PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 1, 1968

INTEGRAL INEQUALITIES INVOLVING SECOND ORDER DERIVATIVES

JAMES CALVERT

An integral inequality involving second order derivatives is derived. A most important consequence of this inequality is that the Dirichlet form

$$D(u, u) = \int_D \sum_{i,k} a_{ik} D_i^2 u D_k^2 \bar{u} = q|u|^2 dx \geqq 0 ,$$

for functions $q(x)$ which are positive and "not too large" in a sense which will be made precise later and for functions $u(x)$ with compact support contained in D. Some examples are given and an application is made to an existence theorem for a fourth order uniformly elliptic P.D.E.

An earlier paper by the author [1] contains some similar results for inequalities involving first derivatives. The following definitions and notations will be used throughout the paper. Let

$$x = (x_1, x_2, \cdots, x_n) \in R^n .$$

Let D be an open domain in R^n which may be unbounded. Let $C^\infty(D)$ denote the set of infinitely differentiable complex valued functions on D and let $C_0^\infty(D)$ denote the subset of $C^\infty(D)$ consisting of functions with compact support contained in D. Let

$$\| u \|_q = \left(\int_D \sum_{i=1}^n | D_i^2 u |^2 + q| u |^2 dx\right)^{1/2}, \text{ where } D_i^2 u = \frac{\partial^2 u}{\partial x_i^2}$$

and q is either equal to 1 or to one of the positive functions to be defined later. Let $H_q(D)$ be the completion of $\{u \in C^\infty(D): \| u \|_q < \infty\}$ with respect to $\| u \|_q$ and let $\mathring{H}_q(D)$ be the completion of $C_0^\infty(D)$ with respect to $\| u \|_q$. The functions u in $H_q(D)$ or $\mathring{H}_q(D)$ have strong L_2 second derivatives which we will denote by the same symbol as for the oridnary derivative. So that

$$\lim_{n\to\infty} \int_D | D_i^2 u - D_i^2 u_n |^2 dx = 0$$

where $\{u_n\}$ is any sequence of elements in $C^\infty(D)$ such that $\| u - u_n \|_q \to 0$. All coefficient functions considered will be real valued. The variable functions u may be complex valued. There do not seem to be any analogues of the basic results with complex valued coefficients.

THEOREM 1. *Suppose that the boundary of D is smooth enough* *to apply Gauss' Theorem. Let* $a_{ik} \in C^1(D)$ *and* (a_{ik}) *be symmetric positive definite. Let* $f_1, f_2, \cdots, f_n \in C^1(D)$, $q_k = (f_k + D_k)f_k$ *and suppose that* $\sum_i a_{ik}q_i \leqq 0$, *for every* $k = 1, 2, \cdots, n$. *Then, for any* $u \in C^1(D)$,

$$\int_D \sum_{i,k} a_{ik}D_i^2uD_k^2\bar{u} - (f_i + D_i)^2(a_{ik}q_k) \mid u \mid^2 dx$$
$$\geqq \int_{\dot{D}} \sum_{i,k} [a_{ik}q_iD_k \mid u \mid^2 - (D_k(a_{ik}q_i) + 2a_{ik}q_if_k) \mid u \mid^2]\nu_k ds$$

where ν_k *is the* k^{th} *component of the normal and the integral on the integral on the right is assumed to exist. Equality holds if and only if* $D_i^2u = q_iu$ *and* $D_iu = f_iu$, *for every* i.

