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TENSOR PRODUCTS OF W^*-ALGEBRAS

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This paper deals primarily with a characterization of the tensor products of a family of W^*-algebras (abstract von Neumann algebras). It is especially concerned with infinite tensor products; the results, however, apply and have interest in the finite case.

A tensor product for a family $(\mathscr{A}_i)$ of W^*-algebras is defined to be a W^*-algebra $\mathscr{A}$ together with injections α_i of $\mathscr{A}_i$ into $\mathscr{A}$ satisfying four conditions: the first two are that the $\alpha_i(\mathscr{A}_i)$ commute and generate $\mathscr{A}$; the last two are conditions on the set of positive normal functionals of $\mathscr{A}$ which are products with respect to the $\alpha_i(\mathscr{A}_i)$. A local tensor product is defined to be a tensor product satisfying a fifth condition—that its tail reduce to the scalars. It is shown that the local tensor products of $(\mathscr{A}_i)$ are precisely the incomplete direct products $\otimes(\mathscr{A}_i, \mu_i)$, and that every tensor product is a direct sum of local tensor products which are not product isomorphic.

Suppose that $(\mathscr{A}_i)_{i\in I}$ is a family of W^*-algebras. We cal $(\mathscr{A}, (\alpha_i)_{i\in I})$ a *product* for the family $(\mathscr{A}_i)_{i\in I}$ if $\mathscr{A}$ is a W^*-algebra if, for each $i \in I, \alpha_i$ is an injection of $\mathscr{A}_i$ into $\mathscr{A}$ with $\alpha_i(1) = 1$ and if the following conditions hold:

(I). $\alpha_i(\mathscr{A}_i)$ commutes with $\alpha_j(\mathscr{A}_j)$ for all $i, j \in I$ with $i \neq j$.

(II). $\mathscr{R}\{\alpha_i(\mathscr{A}_i): i \in I\} = \mathscr{A}$: that is, $\mathscr{A}$ is the smallest W^* subalgebra of $\mathscr{A}$ which contains all $\mathscr{A}_i$ for $i \in I$.

By a *product functional* for $(\mathscr{A}, (\alpha_i))$ we mean a nonzero *norma positive* functional μ on $\mathscr{A}$ for which there exist normal positiv functionals μ_i on $\mathscr{A}_i$ for each $i \in I$ such that:

$$\mu\Big(\prod_{i=I} \alpha_i(A_i)\Big) = \prod_{i\in I} \mu_i(A_i)$$

whenever each $A_i \in \mathscr{A}_i$ and $A_i = 1$ for a.a. $i \in I$. (a.a. $i \in I$ means—here and throughout the paper—*all but a finite number of* $i \in I$) Because of (II), it is evident that the μ_i determine μ uniquely, an we write $\mu = \bigotimes_{i\in I} \mu_i$. We will denote the set of product functional for $(\mathscr{A}, (\mathscr{A}_i))$ by Σ_p.

We call $(\mathscr{A}, (\alpha_i))$ a *tensor product* for $(\mathscr{A}_i)$ if it is a produc for $(\mathscr{A}_i)$ (i.e., if (I) and (II) hold) and if the following conditions hold

(III). Σ_p is separating: i.e., if $A \in \mathscr{A}^+$ and $\mu(A) = 0$ for al $\mu \in \Sigma_p$, then $A = 0$.

(IV). For all $\mu \in \Sigma_p$, (IV-μ) holds.

(IV-μ). $\mu = \bigotimes_{i \in I} \mu_i \in \Sigma_p$, and, if ν_i is a nonzero normal positive functional on $\mathscr{A}_i$ with $\nu_i = \mu_i$ for a.a. $i \in I$, then $\bigotimes_{i \in I} \nu_i$ exists in Σ_p.

We define the tail $\mathscr{T}$ of a product $(\mathscr{A}, (\alpha_i))$ to be the intersection over all finite subsets F of I of the algebras

$$\mathscr{A}_{I-F} = \mathscr{R}\{\alpha_i(\mathscr{A}_i) \colon i \in I - F\} .$$

We call $(\mathscr{A}, (\alpha_i))$ a *local tensor product* if it is a tensor product and the following condition holds:

(V). The tail $\mathscr{T}$ of the product $(\mathscr{A}, (\alpha_i))$ consists of the scalars only.

A local tensor product will be called a *(μ_i)-local tensor product* if $\bigotimes \mu_i \in \Sigma_p$.

We show (Theorem 4.7) that, for every family $(\mathscr{A}_i, \mu_i)_{i \in I}$ with μ_i a normal positive functional on the W^*-algebra $\mathscr{A}_i$ and

$$0 < \prod_{i \in I} \mu_i(1) < \infty ,$$

a (μ_i)-local tensor product exists and is unique up to isomorphism. (An *isomorphism* of a product $(\mathscr{A}, (\alpha_i))$ with a product $(\mathscr{A}, (\beta_i))$ is an isomorphism ψ of $\mathscr{A}$ onto $\mathscr{B}$ such that $\psi \circ \alpha_i = \beta_i$ for all $i \in I$.) In fact, a (μ_i)-local tensor product for $(\mathscr{A}_i)$ can be constructed as follows. For each $i \in I$ let ϕ_i be an isomorphism of $\mathscr{A}_i$ onto a von Neumann algebra on the Hilbert space H_i and let $x_i \in H_i$ induce μ_i:

$$\mu_i(A_i) = ((\phi_i(A_i))x_i \mid x_i) \qquad \text{for all } A_i \in \mathscr{A}_i .$$

Let $\mathscr{A}$ be $\bigotimes_{i \in I} (\phi_i(\mathscr{A}_i), x_i)$, i.e., von Neumann's incomplete direct product of $(\phi(\mathscr{A}_i))_{i \in I}$ with respect to the C_0-sequence (x_i) (see [7], [1], [2], or § 2 below); and for each $i \in I$ let $\alpha_i = \gamma_i \circ \phi_i$, where γ_i is the natural injection of $\phi_i(\mathscr{A}_i)$ into $\mathscr{A}$. Then $(\mathscr{A}, (\alpha_i))$ is a (μ_i)-local tensor product for $(\mathscr{A}_i)_{i \in I}$. A special consequence of the uniqueness of (μ_i)-local tensor products is, then, roughly that the tensor product of a family of von Neumann algebras depends on their algebraic structure only (see Corollary 3.5, below, for a proper statement). This is an easy result which can be proved also from [9] or directly (see remark in [2, § 3]). For finite I, it is a result due to Misonou [4].

If I is finite, all tensor products of $(\mathscr{A}_i)_{i \in I}$ are local and all are isomorphic. Thus properties (I), (II), (III), and (IV) characterize the finite tensor product. A special case of this result was proved by Nakamura [6]: he showed that (I) and (II) characterize the finite tensor product of *finite factors*. A stronger result of this kind was proved by Takesaki [10]: he showed that (I), (II) and the existence of a nonzero ultraweakly continuous (not necessarily positive) product functional characterize the finite tensor product of *factors* (c.f. Lemma 6.2, below).

In § 5, we determine all possible tensor products for $(\mathscr{A}_i)_{i\in I}$. Let Λ be the set of all families $(\mu_i)_{i\in I}$ where each μ_i is a normal positive functional on $\mathscr{A}_i$ and $0 < \prod_{i\in I}\mu_i(1) < \infty$. Define an equivalence relation R on Λ by writing $(\mu_i) \sim (\nu_i)$ when a (μ_i)-local tensor product is necessarily a (ν_i)-local tensor product. Denote Λ/R by Δ and the natural quotient map $\Lambda \to \Delta = \Lambda/R$ by φ. If Γ is a subset of Δ, we call $(\mathscr{A}, (\alpha_i))$ a *Γ-tensor product* for $(\mathscr{A}_i)_{i\in I}$ if $(\mathscr{A}, (\alpha_i))$ is a tensor product for $(\mathscr{A}_i)_{i\in I}$ and if

$$\{(\mu_i) \in \Lambda: \bigotimes \mu_i \text{ exists on } \mathscr{A}\} = \varphi^{-1}(\Gamma) .$$

Then:

1. Every tensor product for $(\mathscr{A}_i)_{i\in I}$ is a Γ-tensor product for some subset Γ of Δ.
2. For every nonempty subset Γ of Δ a Γ-tensor product exists for $(\mathscr{A}_i)_{i\in I}$.
3. A Γ_1-tensor product is isomorphic (as a *product*) to a Γ_2-tensor product if and only if $\Gamma_1 = \Gamma_2$.
4. A Γ-tensor product is a local tensor product if and only if Γ consists of only one point.
5. A Γ-tensor product is the direct sum of $\{\alpha\}$-tensor products as α runs through Γ.

In case each $\mathscr{A}_i$ is semi-finite, the equivalence relation R may be defined explicitly by using the Kakutani product theorem for W^*-algebras [2]. We obtain $(\mu_i) \sim (\nu_i)$ if and only if

$$\sum_{i\in I} [d(\mu_i, \nu_i)]^2 < \infty ,$$

where $d(\mu, \nu)$ is roughly the infinum of $\|x - y\|$ over all representations of $\mathscr{A}$ as a von Neumann algebra and all x, y inducing μ and ν respectively.

It is not difficult to see that Takeda's infinite direct product of $(\mathscr{A}_i)_{i\in I}$ (see [9]) is a Δ-tensor product for $(\mathscr{A}_i)_{i\in I}$.

Section 6 contains some special results on tensor products of factors. Section 7 contains a few simple counterexamples which demonstrate that conditions (III) and (IV) are necessary.

1. Products and factorizations. If μ is a normal positive functional on a W^*-algebra $\mathscr{A}$, we denote the support of μ by $S(\mu)$, and the central support (the smallest projection of the center of $\mathscr{A}$ larger than $S(\mu)$) by $Z(\mu)$.

Throughout this section $(\mathscr{A}_i)_{i\in I}$ will be a *factorization* of the

W^*-algebra $\mathscr{A}$. By this we mean that each $\mathscr{A}_i$ is a W^*-subalgebra of $\mathscr{A}$ and that, if λ_i denotes the inclusion mapping of $\mathscr{A}_i$ into $\mathscr{A}$, $(\mathscr{A}, (\lambda_i))$ is a product for $(\mathscr{A}_i)_{i \in I}$. $\mathscr{Z}$ will denote the center of $\mathscr{A}$ and $\mathscr{Z}_i$ the center of $\mathscr{A}_i$. For J a subset of I we let $\mathscr{A}_J = \mathscr{R}(\mathscr{A}_i: i \in J)$. We call an element of $\mathscr{A}$ *tail* if it is in $\mathscr{T} = \bigcap_F \mathscr{A}_{I-F}$. For $\mu \in \Sigma_p$, $T(\mu)$ will denote the smallest tail projection larger than $S(\mu)$.

LEMMA 1.1. (i). *If $\mu \in \Sigma_p$ and $x > 0$, then $x\mu \in \Sigma_p$, where $(x\mu)(A) = x(\mu(A))$ for all $A \in \mathscr{A}$.*

(ii). *Suppose that μ is a normal positive functional on $\mathscr{A}$ with $\mu(1) = 1$. Then $\mu \in \Sigma_p$ if and only if the family $(\mathscr{A}_i)_{i \in I}$ is independent with respect to μ: i.e., if and only if*

$$u\Big(\prod_{i \in F} A_i\Big) = \prod_{i \in F} \mu(A_i) \tag{1.1}$$

for all $A_i \in \mathscr{A}_i$ and all finite subsets F of I.

