PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 2, 1968

ON INTERCHANGE GRAPHS

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The interchange graph $I(G)$ for an unoriented graph G has been defined by Ore as follows: The vertices of $I(G)$ are the edges of G; and two vertices of $I(G)$ are connected by an edge if and only if they are adjacent (i.e., have a vertex in common) in G. In 1962 Ore raised the problem of determining those graphs for which

$$I(G) = G . \tag{1}$$

This paper solves this problem for finite connected graphs with loops and parallel edges, extending earlier work on the problem.

A loop (an edge whose two ends coincide in a single vertex) is considered adjacent to itself, and hence generates another loop under the I mapping. If two edges of G connect the same two vertices, the corresponding vertices of $I(G)$ are also connected by two distinct edges.

A natural generalization of (1) is the problem of finding the graphs for which, for some fixed positive integer k,

$$I^k(G) = G . \tag{2}$$

Prior work, has shown that for graphs without loops, (1) and (2) are equivalent, and hold if and only if G is a cycle. When loops are present, the result is different and new methods are required. A typical noncycle graph satisfying (1) and (2) is a loop with a line adjoined. It is proved below that the only finite solutions to (2) are either graphs of this type or cycles.

The first step toward solving Ore's original problem [7] and its generalization seems to have been made by Anna Maria Ghirlanda [3], who solved the case in which G has no loops and no multiple edges. Menon [5], using different methods, removed the latter restriction to show that if multiple edges are admitted the same conclusion holds: viz., G must be a cycle. Harary [4] remarked in 1967 that this result "is now well known".

This paper comprehends both these earlier results and extends them to include the case of graphs with loops, for which the new answer does not appear to be so well known, (although B. Clark [2] discovered in 1964 that noncycle solutions to (1) exist). Moreover, whereas the above named earlier studies use techniques that are to some extent advanced, one of the most striking aspects of the demonstration that follows is that in extending and unifying these earlier results, it has been possible to limit the methods entirely to elementary procedures. At present further extensions, e.g., to oriented and/or disconnected graphs, appear to pose a more challenging problem. Some partial results occur in [6], but the difficulties are illustrated by the fact that (1) and (2) are not equivalent for oriented graphs.

2. Notations. Terminology in graph theory is not standardized. This paper will use primarily that of [1]. Upper case Latin letters will be used for graphs, Greek for their edges and vertices, and lower case Latin for nonnegative integers. Graphs with more than one edge allowed between the same two vertices are called *s*-graphs. Figures 1 and 2 are examples of a graph and an *s*-graph respectively, each

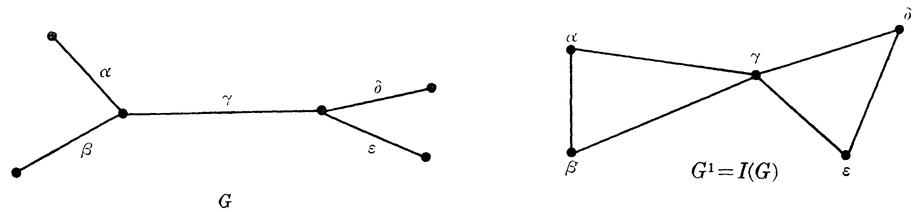

FIGURE 1

FIGURE 2

with its interchange graph. A *k-vertex* of a graph means a vertex of degree k, i.e., a vertex at which k edges meet. (By convention, a loop contributes 1 to the degree of its vertex.) If the highest degree vertex of a graph G is of degree k, we shall say that G is of degree k.

For brevity the term *line* will be used to denote an elementary path, i.e., a connected chain of edges that does not meet the same vertex twice. The number of edges in a line will be called its *length*. By convention, a line of length zero is an isolated vertex; and a line of negative length is the empty graph. It is easy to show that for length greater than 1, an equivalent definition is that a line is a tree of degree 2.

3. Preliminaries. In this section let G be an arbitrary nonempty unoriented finite connected *s*-graph of m edges.

LEMMA 1. *If G has a p-vertex, α, with $p \geqq 3$, then $I(G)$ contains* *a cycle generated by* α. This is immediate from the definition of the I mapping.

LEMMA 2. *If G contains a cycle C, then $I(G)$ contains a cycle generated by C.* This is also trivial from the definition of I.

COROLLARY 1. *If G contains a cycle, then so does $I^k(G)$ for $k > 0$.*

THEOREM 1. *If G is a tree, then $I^k(G) \neq G$ for any $k > 0$.*

Proof. a. For graphs of degree 0 and 1 the assertion is trivial.

b. If G is of degree 2, it is a line of length m. By definition of the I mapping, $I^k(G)$ is a line of length $m - k \neq m$.

c. If G is of degree $p > 2$, then by Lemma 1, $I(G)$ has a cycle. By Corollary 1, $I^k(G)$ also has a cycle and hence is not a tree.

4. **The main results.** In this section let G be an arbitrary nonempty unoriented finite connected s-graph such that $I^k(G) = G$ for some (known, fixed) $k > 0$. Let the number of edges and vertices of $I^j(G)$ be denoted respectively m_j and n_j.

THEOREM 2. $m_j = n_j$.

Proof. Since (2) is true for G, it is also true for $I^j(G)$ for all j. Hence G and $I^j(G)$ are all not trees, by Theorem 1. Now it is well known that $m_j + 1 \geqq n_j$ for any graph K [1; p. 28]. But equality holds if and only if K is a tree [1; p. 152]. Hence, in the present case $m_j \geqq n_j$.

But by definition of the I operation, $m_{j-1} = n_j$. Hence $\{m_j\}$ is a monotone nondecreasing sequence. But since $m_k = m_0$, we can conclude that $\{m_j\}$ is actually a constant. Hence, in particular, $m_j = m_{j-1} = n_j$.

COROLLARY 2. *G and $I^j(G)$ each contain exactly one cycle* [1; p. 29].

THEOREM 3. *The degree of G is $\leqq 2$.*

Proof. Since G contains a cycle, so does $I(G)$, by Lemma 2. But if G also had a vertex of degree $p, p \geqq 3$, then $I(G)$ would have *another* cycle, by Lemma 1, contrary to Corollary 2.

THEOREM 4. *If G contains no loops, then G is a cycle.* This follows immediately from Theorem 3 and Corollary 2. This is Menon's previous result [5].

THEOREM 5. *If G contains loops, then it contains exactly one loop and consists of that loop with a line adjoined to its* (*only*) *vertex*. See Figure 3. (The line may be of length zero.)

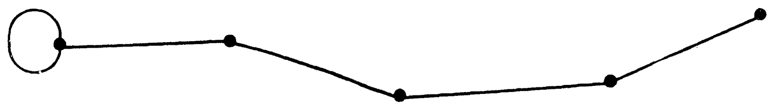

FIGURE 3

Proof. A loop is a cycle. Hence by Corollary 2, there can be only one. By Theorem 3, any additional part of G can attach to the vertex of the loop by only one other edge, and all other vertices must be of degree 2 or 1. Hence the additional part can only be a line.

It follows that only graphs of the type described are eligible candidates to satisfy (2). By direct calculation, we verify that for such graphs, (1) holds, and hence (2) follows.

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Received June 26, 1967.

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