_{j=1}^{T} n_j$, and we denote them C_s, $s = 1 \cdots S$. If $n \equiv d \cdot C_s$, we get by (2.B)

$$(3.12) \qquad \sum_{i-1}^{\infty} H(i, n) \leqq \frac{2}{d^{1/2}} \sum_{i=j}^{\infty} H(i, C_s) .$$

However only a finite number of q_i appear in any C_s so that the sum is bounded by, say $M_s > 0$. Therefore (3.12) is less $2M_s/d^{1/2}$, and we can pick d_s large enough so that $d \geqq d_s$ implies (3.12) is less than ε.

Thus if $n > \mathrm{Max}\,[q_{i_*}^{\alpha(i_*,1)}, q_1^{\alpha(1,L)}, d_1c_1 \cdots d_Sc_S]$, $\sum_{i=1}^{\infty} H(i, n) < \varepsilon$, implying $VB_nf < \varepsilon$ by (3.2).

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