Proof. (i) is obvious. Suppose that μ is a normal positive functional on $\mathscr{A}$ with $\mu(1) = 1$. If (1.1) holds let μ_i be the restriction of μ to $\mathscr{A}_i$; then $\mu = \bigotimes_{i \in I} \mu_i \in \Sigma_p$. Suppose, on the other hand, that $\mu \in \Sigma_p$. Then $\mu = \bigotimes_{i \in I} \mu_i$ for normal positive functionals μ_i on $\mathscr{A}_i$. We have $\mu(1) = \prod_{i \in I} \mu_i(1)$, so that $\mu = \bigotimes_{i \in I} \mu'_i$ where $\mu'_i = (\mu_i(1))^{-1}\mu_i$ and $\mu'_i(1) = 1$. Evidently μ'_i is the restriction of μ to $\mathscr{A}_i$, and (1.1) follows.

LEMMA 1.2. (i) $\mathscr{T} \subset \mathscr{Z}$.

(ii) *$Z(\mu) \leqq T(\mu)$ for all $\mu \in \Sigma_p$.*

Proof. $\mathscr{T}$ commutes with each $\mathscr{A}_i$ because $\mathscr{T} \subset \mathscr{A}_{I-\{i\}}$; therefore $\mathscr{T}$ commutes with $\mathscr{A} = \mathscr{R}\{\mathscr{A}_i: i \in I\}$.

LEMMA 1.3. (i) *$\mathscr{Z} \supset \mathscr{Z}_i$ for each $i \in I$*

(ii) *If $\mathscr{A}$ is a factor then each $\mathscr{A}_i$ is a factor.*

LEMMA 1.4. *For all $\mu = \bigotimes_{i \in J} \mu_i \in \Sigma_p$:*

$$S(\mu) \leqq \prod_{i \in I} S(\mu_i) \tag{1.2}$$

and

$$Z(\mu) \leqq \prod_{i \in I} Z(\mu_i)\ . \tag{1.3}$$

Proof.

$$\mu\Big(\prod_{i\in I} S(\mu_i)\Big) = \prod_{i\in I} \mu_i(S(\mu_i)) = \prod_{i\in I} \mu_i(1) = \mu(1) .$$

Therefore (1.2) holds. (1.3) holds because $\prod_{i\in I} Z(\mu_i)$ is a projection of $\mathscr{Z}$ larger than $\prod_{i\in I} S(\mu_i)$ and, hence, by (1.2), larger than $S(\mu)$.

REMARK. The two propositions which follow are stated now for convenience in referring to them later. For the moment, we need only parts (i) and (ii) of Proposition 1.6.

PROPOSITION 1.5. Suppose that J is a subset of I. Then:

(i). $(\mathscr{A}_i)_{i\in I}$ is a factorization of $\mathscr{A}_J$.

(ii). If $\mu = \bigotimes_{i\in I} \mu_i \in \Sigma_p$, then the restriction μ' of μ to $\mathscr{A}_J$ is a product functional on $\mathscr{A}_J$ for the factorization $(\mathscr{A}_i)_{i\in J}$, and μ' is a scalar multiple of $\mu_J = \bigotimes_{i\in J} \mu_i$.

(iii). If Σ is a separating subset of Σ_p, then $\{\mu_J\colon \bigotimes_{i\in I} \mu_i \in \Sigma\}$ is separating on $\mathscr{A}_J$.

(iv). If (III) holds for $(\mathscr{A}_i)_{i\in I}$ then (III) holds for $(\mathscr{A}_i)_{i\in J}$.

(v). If (III) and (IV) hold for $(\mathscr{A}_i)_{i\in I}$, then (IV-μ) holds for $(\mathscr{A}_i)_{i\in J}$ for μ in a separating subset of product functionals on $\mathscr{A}_J$ for $(\mathscr{A}_i)_{i\in J}$.

(vi). If (V) holds for $(\mathscr{A}_i)_{i\in I}$ then (V) holds for $(\mathscr{A}_i)_{i\in J}$.

Proof. (i) and (ii) are obvious, (iii) follows from (ii) and (iv) from (iii). To prove (v) observe that (IV-μ_J) clearly holds for all $\mu \in \Sigma_p$. To prove (vi) let $\mathscr{T}_J$ be the tail of the factorization $(\mathscr{A}_i)_{i\in J}$. For every finite subset F of I:

$$\mathscr{A}_{J-F\cap J} \subset \mathscr{A}_{I-F} .$$

Taking the intersection as F runs over all finite subsets of I, since $F \cap J$ runs over all finite subsets of J, we obtain $\mathscr{T}_J \subset \mathscr{T}$.

PROPOSITION 1.6. Suppose that $(I(j))_{j\in J}$ is a mutually disjoint family of subsets of I whose union is I. Then:

(i). $(\mathscr{A}_{I(j)})_{j\in J}$ is a factorization of $\mathscr{A}$.

(ii). If $\mu = \bigotimes_{i\in I} \mu_i \in \Sigma_p$ then μ is a product functional for the factorization $(\mathscr{A}_{I(j)})_{j\in J}$ and $\mu = \bigotimes_{j\in J} (\bigotimes_{i\in I(j)} \mu_i)$.

(iii). If (III) holds for the factorization $(\mathscr{A}_i)_{i\in I}$ then (III) holds for the factorization $(\mathscr{A}_{I(j)})_{j\in J}$.

(iv). If (V) holds for the factorization $(\mathscr{A}_i)_{i\in I}$ then (V) holds for the factorization $(\mathscr{A}_{I(j)})_{j\in J}$.

REMARK. (IV) holding for $(\mathscr{A}_i)_{i\in I}$ does not necessarily mean that (IV) holds for $(\mathscr{A}_{I(j)})_{j\in J}$: see Example 7.3.

PROPOSITION 1.7. (Zero-one law). For all $\mu \in \Sigma_p$ with $\mu(1) = 1$ and all tail projections T:

$$\mu(T) = 0 \quad \text{or} \quad 1 .$$

Proof. Let F be a finite subset of I. Then μ is a product functional for the factorization $\{\mathscr{A}_F, \mathscr{A}_{I-F}\}$ of $\mathscr{A}$ (Proposition 1.6. (i)) and $T \in \mathscr{A}_{I-F}$; therefore (Lemma 1.1), for all $A \in \mathscr{A}_F$:

$$\mu(AT) = \mu(A)\mu(T) . \tag{1.4}$$

Now $\bigcup_F \mathscr{A}_F$ is ultraweakly dense in $\mathscr{A}$, so (1.4) holds for all $A \in \mathscr{A}$. Putting $A = T \in \mathscr{A}$, we obtain:

$$\mu(T) = (\mu(T))^2 .$$

COROLLARY 1.8. *If $\mu \in \Sigma_p$ and T is a tail projection*:

$$\mu(T) \neq 0 \quad \textit{implies} \quad S(\mu) \leqq T .$$

PROPOSITION 1.9. For every $\mu \in \Sigma_p$, $T(\mu)$ is an atomic projection of $\mathscr{T}$.

Proof. Suppose that T is a projection of $\mathscr{T}$ with $0 \leqq T \leqq T(\mu)$. Then either $\mu(T) = 0$ or $S(\mu) \leqq T$, by Corollary 1.8. If $S(\mu) \leqq T$ then $T = T(\mu)$ by definition. If $\mu(T) = 0$ then $T \leqq 1 - S(\mu)$ and $T(\mu) - T \geqq S(\mu)$; that implies $T = 0$.

COROLLARY 1.10. *For all $\mu, \nu \in \Sigma_p$*:

$$\textit{either} \quad T(\mu) = T(\nu) \quad \textit{or} \quad [T(\mu)][T(\nu)] = 0 .$$

COROLLARY 1.11. *If condition* (III) *holds, then $\mathscr{T}$ is an atomic W^*-algebra.*

LEMMA 1.12. *Suppose that conditions* (III) *and* (IV) *hold and that $i \in I$. For all $A_i \in \mathscr{A}_i^+$ and all $T \in \mathscr{T}^+$*:

$$A_i T = 0 \quad \textit{implies} \quad A_i = 0 \quad \textit{or} \quad T = 0 .$$

Proof. Suppose that $T \neq 0$. Then because of (III), there exists $\mu \in \Sigma_p$ with $\mu(T) \neq 0$. By Proposition 1.6, $\{\mathscr{A}_i, \mathscr{A}_{I-\{i\}}\}$ is a factorization for $\mathscr{A}$ and $\mu = \mu_i \otimes \mu'$ is a product functional for this factorization. We have $T \in \mathscr{A}_{I-\{i\}}$ and $\mu'(T) \neq 0$. Now for every nonzero normal positive functional ν_i on $\mathscr{A}_i$, $\nu_i \otimes \mu'$ exists on $\mathscr{A}$ by (IV). Hence $A_i T = 0$ implies that $(\nu_i \otimes \mu')(A_i T) = 0$ or that $\nu_i(A_i) = 0$ for each ν_i. Therefore $A_i T = 0$ implies $A_i = 0$.

DEFINITION 1.13. Suppose that $(\mathscr{A}, (\alpha_i))$ is a product for $(\mathscr{A}_i)_{i \in I}$ and that $\mu = \bigotimes_{i \in I} \mu_i \in \Sigma_p$. Let $E(\mu) = \sup \{S(\nu): \nu \in \Sigma_p$ and $\nu = \bigotimes_{i \in I} \nu_i$ with $\nu_i = \mu_i$ for a.a. $i \in I\}$.

REMARK. It is clear that $E(\mu)$ is well defined: i.e., $E(\mu)$ does not depend on how μ is expressed as $\bigotimes \mu_i$.

DEFINITION 1.14. A product $(\mathscr{A}, (\alpha_i))$ for $(\mathscr{A}_i)_{i \in I}$ will be said to satisfy (VI-(μ_i)), where each μ_i is a normal positive functional on $\mathscr{A}_i$, if the following conditions hold:

(i). $\mu = \bigotimes_{i \in I} \mu_i$ exists on $\mathscr{A}$.
(ii). (IV-μ) holds.
(iii). $E(\mu) = 1$.

PROPOSITION 1.15. For all $\mu \in \Sigma_p$:

$$E(\mu) \leqq T(\mu) .$$

Proof. Suppose that $\nu = \bigotimes_{i \in I} \nu_i \in \Sigma_p$ with $\nu_i = \mu_i$ for a.a. $i \in I$. Let $F = \{i \in I: \nu_i = \mu_i\}$. Then F is finite so that $T(\mu) \in \mathscr{A}_{I-F}$. By Proposition 1.6, $\{\mathscr{A}_F, \mathscr{A}_{I-F}\}$ is a factorization of $\mathscr{A}$ for which μ and ν are product functionals: $\mu = \mu_F \otimes \mu'$ and $\nu = \nu_F \otimes \nu'$. Clearly $\mu' = \nu'$. We have $0 \neq \mu(T(\mu)) = \mu_F(1)\mu'(T(\mu))$, so that $\nu'(T(\mu)) \neq 0$. Hence $\nu(T(\mu)) = \nu_F(1)\nu'(T(\mu)) \neq 0$ and by Corollary 1.8, $S(\nu) \leqq T(\mu)$. Since $E(\mu)$ is the supremum of such $S(\nu)$, $E(\mu) \leqq T(\mu)$.

PROPOSITION 1.16. Condition (VI-(μ_i)) implies conditions (III), (V) and (IV-ν) for ν in a separating subset of Σ_p.

