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FUNCTIONS REPRESENTED BY RADEMACHER SERIES

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A series of the form $\sum_{m=1}^{\infty} a_m r_m(t)$, where $\{a_m\}$ is a sequence of real numbers and $r_m(t)$ denotes the mth Rademacher function, sign $\sin(2^m \pi t)$, is called a Rademacher series (as usual, sign $0 = 0$).

Letting $f(t)$ denote the sum of this series whenever it exists, we shall investigate the effect that various conditions on $\{a_m\}$ have on the continuity, variation, and differentiability properties of f.

2. **Continuity properties.** We now prove

THEOREM (2.1). *If $\sum |a_m| < \infty$, then $f(t)$ is continuous at dyadic irrationals* (i.e., *numbers not of the form $p/2^k$*) *and has right and left hand limits everywhere in* $[0, 1]$.

Proof. Under our hypothesis we have that $\sum a_m r_m(t)$ converges uniformly to $f(t)$, which implies our conclusion since the Rademacher functions are continuous at dyadic irrationals and have right and left hand limits everywhere in $[0, 1]$.

In general, the right and left hand limits of $f(t)$ are unequal at dyadic rationals. We now investigate under what conditions we have equality and prove.

THEOREM (2.2). *If $\sum |a_m| < \infty$, then the following are equivalent*:

(a) $a_k = \sum_{m=k+1}^{\infty} a_m$,

(b) $f(p2^{-k} + \varepsilon_n) \to f(p2^{-k})$ *as* $n \to \infty$,

(c) $f(p2^{-k} + \delta_n) \to f(p2^{-k})$ *as* $n \to \infty$,

(d) $f(p2^{-k} + \varepsilon_n) - f(p2^{-k} + \delta_n) \to 0$ *as* $n \to \infty$,

where $\{\varepsilon_n\}$ and $\{\delta_n\}$ are some positive and negative sequences tending to zero, and p is an odd integer.

Proof.

$$f(p2^{-k} + t) - f(p2^{-k}) = \sum_{m=1}^{k-1} a_m r_m(p2^{-k} + t) - a_k r_k(t) + \sum_{m=k+1}^{\infty} a_m r_m(t) - \sum_{m=1}^{k-1} a_m r_m(p2^{-k}) ,$$

since $r_m(p2^{-k} + t) = r_m(t)$ if $m \geqq k + 1$, and $r_k(p2^{-k} + t) = -r_k(t)$.

Therefore,

$$f(p2^{-k} + \varepsilon_n) - f(p2^{-k}) \to -a_k + \sum_{m=k+1}^{\infty} a_m \text{ as } n \to \infty .$$

This shows the equivalence of (a) and (b). A similar argument establishes the equivalence of (a), (c), and (d).

We have, at once, the following

COROLLARY (2.1). *For absolutely convergent Rademacher series the following are equivalent*:

(i) *$f(t)$ is continuous at $p2^{-k}$ for some odd integer p,*

(ii) *$f(t)$ is continuous at $p2^{-k}$ for all odd integers p,*

(iii) $a_k = \sum_{m=k+1}^{\infty} a_m$.

REMARKS. 1. Notice that, if $a_k = \sum_{m=k+1}^{\infty} a_m$ and $a_{k+1} = \sum_{m=k+2}^{\infty} a_m$, then $a_{k+1} = (a_k)/2$.

2. Theorem (2.2) is false under the hypothesis that $\sum |a_m| = \infty$ and $a_m \to 0$, since under these conditions we have that in every interval $f(t)$ assumes every real number c times [2, p. 234, Th. 2].

This shows that the existence of the limit in the sense of Theorem (2.2) implies no relationship whatever between a_k and $\sum_{m=k+1}^{\infty} a_m$. Also by choosing $\{a_m\}$ such that $\sum (a_m)^2 = \infty$ we see that the existence of the limit in the above sense does not even imply that $\sum a_m r_m(t)$ converges in a set of positive measure [8, p. 212].