Proof. We shall require two integrations by parts.

$$\int_D a_{ik}q_i(uD_k^2\bar{u} + \bar{u}D_k^2u)dx$$
$$= -\int_D [a_{ik}q_iD_ku + uD_k(a_{ik}q_i)]D_k\bar{u}$$
$$+ [a_{ik}q_iD_k\bar{u} + \bar{u}D_k(a_{ik}q_i)]D_kudx$$
$$+ \int_{\dot{D}} a_{ik}q_i(\bar{u}D_ku + uD_k\bar{u})\nu_k ds$$
$$= \int_D D_k^2(a_{ik}q_i) \mid u \mid^2 - 2a_{ik}q_i \mid D_ku \mid^2 dx$$
$$+ \int_{\dot{D}} [a_{ik}q_iD_k \mid u \mid^2 - D_k(a_{ik}q_i) \mid u \mid^2]\nu_k ds$$

and

$$-\int_D a_{ik}q_if_k(\bar{u}D_ku + uD_k\bar{u})dx$$
$$= \int_D D_k(a_{ik}q_if_k) \mid u \mid^2 dx - \int_{\dot{D}} a_{ik}q_if_k \mid u \mid^2 \nu_k ds .$$

$$\sum_{i,k} a_{ik}(D_i^2u - q_iu)(D_k^2\bar{u} - q_k\bar{u}) - 2\sum_i a_{ik}q_i \sum_k \mid D_ku - f_ku \mid^2 \geqq 0$$

$$\int_D \sum_{i,k} a_{ik}D_i^2uD_k^2\bar{u} - (f_i + D_i)^2a_{ik}q_k \mid u \mid^2 dx$$
$$\geqq \int_D \sum_{i,k} - (f_i + D_i)^2a_{ik}q_k \mid u \mid^2 + a_{ik}q_i(uD_k^2\bar{u} + \bar{u}D_k^2u) - a_{ik}q_iq_k \mid u \mid^2$$
$$+ 2a_{ik}q_i(\mid D_ku \mid^2 - f_kuD_k\bar{u} - f_k\bar{u}D_ku + f_k^2 \mid u \mid^2)dx$$
$$= \int_D \sum_{i,k} - [a_{ik}f_i^2q_k + f_iD_i(a_{ik}q_k) + D_i(a_{ik}f_iq_k) + D_i^2(a_{ik}q_k)] \mid u \mid^2$$
$$+ [D_k^2(a_{ik}q_i) - a_{ik}q_iq_k + 2D_k(a_{ik}q_if_k) + 2a_{ik}q_if_k^2] \mid u \mid^2$$
$$- 2a_{ik}q_i \mid D_ku \mid^2 + 2a_{ik}q_i \mid D_ku \mid^2 dx$$

$$+ \int_{\dot{D}} \sum_{i,k} [a_{ik}q_iD_k |u|^2 - D_k(a_{ik}q_i)|u|^2 - 2a_{ik}q_if_k|u|^2]\nu_k ds$$

$$= \int_D \sum_{i,k} [a_{ik}q_i(f_k^2 - q_k) - f_iD_i(a_{ik}q_k) + D_i(a_{ik}f_iq_k)]|u|^2dx$$

$$+ \int_{\dot{D}} [\cdots]ds$$

$$= \int_D \sum_{i,k} [-a_{ik}q_iD_kf_k - f_kD_k(a_{ik}q_i) + D_k(a_{ik}f_kq_i)]|u|^2dx$$

$$+ \int_{\dot{D}} [\cdots]ds$$

$$= \int_{\dot{D}} [a_{ik}q_iD_k|u|^2 - D_k(a_{ik}q_i)|u|^2 - 2a_{ik}q_if_k|u|^2]\nu_k ds ,$$

which was to be shown.

(1) We will reserve the notation $q(x)$ for a positive function of of the form $q(x) = \sum_{i,k} (f_i + D_i)^2 a_{ik}q_k$.

COROLLARY 1. *Suppose that D is any open set. If $a_{ik}(x)$ is uniformly bounded in D, then*

$$\int_D \sum_{i,k} a_{ik}D_i^2uD_k^2\bar{u} - q|u|^2dx \geqq 0 ,$$

for every $u \in \mathring{H}_q(D)$ and equality holds if and only if $D_i^2u = q_iu$ and $D_iu = f_iu$ almost everywhere, for each i.