Proof. Evidently (VI-(μ_i)) implies (IV-ν) for ν in a separating subset of Σ_p, and hence (III). That it implies (V) is a consequence of Proposition 1.9 and Proposition 1.15.

LEMMA 1.17. *Suppose that Z is a projection of $\mathscr{Z}$. Let $\alpha_i: \mathscr{A}_i \rightarrow Z\mathscr{A}$ be defined, for each $i \in I$, by*:

$$\alpha_i(A_i) = ZA_i \qquad \textit{for all } A_i \in \mathscr{A}_i .$$

Let Z_i be the support of α_i. Then $(Z\mathscr{A}, (\alpha'_i))$ is a product for $(Z_i\mathscr{A}_i)$, where α'_i denotes the restriction of α_i to $Z_i\mathscr{A}_i$. Suppose that $\mu' = \bigotimes \mu'_i$ is a product functional for $(Z\mathscr{A}, (\alpha'_i))$. Define μ on $\mathscr{A}$ and μ_i on $\mathscr{A}_i$ by:

$$\mu(A) = \mu'(ZA) \qquad \textit{for all } A \in \mathscr{A}$$
$$\mu_i(A_i) = \mu'_i(Z_iA_i) \qquad \textit{for all } A_i \in \mathscr{A}_i .$$

Then μ *is in* Σ_p, $S(\mu) = S(\mu')$, *and* $\mu = \bigotimes \mu_i$.

Proof. Obviously, since $\alpha'_i(Z_i\mathscr{A}_i) = \alpha_i(\mathscr{A}_i) = Z\mathscr{A}_i$, $(Z\mathscr{A}, (\alpha'_i))$ is a product for $(Z_i\mathscr{A}_i)$. Suppose μ', μ'_i, μ, and μ_i are as in the lemma. Then whenever each $A_i \in \mathscr{A}_i$ and $A_i = 1$ for a.a. $i \in I$:

$$\mu\Big(\prod_{i\in I} A_i\Big) = \mu'\Big(Z\prod_{i\in I} A_i\Big) = \mu'\Big(\prod_{i\in I} \alpha_i(A_i)\Big)$$
$$= \mu'\Big(\prod_{i\in I} \alpha'_i(Z_iA_i)\Big) = \prod_{i\in I} \mu'_i(Z_iA_i) = \prod_{i\in I} \mu_i(A_i) .$$

PROPOSITION 1.18. Suppose that the factorization $(\mathscr{A}_i)_{i\in I}$ satisfies (III) and (IV), and suppose that T is a nonzero tail projection. Let $\alpha_i\colon \mathscr{A}_i \to T\mathscr{A}$ be defined, for each $i \in I$, by:

$$\alpha_i(A_i) = TA_i \qquad \text{for all } A_i \in \mathscr{A}_i .$$

Then:

(i). Each α_i is an isomorphism and $(T\mathscr{A}, (\alpha_i))$ is a tensor product for $(\mathscr{A}_i)$: i.e., $(T\mathscr{A}, (\alpha_i))$ is a product for $(\mathscr{A}_i)$ satisfying (III) and (IV).

(ii). $(T\mathscr{A}, (\alpha_i))$ is a local tensor product if and only if T is atomic in $\mathscr{T}$.

(iii). There is a one-to-one correspondence $\mu' \leftrightarrow \mu$ between product functionals μ' for $(T\mathscr{A}, (\alpha_i))$ and product functionals μ on $\mathscr{A}$ for $(\mathscr{A}_i)$ with $S(\mu) \leqq T$, where μ' is the restriction of μ to $T\mathscr{A}$ and $\mu(A) = \mu'(TA)$ for all $A \in \mathscr{A}$. We have $S(\mu) = S(\mu')$ and $\mu = \bigotimes \mu_i$ if and only if $\mu' = \bigotimes \mu_i$.

Proof. Lemma 1.12 shows that each α_i is an isomorphism. Then Lemma 1.17 shows both that $(T\mathscr{A}, (\alpha_i))$ is a product for $(\mathscr{A}_i)$, and also that, if $\mu' = \bigotimes \mu_i$ is a product functional for $(T\mathscr{A}, (\alpha_i))$, then the μ corresponding to μ' is in Σ_p, $\mu = \bigotimes \mu_i$, and $S(\mu) = S(\mu')$. Suppose that $\mu = \bigotimes \mu_i$ is in Σ_p with $S(\mu) \leqq T$, and let μ' be the restriction of μ to $T\mathscr{A}$. Suppose $A_i \in \mathscr{A}_i$ and $A_i = 1$ for all $i \in I - F$ for a finite subset F of I. Then:

$$\mu'\Big(\prod_{i\in I} \alpha_i(A_i)\Big) = \mu\Big(\prod_{i\in I} TA_i\Big) = \Big[\prod_{i\in F} \mu_i(A_i)\Big][\mu_{I-F}(T)] \tag{1.5}$$

because $T \in \mathscr{A}_{I-F}$. Now $\mu(T) = \mu_F(1)\mu_{I-F}(T)$, and, since $S(\mu) \leqq T$, $\mu(T) = \mu(1) = \mu_F(1)\mu_{I-F}(1)$. Therefore:

$$\mu_{I-F}(T) = \mu_{I-F}(1) = \prod_{i\in I-F} \mu_i(1) . \tag{1.6}$$

Combining (1.5) and (1.6), we conclude that $\mu = \bigotimes \mu_i \in \Sigma_p$. That completes the proof of (iii).

Since (III) holds for the factorization $(\mathscr{A}_i)$, evidently Corollary 1.8 and (iii) demonstrate that (III) holds for $(T\mathscr{A}, (\alpha_i))$. To prove (IV) for $(T\mathscr{A}, (\alpha_i))$, let us assume that $\mu' = \bigotimes \mu_i$ is a product functional for $(T\mathscr{A}, (\alpha_i))$ and that ν_i is a non-zero normal positive functional on $\mathscr{A}_i$ with $\nu_i = \mu_i$ for a.a. $i \in I$. Let μ correspond to μ' as in (iii) so that $\mu = \bigotimes \mu_i$ for $(\mathscr{A}_i)$ and $S(\mu) \leqq T$. Now (IV) holds for $(\mathscr{A}_i)$, so that $\nu = \bigotimes \nu_i$ exists on $\mathscr{A}$. We have $S(\nu) \leqq E(\mu) \leqq T(\mu) \leqq T$ by Propositions 1.15 and 1.9. $\nu' = \bigotimes \nu_i$ exists as a product functional for $(T\mathscr{A}, (\alpha_i))$ by (iii). That demonstrates (IV) and thus (i).

Since T is in $\mathscr{Z}$ and each $\mathscr{A}_{I-F}$, a direct calculation shows that the tail of the product $(T\mathscr{A}, (\alpha_i))$ is precisely $T\mathscr{T}$. Hence (V) holds for $(T\mathscr{A}, (\alpha_i))$ if and only if T is atomic in $\mathscr{T}$. That proves (ii).

2. Direct products of von Neumann algebras. We summarize here the definition and same basic properties of the direct product of a family of von Neumann algebras. For details and omitted proofs, see [7] or [1].

Let I be an arbitrary indexing set. Suppose that $(H_i)_{i \in I}$ is a family of Hilbert spaces and that, for each $i \in I$, x_i is in H_i with $0 < \prod_{i \in I} \|x_i\| < \infty$. Then we denote by $\bigotimes_{i \in I} (H_i, x_i)$ von Neumann's incomplete direct product of the family (H_i) with respect to the C_0-sequence (x_i), (see [7]). Let $\Lambda = \{(y_i)$: each $y_i \in H_i$, $\sum |1 - (x_i \mid y_i)| < \infty$ and $\sum |1 - \|y_i\|| < \infty\}$. Then there is a natural multilinear mapping $(y_i) \to \bigotimes y_i$ from Λ into a dense subset of H with:

$$(\bigotimes y_i \mid \bigotimes z_i) = \prod (y_i \mid z_i) \quad \text{for all} \quad (y_i), (z_i) \in \Lambda\ .$$

LEMMA 2.1. *Suppose that $x_i, y_i \in H_i$ with $0 < \prod \|x_i\|, \prod \|y_i\| < \infty$ and that $\sum |1 - (x_i \mid y_i)| < \infty$. Then $\bigotimes (H_i, x_i) = \bigotimes (H_i, y_i)$.*

LEMMA 2.2. *Suppose that, for each $i \in I$, L_i is a dense linear subset of H_i with $x_i \in L_i$, and suppose that*

$$0 < \prod \|x_i\| < \infty. \text{ Let } L = \{\bigotimes_{i \in I} y_i: y_i \in L_i \text{ for all } i \in I \text{ and } y_i = x_i \text{ for a.a. } i \in I\}$$

Then L is dense in $\bigotimes_{i \in I} (H_i, x_i)$.

LEMMA 2.3. *Let $H = \bigotimes_{i \in I} (H_i, x_i)$. Then, for each $j \in I$, there exists a normal isomorphism α_j of $\mathscr{L}(H_j)$ into $\mathscr{L}(H)$ such that, for all $A_j \in \mathscr{A}_j$ and all $(y_i) \in \Lambda$:*

$$(\alpha_j(A_j))(\bigotimes y_i) = \bigotimes y'_i$$

where $y'_i = y_i$ for $i \neq j$ and $y'_j = A_j y_j$. We call α_j the natural injection of $\mathscr{L}(H_j)$ into $\mathscr{L}(H)$.

DEFINITION 2.4. Suppose that, for each $i \in I$, $\mathscr{A}_i$ is a von Neumann algebra on H_i and $x_i \in H_i$, and suppose that $0 < \prod \|x_i\| < \infty$. Then by $\bigotimes_{i \in I} (\mathscr{A}_i, x_i)$ we will mean $\mathscr{R}(\alpha_i(\mathscr{A}_i): i \in I)$, where α_i is the natural injection of $\mathscr{L}(H_i)$ into $\mathscr{L}(\bigotimes (H_i, x_i))$.

LEMMA 2.5. (i). $\bigotimes_{i \in I} (\mathscr{L}(H_i), x_i) = \mathscr{L}(\bigotimes_{i \in I} (H_i, x_i))$.
(ii). $\bigotimes_{i \in I} (\mathscr{A}_i, x_i)$ *is a factor if and only if each* $\mathscr{A}_i$ *is a factor.*

PROPOSITION 2.6. Suppose that, for each $i \in I, \mathscr{A}_i$ is a von Neumann algebra on H_i and $x_i \in H_i$, and suppose that $0 < \prod \|x_i\| < \infty$. Let $\mu_i(A_i) = (A_i x_i \mid x_i)$. Let $\mathscr{A}$ be $\bigotimes_{i \in I} (\mathscr{A}_i, x_i)$, and let α_i be the natural injection of $\mathscr{A}_i$ into $\mathscr{A}$ for each $i \in I$. Then $(\mathscr{A}, (\alpha_i))$ is a product for $(\mathscr{A}_i)_{i \in I}$ which satisfies (VI-(μ_i)). Furthermore, if $\mu = \bigotimes_{i \in I} \mu_i$, then

$$S(\mu) = \prod_{i \in I} \alpha_i(S(\mu_i)) \ . \tag{2.1}$$

REMARK. (IV) also holds, of course, and is easily proved directly. See Proposition 4.2.

