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ON COMMUTATIVE, NONPOTENT ARCHIMEDEAN SEMIGROUPS

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In this paper we will study commutative, archimedean, nonpotent (i.e., without an idempotent) semigroups, obtaining several results concerning finitely generated ones. The main theorem of this paper is the following: a finitely generated, commutative, archimedean, nonpotent semigroup is power joined. The main theorem is derived by considering the decomposition of the semigroup S into a union of disjoint semilattices; the congruence ρ_b, defined by $x\rho_b y$ if and only if there exist positive integers n and m such that $b^n x = b^m y$, determines the union, whereas congruence classes are semilattices under the partial order $\geqq_b$ defined by $x \geqq_b y$ if and only if $y = b^n x$ or $y = x$. The set of maximal elements relative to $\geqq_b$ generates S. The following is a crucial lemma in the proof of the main theorem: let S be a finitely generated, commutative, nonpotent, archimedean semigroup; then the set of maximal elements of S relative to $\geqq_b$ is a finite set.

Let S be a commutative, nonpotent, archimedean semigroup. We will define a congruence ρ on S and state several results concerning S/ρ and the congruence classes of S modulo ρ. The remarks and definitions which precede Definition 5 will be used in several instances; a complete discussion can be found in [5]. See [6] and [7] for an abstract of these results. Proofs of all other results in this paper are supplied.

DEFINITION 1. Let $b \in S$. The binary relation ρ_b on S is defined by $x\rho_b y$ if and only if there exist positive integers n and m such that $b^n x = b^m y$.

The relation ρ_b is a congruence relation on S and b is called the standard element determining the corresponding decomposition of S. Furthermore, for any b, S/ρ_b is a group; the congruence class modulo ρ_b containing b is the identity element of S/ρ_b and it is a subsemigroup of S. We call S/ρ_b the structure group of S with respect to b.

DEFINITION 2. Let S_α be an arbitrary congruence class of S (mod ρ_b). The following relation, $\geqq_b$ is a partial order on S_α. Let $x, y \in S_\alpha$. We define $\geqq_b$ on S_α by $x \geqq_b y$ if and only if there exists a positive integer n such that $y = b^n x$, or $y = x$.

DEFINITION 3. A discrete tree R is a lower semilattice (i.e., a partially ordered set in which every pair of elements have a greatest lower bound) satisfying:

(a) for all $x, y, z \in R$, $x < z$ and $y < z$ imply $x \leqq y$ or $y \leqq x$, and

(b) the set $\{x \mid x \in R \text{ and } b \leqq x \leqq c\}$ is a finite set for any pair $b, c \in R$.

Let S_α be a congruence class of S modulo ρ_b. Then S_α is a discrete tree with respect to the partial order $\geqq_b$.

DEFINITION 4. An element x of S is called a prime element of S relative to the congruence ρ_b if x is not divisible by b. Or, alternately, x is a prime element if x is a maximal element of a congruence class S_α of S (mod ρ_b) relative to the partial order $\geqq_b$ defined on S_α.

The following two remarks are particularly useful.

REMARK 1. Let $a \in S$. Then

$$\bigcap_{n=1}^{\infty} a^n S = \varnothing \text{ .}$$

REMARK 2. Let $a, b \in S$. Then

$$a \neq ab \text{ .}$$

DEFINITION 5. Let R be an arbitrary semigroup. We define the binary relation $\leqq$ on R by $a \leqq b$ if and only if there exists $x \in R$ such that $a = bx$, or $a = b$. If $a \neq b$ and $a \leqq b$ we generally write $a < b$.

LEMMA 1. *Let S be a finitely generated, commutative, nonpotent, archimedean semigroup. Then the relation $\leqq$ on S is a partial order and S satisfies the ascending chain condition relative to $\leqq$.*

Proof. It follows from the definition that reflexivity is satisfied. Suppose that $a, b \in S$ and $a \leqq b$ and $b \leqq a$. Then either $a = bx$ and $b = ay$, or $a = b$. Consider the former. We conclude that

$$(1) \qquad a = (ay)x = a(yx) \text{ .}$$

But (1) contradicts Remark 2. Therefore $a = b$. Thus, $\leqq$ is antisymmetric. Suppose $a \leqq b$ and $b \leqq c$. We suppose also that $a \neq b$ and $b \neq c$. Then $a = bx$ and $b = cy$ for some $x, y \in S$. Therefore,

$$(2) \qquad a = (cy)x = c(yx) \text{ .}$$

Thus, $a \leqq c$, and now it is obvious that $\leqq$ is transitive.