Proof. It is easy to obtain the inequality for functions in $C_0^\infty(D)$ by integrating around a sphere containing the support of u. The result for $u \in \mathring{H}_q(D)$ can then be obtained by showing that

$$\int_D a_{ik}D_i^2u_mD_k^2\bar{u}_mdx \xrightarrow{m} \int_D a_{ik}D_i^2uD_k^2\bar{u}dx$$

which follows easily from Cauchy's inequality.

COROLLARY 2. *If (a_{ik}) is only positive semidefinite, the same inequality holds but the conditions for equality are not necessarily the same. If $a_{ik} = 1$, the conditions are*

$$\sum_i (D_i^2u - q_iu) = 0 \quad \textit{and} \quad D_iu = f_iu .$$

EXAMPLE 1. Corollary 2 can be used to obtain inequalities whenever a solution u_1 of a plate problem

$$\Delta\Delta u - pu = 0 \quad \text{in } D$$

$$\left.\begin{aligned} u &= 0 \\ \Delta u &= 0 \end{aligned}\right\} \quad \text{on } \dot{D}$$

is known. Here $\Delta u = \sum_k D_k^2 u$ and $\Delta\Delta u = \sum_{i,k} D_i^2 D_k^2 u$. Suppose that $u_1 > 0$ in D and let $f_k = (D_k u_1)/u_1$. We must show that

$$\sum_k (f_k + D_k) f_k = \sum_k \frac{D_k^2 u_1}{u_1} \leqq 0 .$$

For this it suffices to show that $\sum_k D_k^2 u_1 \leqq 0$. If $\sum_i D_i^2 v = p u_1 \geqq 0$ and $v = 0$ on $\dot{D}$, then $v \leqq 0$ by the maximum principle. Set $v = \sum_k D_k^2 u_1$, then v satisfies the problem and hence $\sum_k D_k^2 u_1 \leqq 0$ in D. Calculate

$$\sum_{i,k} (f_i + D_i)^2 (f_k + D_k) f_k = \sum_{i,k} \frac{D_i^2 D_k^2 u_1}{u_1} = p .$$

So

$$\int_D \sum_{i,k} D_i^2 u D_k^2 \bar{u} - p \mid u \mid^2 dx \geqq 0, \quad \text{for} \quad u \in C_0^\infty(D) .$$

In particular, for $p = \lambda q$, where λ is the first eigenvalue of the plate problem and u_1 is the corresponding eigenfunction, the inequality becomes Rayleigh's characterization of the first eigenvalue. In this case, the conditions for equality become $u = k u_1$.

EXAMPLE 2. Suppose $n \geqq 5$, then

$$\int_D \sum_{i,k} D_i^2 u D_k^2 \bar{u} - \frac{n^2(n-4)^2}{16} \Big(\sum_{i=1}^n x_i^2\Big)^{-1} \mid u \mid^2 dx \geqq 0 ,$$

for every $u \in \mathring{H}_q$. To apply Corollary 1, let $f_k = (a/s) x_k$, where $s = \sum_{i=1}^n x_i^2$, then

$$q_k = \frac{a(a-2)}{s^2} x_k^2 + \frac{a}{s} ,$$

$$\sum_{k=1}^n q_k = \frac{a(a + (n-2))}{s} \leqq 0 \quad \text{if} \quad 2 - n \leqq a < 0$$

(the other possibility leads to nothing of interest). Calculate,

$$q = \frac{1}{s^2} a(a-2)[a - (4-n)][a - (2-n)] .$$

Then $q > 0$, if $a < 0, a - 2 > 0$, and $a \geqq 4 - n$. If we choose $a = (4-n)/2$, then q is maximal and equal to $(n^2(n-4)^2)/16$.

It is unfortunate that the preceding example is only good for dimensions larger than five. The following inequality, though unappealing, does yield an example for every dimension.