Proof. Obviously $(\mathscr{A}, (\alpha_i))$ is a product for $(\mathscr{A}_i)$ and $\mu = \bigotimes_{i \in I} \mu_i$ exists in Σ_p: in fact, if $x = \bigotimes_{i \in I} x_i$ then $\mu(A) = (Ax \mid x)$ for all $A \in \mathscr{A}$. By Lemma 1.4,

$$S(\mu) \leqq \prod_{i \in I} \alpha_i(S(\mu_i)) \ . \tag{2.2}$$

Now $S(\mu) = \mathrm{pr}\,[\mathscr{A}'x]$ (By $[L]$ we mean the closure of L; by $\mathrm{pr}\,[L]$ we mean the orthogonal projection onto $[L]$). Because $\mathscr{A}'$ contains each $\alpha_i(\mathscr{A}_i')$, $[\mathscr{A}'x]$ contains the closure of

$$\{\bigotimes A_i' x_i \colon \text{each } A_i' \in \mathscr{A}_i' \text{ and } A_i' = 1 \text{ for a.a. } i \in I\} \ .$$

Thus (Lemma 2.2) $[\mathscr{A}'x]$ contains $\bigotimes ([\mathscr{A}_i' x_i], x_i)$. The projection onto this last subspace of $H = \bigotimes (H_i, x_i)$ is $\prod \alpha_i(S(\mu_i))$. Hence $S(\mu) \geqq \prod \alpha_i(S(\mu_i))$ and (2.1) follows from (2.2).

To prove (VI-(μ_i)), let us assume first that every normal positive functional on $\mathscr{A}_i$ is induced by a vector of H_i. Let

$$L = \{\bigotimes y_i \colon y_i \in H_i,\ y_i = x_i \text{ for a.a. } i \in I\} \ .$$

Then L is dense in H by Lemma 2.2. For each nonzero $y \in L$, let ν_y be the functional induced by y:

$$\nu_y(A) = (Ay \mid y) \qquad \text{for all } A \in \mathscr{A} \ .$$

Then a direct calculation shows that $\nu_y = \bigotimes \nu_i$ where ν_i is induced by y_i and $\nu_i = \mu_i$ for a.a. $i \in I$. We have $(S(\nu_y))y = y$. Since every normal positive functional on $\mathscr{A}_i$ is induced by a vector, as y runs through L, ν_y runs through

$$\Sigma = \{\bigotimes \nu_i\colon \nu_i = \mu_i \quad \text{for a.a. } i \in I\}\ .$$

Thus (IV-μ) holds, and

$$E(\mu) = \sup\{S(\nu)\colon \nu \in \Sigma\} \geqq \mathrm{pr}\,[L] = 1\ .$$

To prove (VI-(μ_i)) in the general case we will show that there exist von Neumann algebras $\mathscr{B}_i$ on G_i and vectors $z_i \in G_i$, and that there exists an isomorphism ψ of $\mathscr{A}$ onto $\mathscr{B} = \bigotimes(\mathscr{B}_i, z_i)$ such that:

(2.3) Every normal positive functional on $\mathscr{B}_i$ is induced by a vector.

(2.4) $\psi(\alpha_i(\mathscr{A}_i)) = \beta_i(\mathscr{B}_i)$ where β_i is the natural injection of $\mathscr{L}(G_i)$ into $\mathscr{L}(G)$ and $G = \bigotimes(G_i, z_i)$.

(2.5) If $z = \bigotimes z_i$ then

$$(\psi(A)z \mid z) = \mu(A) \qquad \text{for all } A \in \mathscr{A}\ .$$

Then by the preceding paragraph (VI-(μ_i)) will hold for the product $(\mathscr{B}, (\beta_i))$ and thus for the product $(\mathscr{A}, (\alpha_i))$.

For each $i \in I$, let H'_i be a Hilbert space of infinite dimension, let $x'_i \in H'_i$ with $\|x'_i\| = 1$, and let $\mathscr{C}_i$ be the algebra of scalars on H'_i. Let $\mathscr{B}_i = \mathscr{A}_i \bigotimes \mathscr{C}_i$ on $G_i = H_i \bigotimes H'_i$ and let $z_i = x_i \bigotimes x'_i$. Let $G = \bigotimes(G_i, z_i) = \bigotimes(H_i \otimes H'_i, x_i \otimes x'_i)$ and let $H' = \bigotimes(H'_i, x'_i)$. Then [7] it is easy to construct a natural isometry ϕ from $H \otimes H'$ onto G such that:

$$\phi(\alpha_i(T_i) \otimes 1_{H'})\phi^{-1} = \beta_i(T_i \otimes 1_{H'})$$

for all $T_i \in \mathscr{L}(H_i)$ and all $i \in I$. Define $\psi\colon \mathscr{A} \to \mathscr{L}(G)$ by:

$$\psi(A) = \phi(A \otimes 1_{H'})\phi^{-1} \qquad \text{for all } A \in \mathscr{A}\ .$$

Then (2.4), (2.5), and $\psi(\mathscr{A}) = \mathscr{B}$ follow immediately.

COROLLARY 2.7. *Suppose that $(\mathscr{A}_i)_{i\in F}$ is a finite family of von Neumann algebras. Let $\mathscr{A} = \bigotimes_{i\in F}\mathscr{A}_i$ and let α_i be the natural injection of $\mathscr{A}_i$ into $\mathscr{A}$. Then $(\mathscr{A}, (\alpha_i))$ is a tensor product for $(\mathscr{A}_i)_{i\in F}$ which satisfies* (V). *In particular $\bigotimes \mu_i$ exists in Σ_p for every nonzero normal positive functional μ_i of $\mathscr{A}_i$.*

LEMMA 2.8. *Suppose that $(H_i)_{i\in I}$ is a family of Hilbert spaces and that, for each $i \in I, H_i = \bigoplus_{j\in J(i)} H_i^j$ where $0 \in J(i)$ (by $H_i = \bigoplus_{j\in J(i)} H_i^j$ we mean that the H_i^j are mutually orthogonal subspaces of H_i which span H_i). Suppose that, for each $i \in I$ and $j \in J(i)$, x_i^j*

is a nonzero vector of H_i^j, *and suppose that* $0 < \prod_{i \in I} \| x_i^0 \| < \infty$. *Denote by* J *the set of families* $(j(i))_{i \in I}$ *with each* $j(i) \in J(i)$ *and* $j(i) = 0$ *for* a.a. $i \in I$. *If* $j = (j(i)) \in J$ *let* $H^j = \bigotimes_{i \in I} (H_i^{j(i)}, x_i^{j(i)})$. *Then each* H^j *is a subspace of* $H = \bigotimes_{i \in I} (H_i, x_i^0)$ *and* $H = \bigoplus_{j \in J} H^j$. *Furthermore, if* α_i^j *denotes, for each* $j = (j(i)) \in J$, *the natural injection of* $\mathscr{L}(H_i^{j(i)})$ *into* $\mathscr{L}(H^j)$, *then*:

(2.6) $\alpha_i[\bigoplus_{j \in J(i)} T_i^j] = \bigoplus_{j=(j(i)) \in J} [\alpha_i^j(T_i^{j(i)})]$ *for all* $(T_i^j)_{j \in J(i)}$ *with each* $T_i^j \in \mathscr{L}(H_i^j)$. *(Here* $\bigoplus T_i^j \colon \bigoplus x_i^j \rightarrow \bigoplus T_i^j x_i^j$*).*

Proof. The H^j are clearly mutually orthogonal, and $[H^j \colon j \in J]$ is H by Lemma 2.2. Formula (2.6) can be confirmed by a direct calculation.

3. The basic isomorphism theorems. By a *representation* ψ of a W^*-algebra $\mathscr{A}$ on a Hilbert space H we mean a normal homomorphism $\mathscr{A}$ onto a von Neumann algebra on H (Notice that $\psi(1)$ is the identity on H). If ψ is a representation of $\mathscr{A}$ on H and μ is a normal positive functional on $\mathscr{A}$, a vector $x \in H$ will be called a *μ-cyclic vector* for ψ if $[\psi(\mathscr{A})x] = H$ and

$$\mu(A) = (\psi(A)x \mid x) \qquad \text{for all } A \in \mathscr{A} .$$

Given $\mathscr{A}$ and μ it is well known (see [3, p. 51], for example) that a representation ψ with a μ-cyclic vector exists (and is essentially unique), and that such a ψ acts isomorphically on $(Z(\mu))\mathscr{A}$ and takes $(1 - Z(\mu))\mathscr{A}$ into 0.

PROPOSITION 3.1. Suppose that $(\mathscr{A}_i)_{i \in I}$ is a factorization of the W^*-algebra $\mathscr{A}$ and that $\mu = \bigotimes_{i \in I} \mu_i$ is a product functional for this factorization. Suppose that ψ is a representation of $\mathscr{A}$ on H with μ-cyclic vector x. Suppose that, for each $i \in I$, ψ_i is a representation of $\mathscr{A}_i$ on H_i with μ_i-cyclic vector x_i. Then there exists an isometry ϕ of H onto $\bigotimes_{i \in I} (H_i, x_i)$ such that:

(i). $\phi(x) = \bigotimes_{i \in 1} x_i$.

(ii). $\phi(\psi(\mathscr{A}))\phi^{-1} = \bigotimes_{i \in I} (\psi_i(\mathscr{A}_i), x_i)$.

(iii). For all $A_i \in \mathscr{A}_i$ and each $i \in I$:

$$\phi(\psi(A_i))\phi^{-1} = \alpha_i(\psi_i(A_i))$$

where α_i denotes the natural injection of $\mathscr{L}(H_i)$ into $\mathscr{L}(\bigotimes_{i \in I} (H_i, x_i))$.

Proof. Let $\mathscr{K}$ denote the set of families $(A_i)_{i \in I}$ with each $A_i \in \mathscr{A}_i$ and $A_i = 1$ for a.a. $i \in I$. Let

$$M = \left\{ \left[\psi\left(\prod_{i \in I} A_i \right) \right] x \colon (A_i) \in \mathscr{K} \right\}$$

and

$$N = \left\{ \bigotimes_{i \in I} [(\psi_i(A_i))x_i] \colon (A_i) \in \mathscr{K} \right\} .$$

First we claim that M is a dense subset of H. For $\mathscr{S}$, the $*$-algebra $\{\prod_{i \in I} A_i \colon (A_i) \in \mathscr{K}\}$, is ultrastrongly dense in $\mathscr{A}$ (a corollary of the double-commutant theorem); hence $\psi(\mathscr{S})$ is strongly dense in $\psi(\mathscr{A})$ and $[\psi(\mathscr{S})x] = [\psi(\mathscr{A})x] = H$ because x is a cyclic vector for $\psi(\mathscr{A})$.

Secondly, N is a dense subset of $\bigotimes_{i \in I}(H_i, x_i)$ by Lemma 2.2, since $x_i \in [\psi_i(\mathscr{A}_i)x_i] = H_i$ for each $i \in I$.

Fix $(A_i) \in \mathscr{K}$. Then:

$$\begin{aligned} \left\| \left(\psi\left(\prod_{i \in I} A_i\right)\right)x \right\|^2 &= \mu\left(\prod_{i \in I} A_i^* A_i\right) = \prod_{i \in I} \mu_i(A_i^* A_i) \left\| \bigotimes_{i \in I} [(\psi_i(A_i))x_i] \right\|^2 \\ &= \prod_{i \in I} \| (\psi_i(A_i))x_i \|^2 = \prod_{i \in I} \mu_i(A_i^* A_i) . \end{aligned}$$

Therefore, since M is dense in H and N is dense in $\bigotimes_{i \in I}(H_i, x_i)$, there exists a (unique) isometry ϕ of H onto $\bigotimes_{i \in I}(H_i, x_i)$ such that, for all $(A_i) \in \mathscr{K}$:

$$\phi\left[\left(\psi\left(\prod_{i \in I} A_i\right)\right)x\right] = \bigotimes_{i \in I} [(\psi_i(A_i))x_i] .$$

Now (i) follows immediately, (iii) by a direct calculation, and (ii) from (iii).