3. If $f(t) = \sum a_m r_m(t)$ is essentially bounded, then $\sum |a_m| < \infty$ (see [3]).

We now omit the condition that $\sum |a_m| < \infty$ and prove

THEOREM (2.3) $a_k = (a_{k-1})/2$, $k > 1$, *if either*

$$(1)\quad \begin{aligned} &\lim_{n\to\infty} [f(2^{-k} + p2^{-k+2} + \varepsilon_n) - f(2^{-k+1} + p2^{-k+2} + \varepsilon_n)] \\ &= \lim_{n\to\infty} [f(2^{-k} + p2^{-k+2} + \delta_n) - f(2^{-k+1} + p2^{-k+2} + \delta_n)] \end{aligned}$$

or

$$(2)\quad \begin{aligned} &\lim_{n\to\infty} [f(2^{-k+1} + p2^{-k+2} + \varepsilon_n) = f(3\cdot 2^{-k} + p2^{-k+2} + \varepsilon_n)] \\ &= \lim_{n\to\infty} [f(2^{-k+1} + p2^{-k+2} + \delta_n) - f(3\cdot 2^{-k} + p2^{-k+2} + \delta_n)] \end{aligned}$$

where $\varepsilon_n > 0$, $\delta_n < 0$, $\lim \varepsilon_n = \lim \delta_n = 0$ and p is an interger.

Proof. If $k > 1$, $\Delta(t)$

$$\begin{aligned}&\equiv f(2^{-k}+p2^{-k+2}+t)-f(2^{-k+1}+p2^{-k+2}+t)\\&=a_1[r_1(2^{-k}+p2^{-k+2}+t)-r_1(2^{-k+1}+p2^{-k+2}+t)]+\cdots\\&\quad+a_{k-2}[r_{k-2}(2^{-k}+p2^{-k+2}+t)-r_{k-2}(2^{-k+1}+p2^{-k+2}+t)]\\&\quad+a_{k-1}[r_{k-1}(2^{-k}+t)+r_{k-1}(t)]+a_k[-r_k(t)-r_k(t)]\,.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\Delta(\varepsilon_n)=2a_{k-1}-2a_k \quad\text{and}\quad \lim_{n\to\infty}\Delta(\delta_n)=2a_k\,.$$

In view of (1) we have then $2a_k = a_{k-1}$.
A similar proof will suffice if equation (2) is valid.

REMARK. In much the same way we can prove a more general result, namely that if $\{c_k\}$ has the property that

$$\sum_{m=1}^{\infty} 1/\prod_{k=1}^{m}(1+c_k)=c^{-1}\neq 0$$

is absolutely convergent, then

$$f(t)=cf(0+)\sum_{m=1}^{\infty} r_m(t)/\prod_{k=1}^{m}(1+c_k)$$

if and only if for every $k>1$ we have that in (1) the first limit equals c_k times the second.

We now utilize the concepts of approximate limits and approximately continuous functions (see [5, pp. 132, 219]). From Theorem (2.3), we deduce immediately.

COROLLARY 2.2. *If the approximate limit of $f(t)$ exists at either $2^{-k}+p2^{-k+2}$ and $2^{-k+1}+p2^{-k+2}$ or $2^{-k+1}+p2^{-k+2}$ and $3\cdot 2^{-k}+p2^{-k+2}$ (where $k>1$ and p is any integer), then $a_k=(a_{k-1})/2$.*

We now prove

COROLLARY (2.3). *If $F(t)$ is approximately continuous in $[0,1]$ and $\sum a_m r_m(t)$ converges* a.e. *in $[0,1]$ to $F(t)$, then*

$$F(t)=F(0)\cdot(1-2t),\ a_m=F(0)/2^m(m=1,2,\cdots)\,.$$

Proof. Since $F(t)$ is approximately continuous in $[0,1]$, we have that $f(t)$ has approximate limits everywhere. Thus

$$F(t)=C\sum r_m(t)/2^m \text{ a.e., } C \text{ being a constant.}$$

But, since $\sum r_m(t)/2^m=1-2t$ a.e. (see [7, p. 220]), this implies that

$$F(t)=C(1-2t) \text{ a.e.}$$

which concludes our proof since $F(t)$ is approximately continuous.

REMARKS. 1. Corollary (2.2) shows that, if the approximate limits of $f(t)$ exist at certain dyadic rationals, then $a_m = C/2^m$ for $m \geqq m_0$ (where m_0, C are constants).

2. The conclusion of Corollary (2.3) was proved by Wang Si-Lei ([6, p. 704]; cf. [7, p. 221]) under the stronger hypothesis that $F(t)$ be continuous in $[0, 1]$. Wang's result can also be obtained from Theorem (2.2) and Remarks (1) and (3) following it.