THEOREM 2. *Let* $f_1, f_2, \cdots, f_n \in C^1(D)$ *and suppose that*

$$\sum_{i,k} D_i f_i \,|\, \xi_k \,|^2 + (D_k f_i + D_i f_k + f_i f_k)\xi_i \bar{\xi}_k \leqq 0$$

for every vector $(\xi_1, \xi_2, \cdots, \xi_n)$. *Then*

$$\begin{aligned}\int_D \sum_{i,k} D_i^2 u D_k^2 \bar{u} &- [D_k^2 D_i f_i + 3 f_k D_k D_i f_i + f_i D_k^2 f_i \\ &+ 2 f_k^2 D_i f_i + 4 f_i f_k D_k f_i + (D_k f_i)^2 + (D_k f_i)(D_i f_k) \\ &+ (D_k f_k)(D_i f_i) + f_i^2 f_k^2] \,|\, u \,|^2 \, dx \\ &\geqq \int_{\dot{D}} \sum_{i,k} [2 \operatorname{Re} D_i(f_i u) D_k \bar{u} - f_i \,|\, D_k u \,|^2 \\ &- (D_k D_i f_i + 2 f_k D_i f_i + f_i D_k f_i \\ &+ f_i D_i f_k + f_i^2 f_k) \,|\, u \,|^2] \nu_k ds \,.\end{aligned}$$

Proof.

$$\begin{aligned}(2) \qquad \sum_{i,k} [D_i^2 u - D_i(f_i u)][D_k^2 \bar{u} - D_k(f_k \bar{u})] - D_i f_i \,|\, D_k u - f_k u \,|^2 \\ - (D_k f_i + D_i f_k + f_i f_k)(D_i u - f_i u))D_k \bar{u} - f_k \bar{u}) \geqq 0\end{aligned}$$

when expanded the first term in (2) contains the following two terms which we integrate by parts:

$$-D_i(f_i u) D_k^2 \bar{u} - D_i(f_i \bar{u}) D_k^2 u \quad \text{and} \quad D_i(f_i u) D_k(f_k \bar{u}) \,.$$

Notice that the order of summation has been changed in the first term.

$$\begin{aligned}&-\int_D D_i(f_i u) D_k^2 \bar{u} + D_i(f_i \bar{u}) D_k^2 u dx \\ &= \int_D D_k D_i(f_i u) D_k \bar{u} + D_k D_i(f_i \bar{u}) D_k u dx \\ &\quad - \int_{\dot{D}} [D_i(f_i u) D_k \bar{u} + D_i(f_i \bar{u}) D_k u] \nu_k ds \\ &= \int_D (u D_k D_i f_i + D_i f_i D_k u + D_k f_i D_i u + f_i D_k D_i u) D_k \bar{u} \\ &\quad + (\bar{u} D_k D_i f_i + D_i f_i D_k \bar{u} + D_k f_i D_i \bar{u} + f_i D_k D_i \bar{u}) D_k u \\ &\quad - \int_{\dot{D}} [\cdots] \nu_k ds \\ &= \int_D D_k D_i f_i D_k \,|\, u \,|^2 + f_i D_i \,|\, D_k u \,|^2 + 2 D_i f_i \,|\, D_k u \,|^2 \\ &\quad + D_k f_i D_i u D_k \bar{u} + D_k f_i D_i \bar{u} D_k u dx \\ &\quad - \int_{\dot{D}} [\cdots] \nu_k ds\end{aligned}$$