THEOREM 3.2. *Suppose that $(\mathscr{A}_i)_{i \in I}$ is a factorization of the W^*-algebra $\mathscr{A}$. Suppose that $\mu = \bigotimes_{i \in I} \mu_i$ is a product functional for this factorization, and suppose that* (IV-μ) *holds. Then there exist, for each $i \in I$, a faithful representation Δ_i of $\mathscr{A}_i$ on H_i and a vector $x_i \in H_i$, and there exists a representation ψ of $\mathscr{A}$ on $H = \bigotimes_{i \in I}(H_i, x_i)$ such that:*

(i) *ψ maps $(1 - E(\mu))\mathscr{A}$ into 0 and maps $(E(\mu))\mathscr{A}$ isomorphically onto $\psi(\mathscr{A}) = \bigotimes_{i \in I}(\Delta_i(\mathscr{A}_i), x_i)$.*

(ii) *For each $i \in I$ and all $A_i \in \mathscr{A}_i$:*

$$\psi(A_i) = \alpha_i(\Delta_i(A_i)) ,$$

where α_i denotes the natural injection of $\mathscr{L}(H_i)$ into $\mathscr{L}(H)$.

(iii). *For each $i \in I$ and all $A_i \in \mathscr{A}_i$:*

$$((\Delta_i(A_i))x_i \mid x_i) = \mu_i(A_i) .$$

(iv). *If x denotes $\bigotimes_{i \in I} x_i$, then, for all $A \in \mathscr{A}$:*

$$((\psi(A))x \mid x) = \mu(A) .$$

Proof. For each $i \in I$, select (by Zorn's lemma) a family $(\mu_i^j)_{j \in J(i)}$ of normal nonzero positive functionals on $\mathscr{A}_i$ such that $\sum_{j \in J(i)} Z(\mu_i^j) = 1$ and $0 \in J(_i)$ with $\mu_i^0 = \mu_i$. Let J be the subset of $\prod_{i \in I} J(_i)$ consisting of $(j(i))$ with $j(i) = 0$ for a.a. $i \in I$. Since (IV-μ) holds, each $j = (j(i)) \in J$ the product functional $\mu^j = \bigotimes_{i \in I} \mu_i^{j(i)}$ exists on $\mathscr{A}$. We have $Z(\mu^j) \leqq \prod_{i \in I} Z(\mu_i^{j(i)})$ by Lemma 1.4, so that $(Z(\mu^j))_{j \in J}$ is a mutually orthogonal family of central projections of $\mathscr{A}$. Let $Z = \sum_{j \in J} Z(\mu^j)$.

For each $j \in J$ let Γ^j be a representation of $\mathscr{A}$ on G^j with a μ^j-cyclic vector y^j. Let Γ be the direct sum representation $\bigoplus_{j \in J} \Gamma^j$ of $\mathscr{A}$ on $G = \bigoplus_{j \in J} G^j$:

$$(3.1) \qquad \Gamma(A) = \bigoplus_{j \in J} \Gamma^j(A) \qquad \text{for all } A \in \mathscr{A} .$$

Then Γ maps $(1 - Z)\mathscr{A}$ into 0 and maps $Z\mathscr{A}$ isomorphically onto $\Gamma(\mathscr{A})$.

For each $i \in I$ and each $j \in J(i)$, let Δ_i^j be a representation of $\mathscr{A}_i$ on H_i^j with μ_i^j-cyclic vector x_i^j. Let Δ_i be the direct sum representation $\bigoplus_{j \in J(i)} \Delta_i^j$ of $\mathscr{A}_i$ on $H_i = \bigoplus_{j \in J(i)} H_i^j$:

$$(3.2) \qquad \Delta_i(A_i) = \bigoplus_{j \in J(i)} \Delta_i^j(A_i) \qquad \text{for all } A_i \in \mathscr{A}_i .$$

Then each Δ_i is faithful.

Fix $j = (j(i))$ in J. We know that $\mu^j = \bigotimes_{i \in I} \mu_i^{j(i)}$, that Γ^j is a representation of $\mathscr{A}$ on G^j with μ^j-cyclic vector y^j, and that $\Delta_i^{j(i)}$, for each $i \in I$, is a representation of $\mathscr{A}_i$ on $H_i^{j(i)}$-cyclic vector $x_i^{j(i)}$. Therefore Proposition 3.1 demonstrates the existence of an isometry ϕ^j from G^j onto $H^j = \bigotimes_{i \in I} (H_i^{j(i)}, x_i^{j(i)})$ such that:

$$(3.3) \qquad \phi^j(y^j) = \bigotimes_{i \in I} x_i^{j(i)}$$

and

$$(3.4) \qquad \phi^j(\Gamma^j(A_i))(\phi^j)^{-1} = \alpha_i^j(\Delta_i^{j(i)}(A_i)) \qquad \text{for all } A_i \in \mathscr{A}_i ,$$

where α_i^j denotes the natural injection of $\mathscr{L}(H_i^{j(i)})$ into $\mathscr{L}(H^j)$.

Let x_i denote x_i^0 for each $i \in I$. Let $H = \bigotimes_{i \in I} (H_i, x_i)$, and denote by α_i the natural injection of $\mathscr{L}(H_i)$ into $\mathscr{L}(H)$. Then (Lemma 2.8), $H = \bigoplus_{j \in J} H^j$, and, for each $i \in I$ and all operators $T_i \in \mathscr{L}(H_i)$ with $T_i = \bigoplus_{j \in J(i)} T_i^j$ and with each $T_i^j \in \mathscr{L}(H_i^j)$:

$$(3.5) \qquad \alpha_i(T_i) = \bigoplus_{j = (j(i)) \in J} (\alpha_i^j(T_i^{j(i)})) .$$

Define the isometry ϕ of G onto H by:

$$\phi\Big(\bigoplus_{j \in J} f^j\Big) = \bigoplus_{j \in J} \phi^j(f^j) \qquad \text{for all } f^j \in G^j .$$

Let ψ be the representation of $\mathscr{A}$ on H defined by:

$$\psi(A) = \phi(\Gamma(A))\phi^{-1} \qquad \text{for all } A \in \mathscr{A} .$$

Evidently ψ has the same kernel as Γ: ψ maps $(1 - Z)\mathscr{A}$ into 0 and $Z\mathscr{A}$ isomorphically onto $\psi(\mathscr{A})$.

Now fix $i \in I$ and $A_i \in \mathscr{A}_i$. In view of (3.2), applying (3.5) to $\Delta_i(A_i)$ we obtain:

$$\text{(3.6)} \qquad \alpha_i(\Delta_i(A_i)) = \bigoplus_{j=(j(i)) \in J} [\alpha_i^j(\Delta_i^{j(i)}(A_i))] .$$

Using (3.1), the definitions of ψ and ϕ, and (3.4), we get:

$$\text{(3.7)} \qquad \begin{aligned} \psi(A_i) &= \phi\Big[\bigoplus_{j \in J} \Gamma^j(A_i)\Big]\phi^{-1} = \bigoplus_{j \in J} \phi^j(\Gamma^j(A_i))(\phi^j)^{-1} \\ &= \bigoplus_{j=(j(i)) \in J} [\alpha_i^j(\Delta_i^{j(i)}(A_i))] . \end{aligned}$$

We conclude, from (3.6) and (3.7), that:

$$\text{(3.8)} \qquad \psi(A_i) = \alpha_i(\Delta_i(A_i)) \qquad \text{for all } A_i \in \mathscr{A}_i \text{ and all } i \in I .$$

Hence ψ maps $\mathscr{A} = \mathscr{B}(\mathscr{A}_i\colon i \in I)$ onto

$$\mathscr{R}(\alpha_i(\Delta_i(\mathscr{A}_i))\colon i \in I) = \bigotimes_{i \in I} (\Delta_i(\mathscr{A}_i), x_i) .$$

Assertion (ii) of the theorem is precisely (3.8). (iii) holds because $x_i = x_i^0$ is a μ_i-cyclic vector for Δ_i. (iv) holds because of (3.4) and the choice of y^0 to be a μ-cyclic vector for Γ^0. To complete the proof of the theorem, then, we need to show only that $Z = E(\mu)$.

Evidently $Z \leqq E(\mu)$. Let $\beta_i\colon \mathscr{A}_i \to Z\mathscr{A}$ be defined by $\beta_i(A_i) = ZA_i$ for all $A_i \in \mathscr{A}_i$. Then we have just proved that $(Z\mathscr{A}, (\beta_i))$ is isomorphic to the product $(\bigotimes (\Delta_i(\mathscr{A}_i), x_i), (\alpha_i \cdot \Delta_i))$, which satisfies (VI-$(\mu_i)$) by Proposition 2.6. Hence $(Z\mathscr{A}, (\beta_i))$ is a product for $(\mathscr{A}_i)$ which satisfies (VI-(μ_i)). Now suppose that each ν_i is a nonzero normal positive functional on $\mathscr{A}_i$ and that $\nu_i = \mu_i$ for a.a. $i \in I$. Then $\nu' = \bigotimes \nu_i$ exists as a product functional for $(Z\mathscr{A}, (\beta_i))$. Hence, by Lemma 1.17, $\nu = \bigotimes \nu_i$ exists in Σ_p with $S(\nu) = S(\nu') \leqq Z$. Since $E(\mu)$ is the supremum of such $S(\nu)$, $E(\mu) \leqq Z$. This completes the proof.

COROLLARY 3.3. *Suppose that* $(\mathscr{A}_i)_{i \in I}$ *is a factorization of the W^*-algebra* $\mathscr{A}$, *that* $\mu = \bigotimes_{i \in I} \mu_i$ *is a product functional for this factorization, and that* (IV-μ) *holds. Then*

$$S(\mu) = [E(\mu)] \prod_{i \in I} S(\mu_i) .$$

Proof. Use Theorem 3.2 and (2.1) of Proposition 2.6.

COROLLARY 3.4. *Suppose that* $(\mathscr{A}_i)_{i \in I}$ *is a family of W^*-algebras,* *and that, for each* $i \in I$, μ_i *is a normal positive functional of* $\mathscr{A}_i$. *Suppose that* $(\mathscr{A}, (\alpha_i))$ *and* $(\mathscr{B}, (\beta_i))$ *are products for* $(\mathscr{A}_i)$ *which satisfy* (VI-(μ_i)). *Then* $(\mathscr{A}, (\alpha_i))$ *and* $(\mathscr{B}, (\beta_i))$ *are isomorphic: i.e., there exists an isomorphism* ψ *of* $\mathscr{A}$ *onto* $\mathscr{B}$ *such that* $\psi \circ \alpha_i = \beta_i$ *for all* $i \in I$.