Suppose there exists a sequence of elements of S, $\{a_n \mid n > 0\}$, satisfying

$$a_1 < a_2 < a_3 < \cdots < a_n \cdots . \tag{3}$$

Let $T = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a finite generating set of S. The sequence (3) reduces to the set of equations:

$$a_n = a_{n+1}x_{n+1} , \qquad \text{for all } n \geqq 1, \text{ where } x_{n+1} \in S . \tag{4}$$

The set of equations (4) leads to the set

$$a_1 = a_2x_2 = a_3x_2x_3 = \cdots = a_nx_2x_3 \cdots x_n = \cdots . \tag{5}$$

For all k, x_k can be expressed as

$$x_k = \alpha_1^{r_{k1}}\alpha_2^{r_{k2}} \cdots \alpha^{r_{kn}} , \tag{6}$$

and $r_{kj} \geqq 0$ for $j = 1, 2, \cdots, n$, and there exists $j_0, 1 \leqq j_0 \leqq n$ such that $r_{kj_0} > 0$. Multiplying the x_k's, we arrive at

$$\begin{aligned} x_2 &= \alpha_1^{r_{21}}\alpha_2^{r_{22}} \cdots \alpha_n^{r_{2n}} \\ x_2x_3 &= \alpha_1^{p_{31}}\alpha_2^{p_{32}} \cdots \alpha_n^{p_{3n}} \\ x_2x_3 \cdots x_k &= \alpha_1^{p_{k1}}\alpha_2^{p_{k2}} \cdots \alpha_n^{p_{kn}} , \end{aligned} \tag{7}$$

where $0 \leqq r_{2j} \leqq p_{3j} \leqq p_{4j} \leqq \cdots \leqq p_{kj} \leqq \cdots$, for j satisfying $1 \leqq j \leqq n$.

If for some $j, 1 \leqq j \leqq n$, we have

$$\lim_{n\to\infty} p_{nj} = +\infty , \tag{8}$$

then we can write

$$a_1 = \alpha_jy_1 = (\alpha_j)^2y_2 = (\alpha_j)^3y_3 = \cdots = (\alpha_j)^ky_k = \cdots , \tag{9}$$

and we conclude that

$$a_1 \in \bigcap_{n=1}^{\infty} (\alpha_j)^nS .$$

This contradicts Remark 1. We set

$$R_j = \lim_{n\to\infty} P_{nj} , \tag{10}$$

and

$$M = \max\{R_1, R_2, \cdots, R_n\} . \tag{11}$$

The number M is finite and it is now obvious that there exists an integer $N \geqq M$ such that

$$x_2x_3 \cdots x_N = x_2x_3 \cdots x_Nx_{N+1} . \tag{12}$$

This contradicts Remark 2, and the contradiction establishes that S

satisfies the ascending chain condition relative to the relation $\leqq$.

LEMMA 2. *Let S be a commutative, nonpotent, archimedean semigroup and let $\leqq$ be the partial order on S defined above* (see Definition 4). *Let S satisfy the ascending chain condition relative to $\leqq$. Then the set of maximal elements relative to $\leqq$ is a generating set for S and is contained in every other generating set of S.*

Proof. Let S' be an arbitrary generating set for S and let S'' be the set of all the maximal elements of S relative to $\leqq$. Let $\beta \in S''$. Then

$$\beta = \alpha_1\alpha_2 \cdots \alpha_n \,, \quad \text{where} \quad \alpha_i \in S' \tag{12}$$

Suppose $n > 1$. Then $\beta = \alpha_1(\alpha_2 \cdots \alpha_n)$. This implies that $\beta < \alpha_1$. Since this is impossible, $n = 1$ and $\beta \in S'$. That is, $S'' \subseteqq S'$.