$$
\begin{aligned}
(3) \qquad &= \int_D -(D_k^2 D_i f_i) \mid u \mid^2 + (D_i f_i) \mid D_k u \mid^2 \\
&\quad + D_k f_i D_i u D_k \bar{u} + D_k f_i D_i \bar{u} D_k u dx \\
&\quad + \int_{\dot{D}} [-2 \operatorname{Re} D_i(f_i u) D_k \bar{u} + D_k D_i f_i \mid u \mid^2 + f_i \mid D_k u \mid^2] \nu_k ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_D \sum_{i,k} D_i(f_i u) D_k(f_k \bar{u}) dx \\
&\quad = \int_D \sum_{i,k} (D_i f_i)(D_k f_k) \mid u \mid^2 + f_k(D_i f_i)(u D_k \bar{u} + \bar{u} D_k u) \\
&\quad\quad + f_i f_k (D_i u)(D_k \bar{u}) dx \\
&\quad = \int_D \sum_{i,k} (D_i f_i)(D_k f_k) \mid u \mid^2 - D_k[f_k(D_i f_i)] \mid u \mid^2 + f_i f_k (D_i u)(D_k \bar{u}) dx \\
&\quad\quad + \int_{\dot{D}} \sum_{i,k} f_k (D_i f_i) \mid u \mid^2 \nu_k ds \\
(4) \quad &\quad = \int \sum_{i,k} - f_k(D_k D_i f_i) \mid u \mid^2 + f_i f_k (D_i u)(D_k \bar{u}) dx \\
&\quad\quad + \int_{\dot{D}} \sum_{i,k} f_k (D_i f_i) \mid u \mid^2 \nu_k ds .
\end{aligned}
$$

The second term in (2) contains

$$
\begin{aligned}
(5) \qquad &\int_D f_k D_i f_i (u D_k \bar{u} + \bar{u} D_k u) dx \\
&\quad = -\int_D [(D_k f_k)(D_i f_i) + f_k D_k D_i f_i] \mid u \mid^2 dx \\
&\quad\quad + \int_{\dot{D}} f_k D_i f_i \mid u \mid^2 \nu_k ds .
\end{aligned}
$$

The third term in (2) contains

$$
\begin{aligned}
&\int_D \sum_{i,k} (D_k f_i + D_i f_k + f_i f_k)(f_i u D_k \bar{u} + f_k \bar{u} D_i u) dx \\
&\quad = \int_D \sum_{i,k} f_i (D_k f_i + D_i f_k + f_i f_k) D_k \mid u \mid^2 dx \\
(6) \qquad &\quad = -\int_D \sum_{i,k} D_k (f_i D_k f_i + f_i D_i f_k + f_i^2 f_k) \mid u \mid^2 dx \\
&\quad\quad + \int_{\dot{D}} \sum_{i,k} f_i (D_k f_i + D_i f_k + f_i f_k) \mid u \mid^2 \nu_k ds .
\end{aligned}
$$

Expanding (2) and making use of (3), (4), (5) and (6), one can obtain the advertised result.

COROLLARY. *Suppose that f_i is a function of x_i alone. Then*

$$\int_D \sum_{i,k} D_i^2 u D_k^2 \bar{u} - \sum_i [D_i^3 f_i + 4f_i D_i^2 f_i + 4f_i^2 D_i f_i + 2(D_i f_i)^2] \, |u|^2$$
$$- \sum_{i,k} [2f_k^2 D_i f_i + (D_k f_k)(D_i f_i) + f_i^2 f_k^2] \, |u|^2 \, dx$$
$$\geqq 0, \text{ for every } u \in \overset{\circ}{H}_q$$

where q is the coefficient of $|u|^2$.

EXAMPLE 3. Let $f_i = a/x_i$. Then

$$\sum_{i,k} D_i f_i \, |\xi_k|^2 + (D_k f_i + D_i f_k + f_i f_k)\xi_i \bar{\xi}_k$$
$$= \sum_{i,k} - \frac{a}{x_i^2} |\xi_k|^2 + \frac{a^2}{x_i x_k} \xi_i \bar{\xi}_k + \sum_i - \frac{2a}{x_i^2} |\xi_i|^2$$
$$= a^2 \sum_{i,k} \frac{\xi_i \bar{\xi}_k}{x_i x_k} - a \sum_i \frac{1}{x_i^2} \sum_i |\xi_i|^2 - 2a \sum_i \frac{|\xi_i|^2}{x_i^2}$$
$$\leqq (a^2 - a) \sum_i |\xi_i|^2 \sum_i \frac{1}{x_i^2} - 2a \sum \frac{|\xi_i|^2}{x_i^2} \, .$$