COROLLARY 3.5. *Suppose that, for each* $i \in I$, $\mathscr{A}_i$ *and* $\mathscr{B}_i$ *are von Neumann algebras on* H_i *and* G_i *respectively, that* $x_i \in H_i$ *and* $y_i \in G_i$ *with*

$$0 < \prod \|x_i\|, \quad \prod \|y_i\| < \infty ,$$

and that ψ_i *is an isomorphism of* $\mathscr{A}_i$ *onto* $\mathscr{B}_i$ *such that:*

$$((\psi_i(A_i))y_i \mid y_i) = (A_i x_i \mid x_i) \qquad \text{for all } A_i \in \mathscr{A}_i .$$

Then there exists an isomorphism ψ *of* $\mathscr{A} = \bigotimes (\mathscr{A}_i, x_i)$ *onto* $\mathscr{B} = \bigotimes (\mathscr{B}_i, y_i)$ *such that* $\psi \circ \alpha_i = \beta_i \circ \psi_i$ *for each* $i \in I$, *where* α_i *is the natural injection of* $\mathscr{A}_i$ *into* $\mathscr{A}$ *and* β_i *is the natural injection of* $\mathscr{B}_i$ *into* $\mathscr{B}$.

Proof. Use Corollary 3.4 and Proposition 2.6.

THEOREM 3.6. *Suppose that* $(\mathscr{A}_i)_{i \in F}$ *is a finite family of* W^*-*algebras. Suppose that* $(\mathscr{A}, (\alpha_i))$ *is a product for* $(\mathscr{A}_i)_{i \in F}$ *satisfying* (III) *and* (IV-μ) *for same product functional* μ. *Then there exists an isomorphism* ψ *of* $\mathscr{A}$ *onto* $\bigotimes_{i \in F} \mathscr{A}_i$ *such that:*

$$\psi\Big(\prod_{i \in F} \alpha_i(A_i)\Big) = \bigotimes_{i \in F} A_i \qquad \text{for all } A_i \in \mathscr{A}_i .$$

(We write $\bigotimes_{i \in F} A_i$ *for* $\prod_{i \in F} \lambda_i(A_i)$, *where* λ_i *is the natural injection of* $\mathscr{A}_i$ *into* $\bigotimes_{i \in F} \mathscr{A}_i$.) *Furthermore, for every product functional* $\nu = \bigotimes \nu_i$ *for* $(\mathscr{A}_i, (\alpha_i))$:

$$S(\nu) = \prod_{i \in F} \alpha_i(S(\nu_i)) .$$

Proof. If $\mu = \bigotimes_{i \in F} \mu_i$, $E(\mu) = 1$ because (III) holds and F is finite. Hence (VI-(μ_i)) holds, and Corollory 3.4 and Proposition 2.6 complete the proof.

4. Local tensor products.

LEMMA 4.1. *Suppose that* $(\mathscr{A}_i)_{i=1,2}$ *is a factorization of the* W^*-*algebra* $\mathscr{A}$, *and that* μ_2 *is a normal positive functional on* $\mathscr{A}_2$. *Let* Σ_1 *be the set of normal positive functionals* μ_1 *on* $\mathscr{A}_1$ *such that*

$\mu_1 \otimes \mu_2$ exists as a product functional on $\mathscr{A}$ for the factorization $(\mathscr{A}_i)$. Then:

(i). *If $\mu_1 \in \Sigma_1$ and $x > 0$, then $x\mu_1 \in \Sigma_1$.*

(ii). *If $\mu_1^n \in \Sigma_1$ and $\sum_n \mu_1^n(1) < \infty$, then $\sum_n \mu_1^n \in \Sigma_1$.*

(iii). *If $\mu_1 \in \Sigma_1$ and $A_1 \in \mathscr{A}_1$, with $\mu_1(A_1^* A_1) \neq 0$, then $(\mu_1)_{A_1} \in \Sigma_1$ ($(\mu_1)_{A_1}$ is defined by $((\mu_1)_{A_1})(B_1) = \mu_1(A_1^* B_1 A_1)$ for all $B_1 \in \mathscr{A}_1$).*

(iv). *If ν_1 is a nonzero normal positive functional on $\mathscr{A}_1$ with $S(\nu_1) \leqq Z(\mu_1)$ and $\mu_1 \in \Sigma_1$, then $\nu_1 \in \Sigma_1$.*

(v). *If Σ_1 is separating then Σ_1 is the set of all nonzero normal positive functionals on $\mathscr{A}_1$.*

Proof. (i), (ii), and (iii) are obvious by direct calculation. To prove (iv) suppose that ν_1 is a normal positive functional on $\mathscr{A}_1$ and that $S(\nu_1) \leqq Z(\mu_1)$ with $\mu_1 \in \Sigma_1$. Then, by Proposition 3.1, there exists a normal homomorphism ψ from $\mathscr{A}$ onto $(Z(\mu_1))\mathscr{A}_1 \otimes (Z(\mu_2))\mathscr{A}_2$ such that:

$$\psi(A_1A_2) = (Z(u_1)A_1) \otimes (Z(\mu_2)A_2) \qquad \text{for all } A_1 \in \mathscr{A}_1 \text{ and } A_2 \in \mathscr{A}_2 .$$

Now, since $S(\nu_1) \leqq Z(\mu_1)$, by Corollary 2.7 there exists a normal positive functional $\omega = \nu_1' \otimes \mu_2'$ on $(Z(\mu_1))\mathscr{A}_i \otimes (Z(\mu_2))\mathscr{A}_2$ such that

$$\omega((Z(\mu_1)A_1) \otimes (Z(\mu_2)A_2)) = (\nu_1(A_1))(\mu_2(A_2)) \qquad \text{for all } A_1 \in \mathscr{A}_1 \text{ and } A_2 \in \mathscr{A}_2 .$$

Evidently $\omega \circ \psi$ equals $\nu_1 \otimes \mu_2 \in \Sigma_p$ and $\nu_1 \in \Sigma_1$.

To prove (v) assume that Σ_1 is separating. Then, using (iii) and Zorn's lemma, we can choose a family $(\mu_1^j)_{j \in J}$ with each $\mu_1^j \in \Sigma_1$ and $\sum_{j \in J} Z(\mu_1^j) = 1$. Suppose that ν_1 is a normal state of $\mathscr{A}_1$. Then $\sum \nu_1(Z(\mu_1^j)) < \infty$ and therefore $\nu_1(Z(\mu_1^j)) = 0$ for all but a countable number of $j \in J$. Hence a suitable countable linear combination μ_1 of the μ_1^j satisfies $S(\nu_1) \leqq Z(\mu_1)$ and $\mu_1 \in \Sigma_1$ by (ii). Then $\nu_1 \in \Sigma_1$ by (iv).

REMARK. Lemma 4.1 may be proved directly (without using Proposition 3.1 or properties of the tensor product) by using Sakai's Radon-Nikodỳm theorem [8] and the weak Radon-Nikodỳm type result of [5, p. 211].

PROPOSITION 4.2. Suppose that $(\mathscr{A}_i)_{i \in I}$ is a factorization of the W^*-algebra $\mathscr{A}$. Let Σ_{IV} be the set of product functionals on $\mathscr{A}$ for which (IV-μ) holds, and suppose that Σ_{IV} is separating. Suppose that F is a finite subset of I. Then there exists an isomorphism ψ of $\mathscr{A}$ onto $\mathscr{A}_F \otimes \mathscr{A}_{I-F}$ such that:

$$\psi(AB) = A \otimes B \qquad \text{for all } A \in \mathscr{A}_F \text{ and } B \in \mathscr{A}_{I-F} .$$

Proof. By Proposition 1.6 $\{\mathscr{A}_F, \mathscr{A}_{I-F}\}$ is a factorization of $\mathscr{A}$ and each $\mu \in \Sigma_{IV}$ is a product functional for this factorization: $\mu = \mu_F \otimes \mu_{I-F}$. Let $\Sigma_2 = \{\mu_{I-F}: \mu \in \Sigma_{IV}\}$ and let Σ_1 be the set of product functionals on $\mathscr{A}_F$ for the factorization $(\mathscr{A}_i)_{i \in F}$. By Proposition 1.5. (iii), Σ_1 is separating on $\mathscr{A}_F$ and Σ_2 is separating on $\mathscr{A}_{I-F}$. Because (IV-μ) holds for all $\mu \in \Sigma_{IV}$, $\nu \otimes \mu_{I-F}$ exists on $\mathscr{A}$ for all $\nu \in \Sigma_1$ and all $\mu_{I-F} \in \Sigma_2$. Hence, by Lemma 4.1 (v), $\nu \otimes \omega$ exists on $\mathscr{A}$ for all $\nu \in \Sigma_1$ and all nonzero normal positive functionals ω on $\mathscr{A}_{I-F}$; and, from there, by the same lemma, $\nu \otimes \omega$ exists on $\mathscr{A}$ for all nonzero normal positive functionals ν of $\mathscr{A}_F$ and ω of $\mathscr{A}_{I-F}$. Thus (IV) holds for the factorization $\{\mathscr{A}_F, \mathscr{A}_{I-F}\}$. (III) obviously holds for $(\mathscr{A}_i)$ and thus for $\{\mathscr{A}_F, \mathscr{A}_{I-F}\}$ (Proposition 1.6. (iii)). Now Theorem 3.6 completes the proof.

REMARK. Proposition 4.2 is false if the hypothesis that F be finite is omitted (see Example 7.3).

COROLLARY 4.3. *If Σ_{IV} is separating then* (III) *and* (IV) *hold.*

Proof. That (III) holds is obvious. To prove (IV) use Proposition 4.2 and Corollary 2.7.

COROLLARY 4.4. *If a product $(\mathscr{A}, (\alpha_i))$ satisfies* (VI-μ_i)), *then it satisfies* (III), (IV), *and* (V): *i:e., it is a (μ_i)-local tensor product.*

Proof. Use Proposition 1.16 and Proposition 4.2.

PROPOSITION 4.5. *Suppose that $(\mathscr{A}, (\alpha_i))$ is a tensor product for $(\mathscr{A}_i)_{i \in I}$: i.e., that* (III) *and* (IV) *hold. Then, for all $\mu = \bigotimes_{i \in I} \mu_i \in \Sigma_p$:*

$$E(\mu) = T(\mu) \tag{4.1}$$

and

$$S(\mu) = [T(\mu)] \prod_{i \in I} \alpha_i(S(\mu_i)) . \tag{4.2}$$

Proof. $E(\mu) \leqq T(\mu)$ by Proposition 1.15. To prove (4.1), then, it suffices to prove that $E(\mu)$ is tail. Let $\Sigma = \{\nu \in \Sigma_p: \nu = \bigotimes \nu_i$ with $\nu_i = \mu_i$ for a.a. $i \in I\}$. Then $E(\mu) = \sup \{S(\nu): \nu \in \Sigma\}$. Suppose that F is a finite subset of I. Then $\{\mathscr{A}_F, \mathscr{A}_{I-F}\}$ is a factorization of $\mathscr{A}$ for which each $\nu \in \Sigma$ is a product functional: $\nu = \nu_F \otimes \nu_{I-F}$. By Proposition 4.2 and (2.1) of Proposition 2.6, for all $\nu \in \Sigma$:

$$S(\nu) = [S(\nu_F)][S(\nu_{I-F})] . \tag{4.3}$$

Thus:

(4.4) $$E(\mu) \leqq \sup\{S(\nu_{I-F}) : \nu \in \Sigma\} .$$

Because of (IV), for fixed ν_{I-F}, ν_F runs through all product functionals for $(\mathscr{A}_F, (\alpha_i)_{i\in F})$ and therefore (Proposition 1.5. (iv)):

$$\sup\{S(\nu_F) : \nu \in \Sigma, \nu_{I-F} \text{ fixed}\} = 1 .$$

Using (4.3), we obtain:

(4.5) $$E(\mu) \geqq \sup\{S(\nu_{I-F}) : \nu \in \Sigma\} .$$

Now (4.4) and (4.5) show that $E(\mu) \in \mathscr{A}_{I\in F}$. Since F was an arbitrary finite subset of I, we have shown that $E(\mu)$ is tail. That proves (4.1).