3. Corollary (2.2) is a generalization of some theorems of Wang [6, Th. 1, 2, 3].

4. In Corollary (2.3), the condition "convergent a.e." cannot be replaced by "convergent in $E \subset [0, 1], |E| < 1$" [6, p. 706].

3. Variational properties. A. I. Rubinstein has shown [4, p. 143] that if $\sum |a_m| 2^m < \infty$, then $f(t) \in \mathrm{Lip}\,(1, 1)$.

In order to strengthen this result we now state the following lemma which follows from Minkowski's inequality:

LEMMA (3.1). *If* $V_p(f_m)$ *denotes the* pth *variation of* $f_m(t)$, *then*

(i) *if* $0 < p \leqq 1$, $V_p^p\left(\sum_{m=1}^{\infty} f_m\right) \leqq \sum_{m=1}^{\infty} V_p^p(f_m)$;

(ii) *if* $p \geqq 1$, $V_p\left(\sum_{m=1}^{\infty} f_m\right) \leqq \sum_{m=1}^{\infty} V_p(f_m)$.

We will now prove

THEOREM (3.1). (i) *If* $0 < p \leqq 1$, *then* $\sum |a_m|^p 2^m < \infty$ *implies* $f(t)$ *is of bounded* pth *variation;*

(ii) *if* $p \geqq 1$, *then* $\sum |a_m| 2^{m/p} < \infty$ *implies* $f(t)$ *is of bounded* pth *variation;*

(iii) *if* $0 < p \leqq 1$, *then* $a_m \downarrow 0, \sum a_m^p 2^m = \infty$ *implies*

$$g(t) = \sum (-1)^m a_m r_m(t)$$

is not of bounded pth *variation.*

Proof. Parts (i) and (ii) are immediate by the lemma.

Also, setting $\{t_i\} = \{2^{-n-1} + i2^{-n}\}_{i=0}^{2^n-1}$ and $b_m = (-1)^m a_m$ we obtain

$$\begin{aligned}\sum_{i=1}^{2^n-1} |g(t_i) - g(t_{i-1})| &= |-2b_1 + \cdots + 2b_n|^p \\ &+ 2|-2b_2 + \cdots + 2b_n|^p + \cdots + 2^{n-2}|-2b_{n-1} + 2b_n|^p \\ &+ 2^{n-1}|2b_n|^p \geqq \sum_{i=1}^{n} 2^{i-1}|2b_i|^p \to \infty \text{ as } n \to \infty .\end{aligned}$$

This demonstrates Part (iii).

4. **Differentiability properties.** With regard to differentiability, L. A. Balasov has shown [1, p. 631] that $f(t)$ has a derivative at least one point if and only if

$$\lim 2^m a_m = A \text{ exists .} \tag{3}$$

Balasov has demonstrated that this condition alone is not sufficient in order to have $f(t)$ differentiable a.e. [1, pp. 633–4]. He then proves that condition (3) and the relation

$$a_k \geqq \sum_{m=k+1}^{\infty} a_m \text{ for every } k \geqq 1$$

implies $f(t)$ is monotone in $[0, 1]$, which of course implies differentiability almost everywhere.

We now prove

THEOREM (4.1). (i) *If* $\sum |a_m| 2^m < \infty$, *then* $f(t)$ *is differentiable almost everywhere;*

(ii) *if* $\{\varepsilon_m\}$ *is any null sequence, then there exists a sequence* $\{a_m\}$ *satisfying*

(a) $\sum |a_m 2^m \varepsilon_m| < \infty$,

(b) $f(t) = \sum a_m r_m(t)$ *is differentiable nowhere.*

Proof. Part (i) follows immediately from Theorem (3.1).

Part (ii). Since $\{\varepsilon_m\}$ is a null sequence, there exists an increasing sequence of positive integers $\{N_m\}$ such that

$$|\varepsilon_{N_m}| < 2^{-m} , \qquad m = 1, 2, \cdots . \tag{4}$$

Now set

$$\begin{aligned} a_m &= 2^{-m}, \text{ if } m = N_i, \qquad i = 2, 4, 6, \cdots \\ &= 0, \text{ otherwise.} \end{aligned}$$

Then (a) follows from condition (4), and (b) follows since Balasov's condition (3) for differentiability is not satisfied.

REMARK. It would be interesting to know if the sum, $f(t)$, of a Rademacher series is of bounded variation whenever $f(t)$ is differentiable almost everywhere (as is the case for lacunary trigonometric series).

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