Let $a = 1 + \varepsilon, \varepsilon > 0$. The right side will be negative when

$$\varepsilon \sum_i |\xi_i|^2 \sum_i \frac{1}{x_i^2} \leqq 2 \sum_i \frac{|\xi_i|^2}{x_i^2} \, .$$

Take $\lambda_i = (|\xi_i|^2)/(\sum_i |\xi_i|^2)$, then $\sum_i \lambda_i = 1$ and the inequality becomes

$$\frac{\varepsilon}{2} \sum_i \frac{1}{x_i^2} \leqq \sum_i \lambda_i \frac{1}{x_i^2} \, .$$

It is always possible to choose an ε so that this inequality holds provided D is bounded and bounded away from the origin. For let $0 < m \leqq x_i^2 \leqq M$, then

$$\sum_i \lambda_i \frac{1}{x_i^2} \geqq \frac{1}{M} \quad \text{and} \quad \sum_i \frac{1}{x_i^2} \leqq \frac{n}{m} \, .$$

Take $\varepsilon/2 \leqq (m/nM)$ and the inequality holds.

Let us compute q using the formula in the corollary.

$$q = \sum_i \frac{-6a + 8a^2 - 4a^3 + 2a^2}{x_i^4} + \sum_{i,k} \frac{-2a^3 + a^2 + a^4}{x_i^2 x_k^2}$$
$$= \sum_i \frac{(-4a^3 + 10a^2 - 6a)}{x_i^4} + a^2(a-1)^2 \sum_{i,k} \frac{1}{x_i^2 x_k^2}$$
$$= (a^4 - 6a^3 + 11a^2 - 6a) \sum_i \frac{1}{x_i^4} + a^2(a-1)^2 \sum_{i \neq k} \frac{1}{x_i^2 x_k^2}$$
$$= a(a-1)(a-2)(a-3) \sum_i \frac{1}{x_i^4} + a^2(a-1)^2 \sum_{i \neq k} \frac{1}{x_i^2 x_k^2} \, .$$

Taking $a = 1 + \varepsilon$

$$q = \varepsilon(1+\varepsilon)(1-\varepsilon)(2-\varepsilon)\sum_i \frac{1}{x_i^4} + \varepsilon^2(1+\varepsilon)^2 \sum_{i\neq k} \frac{1}{x_i^2 x_k^2}$$

which is positive for $\varepsilon < 1$ or for $\varepsilon > 2$.

THEOREM 3. *A fourth order existence theorem.*

Let $q(x)$ be a function of the special form (1) and let $p(x)$ be a continuously differentiable function such that $0 < p(x) \leqq (1-\varepsilon)q(x)$, where $\varepsilon > 0$ and fixed. Let $\int_D q^{-1}|f|^2\,dx < \infty$, $g \in H_q$ and let $Au = \sum_{i,k} D_i^2(a_{ik}D_k^2 u) - pu$ be a uniformly elliptic operator. That is, a_{ik} is positive definite and there exist positive constants M and λ such that

$$|a_{ik}(x)| \leqq M \quad and \quad \lambda \sum_i |\xi_i|^2 \leqq \sum_{i,k} a_{ik}(x)\xi_i\bar{\xi}_k \,,$$

for any $(\xi_1, \xi_2, \cdots, \xi_n)$ and all x in D. Then the Dirichlet problem

$$\begin{aligned} Au &= f && \text{in } D \\ u &= g && \left.\right\} \text{on } \dot{D} \\ \sum_i D_i^2 u &= \sum_i D_i^2 g && \end{aligned}$$

has a unique weak solution.