Let $x \in S$. Then

$$x = \alpha_1\alpha_2 \cdots \alpha_n \,, \quad \text{where } \alpha_i \in S' \text{ for } 1 \leqq i \leqq n \,. \tag{13}$$

Fix $t, 1 \leqq t \leqq n$. Suppose α_t is not a maximal element. Then there exists $\beta_t \in S''$ such that $\alpha_t < \beta_t$ and $\alpha_t = \beta_t x_t$ for some $x_t \in S$. If x_t is not a maximal element then there exists a maximal element β_{t1} such that $x_t = \beta_{t1}x_{t1}$. By definition of $\leqq$, $x_t < x_{t1}$. We continue in this fashion. After N steps we arrive at the equation

$$\alpha_t = \beta_t\beta_{t1}\beta_{t2} \cdots \beta_{t,N-1}x_{t,N-1} \tag{14}$$

and the sequence of inequalities

$$\alpha_t < x_t < x_{t1} < x_{t2} \cdots < x_{t,N-1} \,, \tag{15}$$

where $\beta_{t,k}$ is a maximal element for $1 \leqq k \leqq N-1$. Since S satisfies the ascending chain condition, we conclude that this procedure must lead to a maximal element $x_{t,M}$ in (14) and (15). Setting $M = N - 1$ in (14) and substituting (14) into (13) we express x as a product of maximal elements. We conclude that S'' generates S and that S'' is the smallest generating set of S.

PROPOSITION 3. Let S be a finitely generated, commutative, nonpotent, archimedean semigroup. Then the set of maximal elements (relative to $\leqq$) of S is a finite set.

Proof. According to Lemma 1, S satisfies the ascending chain condition relative to $\leqq$. By Lemma 2, the set of maximal elements of S is a subset of every generating set of S. Since S is finitely generated, the set of maximal elements of S must be a finite set.

PROPOSITION 4. Let S be a finitely generated, commutative, nonpotent, archimedean semigroup. Let $a \in S$. Then the set of prime elements of S, with respect to the standard element a, is a finite set.

Proof. Suppose there are an infinite number of primes. Let $S' = \{b_1, b_2, \cdots, b_k, \cdots\}$ be a countably infinite subset of the set of primes of S. Let $T = \{a_1, a_2, \cdots, a_n\}$ be the set of all maximal elements of S. Every element of S' admits a representation of the form

$$b_j = a_1^{\mu_{j1}} a_2^{\mu_{j2}} \cdots a_i^{\mu_{ji}} \cdots a_n^{\mu_{jn}},\ \mu_{ji} \geqq 0\ . \tag{16}$$

Consider the sequences

$$\mu_{1i}, \mu_{2i}, \mu_{3i}, \cdots, \mu_{ki}, \cdots\ , \quad \text{where} \quad 1 \leqq i \leqq n\ . \tag{17}$$

For at least one i between 1 and n the corresponding sequence (17) will be unbounded. Otherwise we immediately conclude that S' is a finite set. Suppose the sequence for i_0 is unbounded. Choose a subsequence

$$\mu_{t_1,i_0}, \mu_{t_2,i_0}, \cdots, \mu_{t_r,i_0}, \cdots\ , \tag{18}$$

satisfying

$$\mu_{t_1,i_0} < \mu_{t_2,i_0} < \cdots\ . \tag{19}$$

For convenience, we will change the notation. Set

$$r_j = \mu_{t_j,i_0}\ , \qquad \text{for } j \geqq 1\ . \tag{20}$$

We now have

$$b_{j_k} = a_1^{(\ )} \cdots a_{i_0}^{r_k} \cdots a_n^{(\ )}\ , \qquad \text{for } k \geqq 1\ . \tag{21}$$

For all $k, b_{j_k} \neq a x_k$ for any $x_k \in S$. But since S is an archimedean semigroup, there exists an integer $l, l > 0$ such that

$$(a_{i_0})^l = a\mu\ . \tag{22}$$

There exists $k_0 > 0$ such that $r_k > l$ for all $k \geqq k_0$. Therefore we have

$$\begin{aligned} b_{j_k} &= a_1^{(\ )} \cdots a_{i_0}^{r_k} \cdots a_n^{(\ )} \\ &= a_1^{(\ )} \cdots a_{i_0}^{l} a_{i_0}^{r_k - l} \cdots a_n^{(\ )} \\ &= a\mu(a_1^{(\ )} \cdots a_{i_0}^{r_k - l} \cdots a_n^{(\ )})\ , \qquad \text{for all } k \geqq k_0 \end{aligned} \tag{23}$$

This contradiction establishes that the set of primes is a finite set.