(4.2) follows from (4.1) and Corollary 3.3.

COROLLARY 4.6. *A product $(\mathscr{A}, (\alpha_i))$ for $(\mathscr{A}_i)$ is a (μ_i)-local tensor product for $(\mathscr{A}_i)$ if and only if* (VI-(μ_i)) *holds.*

Proof. Corollary 4.4 shows that (VI-(μ_i)) is sufficient. Suppose that $(\mathscr{A}, (\alpha_i))$ is a (μ_i)-local tensor product for $(\mathscr{A}_i)$. Let $\mu = \bigotimes \mu_i$. Then (IV-μ) holds because (IV) does, and, using Proposition 4.5 and (V), we see that $E(\mu) = T(\mu) = 1$.

THEOREM 4.7. *Suppose that, for each $i \in I$, $\mathscr{A}_i$ is a W^*-algebra and μ_i is a normal positive functional on $\mathscr{A}_i$. Suppose that $0 < \prod_{i\in I} \mu_i(1) < \infty$. Then a (μ_i)-local tensor product exists and is unique up to isomorphism.*

Proof. Proposition 2.6, Corollary 3.4, and Corollary 4.6.

5. **Tensor products.** Throughout this section we suppose that $(\mathscr{A}_i)_{i\in I}$ is a family of W^*-algebras. Let Λ be the set of families $(\mu_i)_{i\in I}$, each μ_i a normal positive functional on $\mathscr{A}_i$ and

$$0 < \prod_{i\in I} \mu_i(1) < \infty .$$

Let the relation R on Λ be defined by writing $(\mu_i) \sim (\nu_i) \pmod R$ to mean that a (μ_i)-local tensor product for $(\mathscr{A}_i)_{i\in I}$ is necessarily a (ν_i)-local tensor product for $(\mathscr{A}_i)_{i\in I}$. R is a well defined equivalence relation because a (μ_i)-local tensor product exists and is unique up to isomorphism. The following lemma is a trivial consequence of the definition of (μ_i)-local tensor product.

LEMMA 5.1. *Let $(\mathscr{A}, (\alpha_i))$ be a (μ_i)-local tensor product and let $(\nu_i) \in \Lambda$. Then $(\mu_i) \sim (\nu_i) \pmod R$ if and only if $\bigotimes \nu_i$ exists on $(\mathscr{A}, (\alpha_i))$.*

REMARK. If $\sum_{i \in I} [d(\mu_i, \nu_i)]^2 < \infty$ then $(\mu_i) \sim (\nu_i)$, and the converse holds provided each $\mathscr{A}_i$ is semi-finite [2].

LEMMA 5.2. *If (μ_i) and $(c_i\mu_i)$ are in Λ (where the c_i are positive real numbers), then $(\mu_i) \sim (c_i\mu_i)$.*

Proof. Since $\prod \mu_i(1)$ and $\prod c_i\mu_i(1)$ both converge to a nonzero number, so must $\prod c_i$ converge to $c \neq 0$. If $\mu = \bigotimes \mu_i$ exists as a product for $(\mathscr{A}_i)$, $c\mu$ is a product state equal to $\bigotimes (c_i\mu_i)$ by direct calculation.

REMARK. This lemma shows that we could, without loss of generality, confine ourselves to (μ_i) with each $\mu_i(1) = 1$.

Define Δ to be the quotient set Λ/R and let φ be the quotient map $\Lambda \to \Lambda/R = \Delta$.

DEFINITION 5.3. A tensor product $(\mathscr{A}, (\alpha_i))$ for $(\mathscr{A}_i)$ will be called a Γ-tensor product for $(\mathscr{A}_i)$ when:

$$\{(\mu_i) \in \Lambda: \bigotimes \mu_i \text{ exists on } (\mathscr{A}, (\alpha_i))\} = \varphi^{-1}(\Gamma) .$$

LEMMA 5.4. *Let $\gamma = \varphi((\mu_i))$. Then a (μ_i)-local tensor product is a $\{\gamma\}$-tensor product.*

THEOREM 5.5. *Suppose that $(\mathscr{A}, (\alpha_i))$ is a tensor product for $(\mathscr{A}_i)_{i \in I}$. Let*

$$\Gamma = \varphi\{(\mu_i) \in \Lambda: \bigotimes \mu_i \text{ exists on } (\mathscr{A}, (\alpha_i))\} .$$

Then:

(i). *$(\mathscr{A}, (\alpha_i))$ is a Γ-tensor product for $(\mathscr{A}_i)$.*

(ii). *If $\mu = \bigotimes \mu_i$ and $\nu = \bigotimes \nu_i$ are product functionals for $(\mathscr{A}, (\alpha_i))$ then:*

$$T(\mu) = T(\nu) \quad \text{if and only if} \quad (\mu_i) \sim (\nu_i)$$

and

$$[T(\mu)][T(\nu)] = 0$$

otherwise.

(iii) *If $\mu = \bigotimes \mu_i$ is a product functional for $(\mathscr{A}, (\alpha_i))$ then:*

$$S(\mu) = [T(\mu)] \prod_{i \in I} \alpha_i(S(\mu_i)) \tag{5.1}$$

$$T(\mu) = \sup \{S(\bigotimes \nu_i): (\nu_i) \sim (\mu_i)\} . \tag{5.2}$$

(iv). $(\mathscr{A}, (\alpha_i))$ *is a* (μ_i)*-local tensor product if and only if* $\Gamma = \{\varphi((\mu_i))\}$.

(v). *For each* $\gamma \in \Gamma$, *define* $T(\gamma)$ *to be* $T(\mu)$ *for* $\mu = \bigotimes \mu_i$ *and* $\varphi((\mu_i)) = \gamma$. *Let* $\mathscr{A}(\gamma) = [T(\gamma)]\mathscr{A}$ *and let* $\alpha_i(\gamma)$ *be defined by*:

$$(\alpha_i(\gamma))(A_i) = [T(\gamma)][\alpha_i(A_i)] \qquad \text{for all } A_i \in \mathscr{A}_i \text{ and all } i \in I .$$

Then, for each $\gamma \in \Gamma$, $(\mathscr{A}(\gamma), (\alpha_i(\gamma)))$ *is a* (μ_i)*-local tensor product for* $(\mathscr{A}_i)$ *provided that* $\gamma = \varphi((\mu_i))$.
Furthermore:

$$\mathscr{A} = \bigoplus_{\gamma \in \Gamma} \mathscr{A}(\gamma) \quad \text{and} \quad \alpha_i = \bigoplus_{\gamma \in \Gamma} \alpha_i(\gamma) \qquad \text{for all } i \in I ,$$

with respect to the same direct sum decomposition of $\mathscr{A}$.

Proof. Suppose that $\mu = \bigotimes \mu_i$ is a product functional on $(\mathscr{A}, (\alpha_i))$. For each $i \in I$, define $\beta_i: \mathscr{A}_i \to \mathscr{A}$ by:

$$\beta_i(A_i) = [T(\mu)][\alpha_i(A_i)] \qquad \text{for all } A_i \in \mathscr{A}_i .$$

Then, by Proposition 1.18:

(5.3) $([T(\mu)]\mathscr{A}, (\beta_i))$ is a (μ_i)-local tensor product for $(\mathscr{A}_i)$.

By Lemma 5.1, therefore, for all $(\nu_i) \in \Delta$, $(\nu_i) \sim (\mu_i)$ if and only if $\bigotimes \nu_i$ exists on $([T(\mu)]\mathscr{A}, (\beta_i))$. According to Proposition 1.18. (iii), however, this happens precisely when $\bigotimes \nu_i$ exists on $(\mathscr{A}, (\alpha_i))$ *and* $S(\bigotimes \nu_i) \leqq T(\mu)$. We have shown that, for all $(\nu_i) \in \Delta$, and for all product functionals $\mu = \bigotimes \mu_i$ for $(\mathscr{A}, (\alpha_i))$:

(5.4) $(\nu_i) \sim (\mu_i)$ if and only if $\bigotimes \nu_i$ exists on $(\mathscr{A}, (\alpha_i))$ *and* $S(\bigotimes \nu_i) \leqq T(\mu)$.

(5.4) shows that, if $(\nu_i) \sim (\mu_i)$ and if $\bigotimes \mu_i$ exists on $(\mathscr{A}, (\alpha_i))$, then $\bigotimes \nu_i$ exists on $(\mathscr{A}, (\alpha_i))$; (i) follows. (ii) is an immediate consequence of (5.4) and the fact that $\mathscr{T}$ is atomic (Proposition 1.9). (5.1) of (iii) is just (4.2) of Proposition 4.5, and (5.2) is a consequence of (4.1) of Proposition 4.5 and (5.4). (iv) follows from (ii). (5.3), together with (ii), proves (v).

THEOREM 5.6. *Suppose that* $(\mathscr{A}_i)_{i \in I}$ *is a family of* W^**-algebras and that* Δ *is as defined above. Then*:

(i). *If* Λ *is a nonempty subset of* Δ, *a* Γ*-tensor product for* $(\mathscr{A}_i)$ *exists and is unique up to isomorphism.*

(ii). *Suppose that* $(\mathscr{A}, (\alpha_i))$ *is a* Γ_1*-tensor product for* $(\mathscr{A}_i)$ *and that* $(\mathscr{B}, (\beta_i))$ *is a* Γ_2*-tensor product for* $(\mathscr{A}_i)$. *Then* $\Gamma_1 = \Gamma_2$ *if and only if* $(\mathscr{A}, (\alpha_i))$ *and* $(\mathscr{B}, (\beta_i))$ *are isomorphic*: *i.e., if and only if* *there exists an isomorphism* ψ *of* $\mathscr{A}$ *onto* $\mathscr{B}$ *such that* $\psi \circ \alpha_i = \beta_i$ *for all* $i \in I$.

Proof. Everything but the existence of a Γ-tensor product for $(\mathscr{A}_i)$ follows from Theorem 5.5. For each $\gamma \in \Delta$, a $\{\gamma\}$-tensor product exists by Theorem 4.7. Hence the existence is a result of the following proposition.

PROPOSITION 5.7. Suppose that Γ is a subset of Δ and that, for each $\gamma \in \Gamma$, $(\mathscr{A}(\gamma), \alpha_i(\gamma))$ is a $\{\gamma\}$-tensor product for $(\mathscr{A}_i)$. Let $\mathscr{A} = \bigoplus_{\gamma \in \Gamma} \mathscr{A}(\gamma)$ and $\alpha_i = \bigoplus_{\gamma \in \Gamma} \alpha_i(\gamma)$ for all $i \in I$. Then $(\mathscr{A}, (\alpha_i))$ is a Γ-tensor product for $(\mathscr{A}_i)$.