Proof. We must show that there is a function $u \in H_q$ such that $u - g \in \mathring{H}_q$ and $(u, A\varphi) = (f, \varphi)$, for every φ in C_0^∞. Set $u_0 = u - g$ and consider the equivalent problem of finding $u_0 \in \mathring{H}_q$ such that $(u_0, A\varphi) = (f, \varphi) - (g, A\varphi)$. Let

$$\begin{aligned} B(u, v) &= \int_D \sum_{i,k} a_{ik} D_i^2 u D_k^2 \bar{v} - pu\bar{v}dx \\ &= \int_D \sum_{i,k} u D_i^2(a_{ik} D_k^2 \bar{v} - pu\bar{v}dx \\ &= (u, Av), \quad \text{for} \quad u, v \in C_0^\infty \,. \end{aligned}$$

We will show that there exist $c_1, c_2 > 0$ such that

$$|B(u, v)| \leqq C_1 \|u\|_q \|v\|_q \quad \text{and} \quad B(u, u) \geqq C_2 \|u\|_q^2 \,.$$

$$\begin{aligned} B(u, u) &= \int_D \sum_{i,k} a_{ik} D_i u D_k \bar{u} - p|u|^2 dx \\ &\geqq \int_D \sum_{i,k} a_{ik} D_i u D_k \bar{u} - q|u|^2 dx + \varepsilon \int_D q|u|^2 dx \,. \end{aligned}$$

By Corollary 1, both integrals are positive and hence

$$\left(1+\frac{2}{\varepsilon}\right)B(u,u) \geqq \int_D \sum a_{ik}D_iuD_k\bar{u} + q\,|\,u\,|^2\,dx$$
$$\geqq \int_D \lambda \sum_i |\,D_iu\,|^2 + q\,|\,u\,|^2\,dx$$
$$\geqq \text{const}\,||\,u\,||_q^2\,.$$

The positivity of $B(u,u)$ implies that $|\,B(u,v)\,|^2 \leqq B(u,u)\cdot B(v,v)$ so that we need only show that $B(u,u) \leqq \text{const}\,||\,u\,||_q^2$.

$$B(u,u) \leqq M\int_D \sum_{i,k} |\,D_iuD_k\bar{u}\,| + p\,|\,u\,|^2\,dx$$
$$\leqq \int_D \sum_{i,k} \frac{1}{2}(|\,D_iu\,|^2 + |\,D_ku\,|^2) + p\,|\,u\,|^2\,dx$$
$$\leqq Mn\int_D \sum_i |\,D_iu\,|^2 + p\,|\,u\,|^2\,dx = Mn\,||\,u\,||_q^2\,.$$

Now extend $B(u,v)$ to all of $\mathring{H}_q$ by continuity. We can now apply the Lax-Milgram Theorem which guarantees that any bounded linear functional $F(\varphi)$ on $\mathring{H}_q$ can be represented as $\overline{B(u_0,\varphi)}$ for some $u_0 \in \mathring{H}_q$. Take $F(\varphi) = \overline{(f,\varphi) - B(g,\varphi)}$. Then

$$|\,F\varphi\,| \leqq \left(\int_D q^{-1}\,|\,f\,|^2 dx\right)^{1/2}\left(\int_D q\,|\,\varphi\,|^2 dx\right)^{1/2}$$
$$+\, c_1\,||\,\varphi\,||_q\,||\,g\,||_q \leqq \text{const}\,||\,\varphi\,||_q\,.$$

So $B(u_0,\varphi) = (f,\varphi) - B(g,\varphi)$ as was to be shown. To obtain the uniqueness result, let $Au = 0$, $u \in \mathring{H}_q$, then

$$0 = (u, Au) = B(u,u) \geqq C_2\,||\,u\,||_q^2. \qquad \therefore\ u = 0 \text{ a.e.}$$

Reference

1. James Calvert, *An integral inequality with applications to the Dirichlet problem*, Pacific J. Math. **22** (1967).

Received June 8, 1967.
UNIVERSITY OF CALIFORNIA, DAVIS