PROPOSITION 5. Let S be a finitely generated, commutative, nonpotent, archimedean semigroup. Let ρ_b be the congruence relation of Definition 1. Then S/ρ_b is a finite group.

Proof. The elements of S/ρ_b are congruence classes. Each congruence class is a tree and contains at least one prime element. If S/ρ_b were an infinite group, then S would contain an infinite number of prime elements. This would contradict Proposition 4. Thus S/ρ_b is a finite group.

THEOREM 1. *A finitely generated, commutative, nonpotent, archimedean semigroup is power joined.*

Proof. Let $a \in S$. Consider S/ρ_a. Let $x, y \in S$. Let $\alpha, \beta \in S/\rho_a$ such that $x \in S_\alpha$ and $y \in S_\beta$. Since S/ρ_a is a finite group, there exist positive integers n and m such that $\alpha^n = \varepsilon$, $\beta^m = \varepsilon$, where ε is the identity of S/ρ_a. Therefore, $x^n \in S_\varepsilon$ and $y^m \in S_\varepsilon$. Let $\{P_1, P_2, \cdots, P_r\}$ be the set of prime elements of S which are contained in S_ε. Let

$$P = glb\{P_1, P_2, \cdots, P_r\} ,$$

where the partial order in S_ε is $\geqq_a$ (see Definition 1). Set

$$T = \{Z \mid Z \in S_\varepsilon, Z \geqq {}_a P\} .$$

T is a finite set because S_ε is a discrete tree. Since S is nonpotent, the powers of x^n and y^m are all distinct. Therefore there exist positive integers r, t such that $P \geqq_a (x^n)^r$ and $P \geqq_a (y^m)^t$. Further, for all Z, where $P \geqq_a Z$, there exists a positive integer s such that $Z = a^s$. Therefore,

$$(x^n)^r = a^{\mu_1}, (y^m)^t = a^{\nu_1} , \tag{24}$$

and

$$(x^{nr})^{\nu_1} = (y^{mt})^{\mu_1} . \tag{25}$$

We conclude that S is a power joined semigroup.

THEOREM 2. *Let S be a commutative, nonpotent, archimedean semigroup. Let $a \in S$ and let G_a $(=S/\rho_a)$ be the corresponding structure group. Then, S is power joined if and only if G_a is a periodic group and S_ε is power joined (where S_ε is the congruence class of S* mod ρ_a *which contains a).*

Proof. Let S be power joined. Let $\alpha \in G_a, y \in S_\alpha$. There exist positive integers n and m such that $y^n = a^m$. Since $a^m \in S_\varepsilon$, so is y^n. Therefore $\alpha^n = \varepsilon$ and we conclude that G_a is periodic. The set S_ε is power joined because it is a subsemigroup of S.

To prove the converse, let $x, y \in S$. There exist $\alpha, \beta \in G_a$ such that $x \in S_\alpha$, $y \in S_\beta$ and $\alpha^n = \varepsilon$, $\beta^m = \varepsilon$. Therefore, $x^n \in S_\varepsilon$, $y^m \in S_\varepsilon$. Since

S_ε is power joined, there exist positive integers k and l such that

$$(x^n)^k = (y^m)^l \ . \tag{26}$$

We conclude that S is power joined.

THEOREM 3. *Let S be a commutative, nonpotent, archimedean semigroup. Then S is power joined if and only if every finitely generated subsemigroup is archimedean.*

Proof. Let S be power joined. Let S' be a finitely generated subsemigroup of S. Then S' is also power joined. Let $x, y \in S'$. Then there exist positive integers n, m such that $x^n = y^m$. Set $\mu = y^{m-1}, v = x^{n-1}$. We get $x^n = y\mu$ and $y^m = xv$. The elements μ and v are also in S'. In case n or m equals 1, we can easily arrange the desired equations by multiplying both sides of the equation $x^n = y^m$ by x or y as required. Therefore S' is archimedean.

Let $x, y \in S$. Let S' be the subsemigroup of S generated by x and y. Since S' is finitely generated, it is archimedean. Thus, S' is a finitely generated, commutative, nonpotent, archimedean semigroup, and by Theorem 1 we conclude that S' is power joined. Therefore, there exist positive integers n and m such that $x^n = y^m$. Since x and y were arbitrary elements of S, we now conclude that S is power joined.

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