Proof. Let $\mathscr{A}$ and α_i be defined as above and let $E(\gamma)$ be the projection of $\mathscr{A}$ with $\mathscr{A}(\gamma) = [E(\gamma)]\mathscr{A}$. Let $\mathscr{B} = \mathscr{R}(\alpha_i(\mathscr{A}_i): i \in I)$. Then $(\mathscr{B}, (\alpha_i))$ is a product for $(\mathscr{A}_i)$. If $\varphi((\mu_i)) = \gamma \in \Gamma$, then $\mu' = \bigotimes \mu_i$ exists on $(\mathscr{A}(\gamma), (\alpha_i(\gamma)))$, and, if μ is defined by

$$\mu(B) = \mu'(E(\gamma)B) \qquad \text{for all } B \in \mathscr{B} ,$$

we can see by direct calculation that $\mu = \bigotimes \mu_i$ on $(\mathscr{B}, (\alpha_i))$ with

$$S(\mu) \leqq E(\gamma) \quad \text{where} \quad \gamma = \varphi((\mu_i)) . \tag{5.5}$$

It is clear that such μ form a separating subset Σ of the normal positive functionals on $\mathscr{B}$, and that—since $\nu_i = \mu_i$ for a.a. $i \in I$ implies $(\nu_i) \sim (\mu_i)$—(IV-μ) holds for each $\mu \in \Sigma$. Therefore (Corollary 4.3), $(\mathscr{B}, (\alpha_i))$ is a tensor product for $(\mathscr{A}_i)$. By (5.2) of Theorem 5.5 (iii), and by (5.5):

$$T(\gamma) = E(\gamma) \qquad \text{for all } \gamma \in \Gamma .$$

Hence each $E(\gamma) \in \mathscr{B}$ and $\mathscr{B} = \mathscr{A}$. Furthermore $\sum_{\gamma \in \Gamma} T(\gamma) = 1$, so that, if $\mu = \bigotimes \mu_i$ is a product functional for $(\mathscr{A}, (\alpha_i))$, then $T(\mu) = T(\gamma)$ for some $\gamma \in \Gamma$ (Proposition 1.9) and $\varphi((\mu_i)) = \gamma \in \Gamma$ by Theorem 5.5. (ii). Therefore $(\mathscr{A}, (\alpha_i))$ is a Γ-tensor product for $(\mathscr{A}_i)$.

PROPOSITION 5.8. Suppose that, for each $i \in I$, $\mathscr{A}_i$ is a von Neumann algebra on H_i and every normal positive functional on $\mathscr{A}_i$ is induced by a vector. Let H denote von Neumann's complete direct product [7] of $(H_i)_{i \in I}$ and let α_i be the natural injection of $\mathscr{L}(H_i)$ into $\mathscr{L}(H)$ for each $i \in I$. Let $\mathscr{A} = \mathscr{R}(\alpha_i(\mathscr{A}_i): i \in I)$. Then $(\mathscr{A}, (\alpha_i))$ is a Δ-tensor product for $(\mathscr{A}_i)_{i \in I}$. Furthermore, for every nonempty subset Γ of Δ, there exists a projection $T(\Gamma)$ in the tail of $(\mathscr{A}, (\alpha_i))$ such that $(T(\Gamma)\mathscr{A}, (\beta_i))$ is a Γ-tensor product for $(\mathscr{A}_i)_{i \in I}$, where $\beta_i(A_i) = [T(\Gamma)]A_i$ for all $A_i \in \mathscr{A}_i$.

The proof is easy and is omitted.

6. Tensor products of factors.

LEMMA 6.1. *If* $(\mathscr{A}, (\alpha_i))$ *is a product for* $(\mathscr{A}_i)$ *and if each* $\mathscr{A}_i$ *is a factor, then* (IV) *holds for* $(\mathscr{A}, (\alpha_i))$.

Proof. Use Lemma 4.1. (iv) and mathematical induction.

LEMMA 6.2. *Suppose that* $(\mathscr{A}, (\alpha_i))$ *is a product for* $(\mathscr{A}_i)$ *and that* $\mathscr{A}$ *is a factor. Then* $(\mathscr{A}, (\alpha_i))$ *is a local tensor product for* $(\mathscr{A}_i)$ *if and only if there exists a product state of* $(\mathscr{A}, (\alpha_i))$.

Proof. Each $\mathscr{A}_i$ is a factor by Lemma 1.3, and therefore (IV) holds by Lemma 6.1. (V) holds because $\mathscr{T} \subset \mathscr{Z}$. If μ is a product functional on $(\mathscr{A}, (\alpha_i))$, then $E(\mu) = 1$, for $E(\mu)$ is central by Theorem 3.2. Thus (III) holds if and only if a product state μ exists.

PROPOSITION 6.3. Suppose that $(\mathscr{A}, (\alpha_i))$ is a tensor product for $(\mathscr{A}_i)$ and that each $\mathscr{A}_i$ is a factor. Then $\mathscr{T} = \mathscr{Z}$: the tail of $(\mathscr{A}, (\alpha_i))$ equals the center of $\mathscr{A}$.

Proof. By Theorem 5.5, the family $(T(\gamma))_{\gamma \in \Gamma}$ of atomic projections of $\mathscr{T}$ is such that each $[T(\gamma)\mathscr{A}]$ is a local tensor product for $(\mathscr{A}_i)$. By Lemma 2.5, each $[T(\gamma)\mathscr{A}]$ is a factor. Hence the center of $\mathscr{A} = \bigoplus [T(\gamma)\mathscr{A}]$ is $\mathscr{R}(T(\gamma)\colon \gamma \in \Gamma) = \mathscr{T}$.

COROLLARY 6.4. *Suppose that* $(\mathscr{A}, (\alpha_i))$ *is a tensor product for* $(\mathscr{A}_i)$ *and that each* $\mathscr{A}_i$ *is a factor. Then* $\mathscr{A}$ *is a factor if and only if* $(\mathscr{A}, (\alpha_i))$ *is a local tensor product: i.e., if and only if* (V) *holds.*

PROPOSITION 6.5. Suppose that $\mathscr{A}$ is a finite factor and that $(\mathscr{A}_i)_{i \in I}$ is a factorization of $\mathscr{A}$. Let μ_i be the restriction of the normalized trace on $\mathscr{A}$ to $\mathscr{A}_i$. Let $(\mathscr{B}, (\alpha_i))$ be a (μ_i)-local tensor product for $(\mathscr{A}_i)_{i \in I}$. Then there exists an isomorphism ψ of $\mathscr{A}$ onto $\mathscr{B}$ such that, for each $i \in I$:

$$\psi(A_i) = \alpha_i(A_i) \qquad \text{for all } A_i \in \mathscr{A}_i .$$

Proof. (c.f. [6]). If μ is the normalized trace on $\mathscr{A}$, a direct calculation (see the proof of Theorem 4.3 in [2]) demonstrates that $\mu = \bigotimes \mu_i$ for $(\mathscr{A}_i)$. From there Lemma 6.2, Theorem 4.7, Corollary 4.4 and Proposition 2.6 complete the proof.

7. Some simple counterexamples.

EXAMPLE 7.1. Let $\mathscr{A}$ be a factor of Type II_1 on the Hilbert space H. Then $\{\mathscr{A}, \mathscr{A}'\}$ is a factorization of $\mathscr{L}(H)$ which satisfies (IV) and (V), and for which *no* product functional exists.

See Lemmas 6.1, 6.2 and Theorem 3.6.

EXAMPLE 7.2. For $i = 1$ and 2, let $\mathscr{A}_i$ be a W^*-algebra with central projection $Z_i \neq 0$ or 1. Let

$$Z = (Z_1 \otimes Z_2) + (1 - Z_1) \otimes (1 - Z_2)$$

in $\mathscr{A}_1 \otimes \mathscr{A}_2$, and let $\mathscr{A} = Z(\mathscr{A}_1 \otimes \mathscr{A}_2)$. Let $\alpha_i \colon \mathscr{A}_i \to \mathscr{A}$ be defined by

$$\alpha_1(A_1) = Z(A_1 \otimes 1) \qquad \text{for all } A_1 \in \mathscr{A}_1$$
$$\alpha_2(A_2) = Z(1 \otimes A_2) \qquad \text{for all } A_2 \in \mathscr{A}_2 .$$

Then $(\mathscr{A}, (\alpha_i))$ is a product for $(\mathscr{A}_i)_{i=1,2}$ which satisfies (III) and (V) but not (IV).

EXAMPLE 7.3. Let $I = \{1, 2\} \times J$ where J is *infinite*, and, for each $i \in I$, let $\mathscr{A}_i$ be an abelian W^*-algebra generated by its two atomic projection E_i and $1 - E_i$. Let the states μ_i and ν_i of $\mathscr{A}_i$ be defined by $\mu_i(1) = \nu_i(1) = 1$ and $\mu_i(E_i) = 1$ and $\nu_i(E_i) = 1/2$. Let $\Gamma = \varphi\{(\mu_i), (\nu_i)\}$ and let $(\mathscr{A}, (\alpha_i))$ be a Γ-tensor product for $(\mathscr{A}_i)$. Let

$$\mathscr{A}_\delta = \mathscr{R}(\alpha_i(\mathscr{A}_i) \colon i \in \{\delta\} \times J)$$

and let $\lambda_\delta \colon \mathscr{A}_\delta \to \mathscr{A}$ be the inclusion map, for $\delta = 1$ and 2. Then $(\mathscr{A}, (\lambda_\delta))$ is a product for $(\mathscr{A}_\delta)_{\delta=1,2}$ which satisfies (III) and (V) but *not* (IV). In particular, $(\mathscr{A}, (\lambda_\delta))$ is not isomorphic to $\mathscr{A}_1 \otimes \mathscr{A}_2$.

To make this clearer, let $\mathscr{B} = \mathscr{A}_1 \otimes \mathscr{A}_2$ with λ_δ the natural injection of $\mathscr{A}_\delta$ into $\mathscr{B}$. Let $\beta_i \colon \mathscr{A}_i \to \mathscr{B}$ be defined for each $i = (\delta, j) \in I$ by $\beta_i = \lambda_\delta \circ \alpha_i$. Then $(\mathscr{B}, (\beta_i))$ is a Γ'-tensor product for $(\mathscr{A}_i)$ where Γ' contains *four* points. In fact

$$\Gamma' = \varphi\{(\mu_i), (\nu_i), (\omega_i), (\rho_i)\}$$

where:

$$\omega_i = \mu_i \quad \text{and} \quad \rho_i = \nu_i \qquad \text{for } i \in \{1\} \times J$$

and

$$\omega_i = \nu_i \quad \text{and} \quad \rho_i = \mu_i \qquad \text{for } i \in \{2\} \times J .$$

EXAMPLE 7.4. Let $I = \{1, 2\} \times J$ with J infinite. For each $i \in I$, let H_i be a Hilbert space (of arbitrary dimension $\geqq 2$) and let φ_i and ψ_i be orthogonal unit vectors in H_i. For each $j \in J$, let $H_j = H_{(1,j)} \otimes H_{(2,j)}$ and let $x_j = [\varphi_{(1,j)} \otimes \omega_{(2,j)} + \psi_{(1,j)} \otimes \psi_{(2,j)}]/\sqrt{2}$. Let $H = \bigotimes_{j \in J} (H_j, x_j)$ and let β_j be the natural injection of $\mathscr{L}(H_j)$ into $\mathscr{L}(H)$. Let $\gamma_{(\delta,j)}$ be the natural injection of $\mathscr{L}(H_{(\gamma,j)})$ into $\mathscr{L}(H_j)$. Let $\alpha_{(\gamma,j)} = \beta_j \circ \gamma_{(\gamma,j)}$ for all $(\delta, j) \in I$. Then $(\mathscr{L}(H), (\alpha_i))$ is a product for $(\mathscr{L}(H_i))_{i \in I}$, and there exist *no* product functionals for $(\mathscr{L}(H), (\alpha_i))$.

See [2] or the remark which follows Lemma 5.1.

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