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FURTHER REMARKS ON IDEALS OF THE PRINCIPAL CLASS

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In a previous paper on this subject, the author gave a new proof of the theorem of "analytic independence of systems of parameters". The methods used can be applied to prove a converse result. Hence: An ideal of a Noetherian ring is of the principal class if, and only if, it is generated by an analytically independent set.

The subset $\{z_1, \cdots, z_n\}$ of a ring R—R is commutative with $1 \neq 0$, as are all rings here considered—is said to be *analytically independent* provided that $(z_1, \cdots, z_n)R$ is not the unit ideal, and for every homogeneous f in the polynomial ring $R[X_1, \cdots, X_n]$ such that $f(z_1, \cdots, z_n) = 0$, the coefficients of f lie in the radical of $(z_1, \cdots, z_n)R$. Now let I be a nonunit ideal of R generated by the set $\{x, x_1, \cdots, x_n\}$, with x not a divisor of zero, and let Q denote the kernel of the R-homomorphism $R[X_1, \cdots, X_n] \rightarrow S = R[x_1/x, \cdots x_n/x]$ determined by $X_i \rightarrow x_i/x$. (We regard S as a subring of the full ring of quotients of R.) A completely elementary fact is that $\{x, x_1, \cdots, x_n\}$ is analytically independent if, and only if, for every prime ideal P of R containing I, $PR[X_1, \cdots, X_n]$ contains Q (see preliminary remarks of [1]). Using this fact we gave in [1] a new proof of the theorem of "analytic independence of systems of parameters": *If R is Noetherian and I is of height $n+1$, then $\{x, x_1, \cdots, x_n\}$ is analytically independent.* In considering a question raised by Professor E. Kunz, it come to light that the methods of [1] can be used to prove the converse result:

PROPOSITION 1. *If R is Noetherian and $\{x, x_1, \cdots, x_n\}$ is analytically independent, then height $(I) = n + 1$.*

LEMMA 1. *If P is a prime ideal of R containing I, then PS is a prime ideal of S, $PS \cap R = P$, and S/PS is (R/P)-isomorphic to $(R/P)[X_1, \cdots, X_n]$.*

LEMMA 2. *Let M be a prime ideal of S, and let $P = M \cap R$. Then height $(P) \geqq$ tr.dg. $[S/M : R/P]$ + height (M).*

Proof of the lemmas. The facts stated in Lemma 1 are proved in the preliminary remarks of [1]. Lemma 2 is well known in the case of R an integral domain (see, for example, Appendix 1 of Zariski-

Samuel vol. II [2]); we generalize to the case at hand as follows. Let M_0 be a minimal prime ideal of S contained in M such that height $(M/M_0) =$ height (M), and let $P_0 = R \cap M_0$. Observe that S/M_0 and R/P_0 have the same quotient field, namely the quotient field of K/N_0, where K is the full ring of quotients of R, and N_0 is the unique prime ideal of K lying over M_0 (and P_0). Hence, using the integral domain result, we have:

$$\begin{aligned}\text{height } (P) &\geqq \text{height } (P/P_0) \\ &\geqq \text{tr.dg. } [S/M_0/M/M_0 : R/P_0/P/P_0] + \text{height } (M/M_0) \\ &\geqq \text{tr.dg. } [S/M : R/P] + \text{height } (M) .\end{aligned}$$

Proof of Proposition 1. Let P be a minimal prime component of I. We must show that height $(P) \geqq n + 1$, for it is well known that height $(P) \leqq n + 1$. Now let $M = PS$. Lemma 1 says: M is a prime ideal, $M \cap R = P$, and S/M is a polynomial ring in n variables over R/P. From these facts and Lemma 2 we have:

$$\text{height } (P) \geqq \text{tr.dg. } [S/M : R/P] + \text{height } (M) = n + \text{height } (M) .$$

Since M contains x, which is not a divisor of zero, height $(M) \geqq 1$ (actually height $(M) = 1$). Thus height $(P) \geqq n + 1$.

COROLLARY. *An ideal generated by an analytically independent subset of a Noetherian ring is of height equal to the cardinality of that set.*

Proof. Let A, J and $\{x_1, \cdots, x_n\}$ be the ring, ideal and analytically independent set in question. Let $R = A[x]$, where x is an indeterminate over A, and let $I = (x, x_1, \cdots, x_n)R$. It is clear that height $(I) =$ height $(J) + 1$. Hence it suffices to prove that height $(I) = n + 1$. It is notationally somewhat bothersome, but completely straightforward to verify that $\{x, x_1, \cdots, x_n\}$ is an analytically independent subset of R. Proposition 1 now applies.

REMARK. One can prove this corollary in a somewhat less elementary way. Localize A at a minimal prime component P of J and then complete. The canonical image of the set $\{x_1, \cdots, x_n\}$ is an analytically independent subset of the complete ring and the image of J is primary for the maximal ideal. Then conclude, using Cohen's structure theory for complete local rings, that the complete ring is an integral extension of a ring of dimension at least n. Thus dimension $(A_P) \geqq n$. The "characteristic equal" and "characteristic unequal" cases must be treated separately.

This corollary and the theorem of "analytic independence of systems of parameters" (proved in many places, among them in Corollary 1 to Proposition 1 of [1]) comprise the following theorem.

THEOREM. *An ideal of a Noetherian ring is of the principal class if, and only if, it is generated by an analytically independent set. Furthermore, the height of an ideal of the principal class is equal to the cardinality of any analytically independent set of generators (which then must be a minimal generating set).*

And of special interest is the following criterion.

COROLLARY. *A subset consisting of dimension* (R) *elements of a local ring R is a system of parameters of R if, and only if, that subset is analytically independent.*

In the notation of the discussion immediately preceding the statement of Proposition 1, this proposition combines with Proposition 1 of [1] to say:

PROPOSITION 2. *The following statements are equivalent.* (1) *Height* $(I) = n + 1$. (2) $\{x, x_1, \cdots, x_n\}$ *is analytically independent.* (3) *The radicals of Q and the ideal generated by* $\{xX_1 - x_1, \cdots, xX_n - x_n\}$ *coincide.*

And in the special case that R is an integral domain we have:

PROPOSITION 2′. *Statements* (1) *and* (2) *of Proposition* 2 *are equivalent to:* (3′) *Q is the radical of the ideal generated by* $\{xX_1 - x_1, \cdots, xX_n - x_n\}$.

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