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QUASI-BLOCK-STOCHASTIC MATRICES

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The quasi-block-stochastic matrices are introduced as a generalization of the block-stochastic and the quasi-stochastic matrices. The derivation of theorems is possible which are similar to those derived for block-stochastic matrices by W. Kuich and K. Walk and for quasi-stochastic matrices by Haynsworth. Among other theorems the theorem on the group property, the reduction formula and its application to nonnegative matrices holds in a modified manner. An example illustrates the definitions and theorems.

NOTATION

$A = (a_{ij})$	quasi-block-stochastic matrix
A_{ij}	block of A
$a_{ij}^{(n)}$	element of A^n
$a^{(n)}$	vector of generalized row sums of A^n
$a_i^{(n)}$	i^{th} generalized row sum of A^n
$S_A = (s_{ij})$	matrix of the generalized row sums of the blocks
$s_{ij}^{(n)}$	element of S_A^n
$s^{(n)}$	vector of row sums of S_A^n
$s_i^{(n)}$	i^{th} row sum of S_A^n
$l \times l$	dimension of A
$l_i \times l_j$	dimension of A_{ij}
$k \times k$	dimension of S_A
I_l	identity matrix of order l
P	permutation matrix
$e_j = \begin{pmatrix} 1 \\ d_{n_j+2} \\ \vdots \\ d_{n_{j+1}} \end{pmatrix}$	of dimension l_j
u_i	i^{th} unit vector of dimension l
v_i	i^{th} unit vector of dimension k
$f_j = \sum_{i=n_j+1}^{n_{j+1}} d_i u_i$	
λ	eigenvalue
μ	greatest eigenvalue
$\delta_{ij} = 1$	for $i = j$
$= 0$	otherwise
$\varnothing$	null matrix
$n_j = \sum_{i=1}^{j-1} l_i$	$n_1 = 0$.

1. Introduction. A matrix $A = (a_{ij})$ $(i, j = 1, \cdots, l)$ is called quasi-block-stochastic if it may be partitioned into rectangular blocks (submatrices) A_{ij} with dimension $(l_i \times l_j)$ $(i, j = 1, \cdots, k)$

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \cdots & \cdots & \cdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix} \tag{1.1}$$

and if

$$A_{ij} e_j = s_{ij} e_i \qquad (i, j = 1, \cdots, k) \tag{1.2}$$

where e_j is the vector

$$e_j = \begin{pmatrix} 1 \\ d^1_{n_j+2} \\ \vdots \\ d_{n_{j+1}} \end{pmatrix} \qquad \begin{aligned} n_1 &= 0 \\ n_j &= \sum_{i=1}^{j-1} l_i \qquad (j = 2, \cdots, k+1) \end{aligned} \tag{1.3}$$

with dimension l_j, and e_i the vector (1.3) with dimension l_i; $(i, j = 1, \cdots, k)$

If there exists a permutation matrix P such that $P^{-1}AP$ has the form (1.1) in connection with (1.2), A is called quasi-block-stochastic, too. In the following we restrict our attention to matrices which may be partitioned immediately into blocks.

s_{ij} is some sort of row sum, we call it generalized row sum of the block-matrix A_{ij} $(i, j = 1, \cdots, k)$. Associated with the matrix A is the matrix of the generalized row sums of its blocks $S_A = (s_{ij})$ $(i, j = 1, \cdots, k)$:

$$S_A = \begin{pmatrix} s_{11} & \cdots & s_{1k} \\ \cdots & \cdots & \cdots \\ s_{k1} & \cdots & s_{kk} \end{pmatrix} \tag{1.4}$$

Let f_j $(j = 1, \cdots, k)$ be an $(l \times 1)$ vector with blocks $(l_i \times 1)$ $(i = 1, \cdots, k)$

$$f_j = \sum_{i=n_j+1}^{n_{j+1}} d_i u_i = \begin{pmatrix} 0 \\ \vdots \\ e_j \\ \vdots \\ 0 \end{pmatrix}$$

and F be the $(l \times k)$ matrix whose columns are f_j $(j = 1, \cdots, k)$.

If we let $AF = C = (C_{ij})$ $(i, j = 1, \cdots, k)$, we have

$$C = AF = \begin{pmatrix} A_{11} A_{12} & \cdots & A_{1k} \\ A_{21} A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \cdots \\ A_{k1} A_{k2} & \cdots & A_{kk} \end{pmatrix} \begin{pmatrix} e_1 0 & \cdots & 0 \\ 0 e_2 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 00 & \cdots & e_k \end{pmatrix}.$$

The matrix C has blocks C_{ij} which are the $(l_i \times 1)$ vectors

$$C_{ij} = A_{ij}e_j \qquad (i, j = 1, \cdots, k) .$$

But by (1.2)

$$C_{ij} = e_i s_{ij}$$

which is the block in the (i, j) position of the product FS_A.

Thus we have

$$AF = FS_A \tag{1.5}$$

which is equivalent to (1.2), but can be used to great advantage in shortening the proofs of several of the theorems.

Two square matrices of l-th order, A and B are said to be quasi-block-stochastic in the same manner, if they both may be partitioned into $(l_i \times l_j)$ block matrices A_{ij}, B_{ij}, respectively, which satisfy (1.2):

$$A_{ij}e_j = s_{ij}e_i \text{ and } B_{ij}e_j = t_{ij}e_i \ (i, j = 1, \cdots, k) . \tag{1.6}$$

The quasi-block-stochastic matrices are a generalization of the block-stochastic-matrices considered by Haynsworth [2] and Kuich, Walk [6], as well as of the quasi-stochastic matrices, considered by Haynsworth [3].

Block-stochastic matrices originate from the quasi-block-stochastic ones by specialization of the vectors e_i:

$$e_i = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \qquad (i = 1, \cdots, k) .$$

Quasi-stochastic matrices consist of only one block which is the matrix itself

$$Ae_1 = s_{11}e_1$$

where e_1 is the vector

$$e_1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ p \\ \vdots \\ p \end{pmatrix} .$$

In the following section several results on quasi-block-stochastic matrices are presented which are generalizations of the results obtained by Kuich, Walk [6] and Haynsworth [3].

2. Group properties of quasi-block-stochastic matrices.

THEOREM 1. *The set of all nonsingular matrices which are quasi-block-stochastic in the same manner forms a group.*

Proof. The assumption that $A = (A_{ij})$ and $B = (B_{ij})$ $(i, j = 1, \cdots, k)$ are quasi-block-stochastic is expressed by (1.5):

$$AF = FS_A \quad \text{and} \quad BF = FS_B .$$

Hence

$$(AB)F = A(FS_B) = F(S_A S_B)$$

so that if we let

$$S_A S_B = S_{AB} \tag{2.1}$$

we have AB quasi-block-stochastic in the same manner. Also

$$I_l F = F I_k$$

and

$$F = A^{-1}(AF) = A^{-1}(FS_A)$$

which yields

$$A^{-1}F = FS_A^{-1} .$$

This proves theorem 1.

With (2.1) there follows

THEOREM 2. *The transformation mapping the group of matrices that are quasi-block-stochastic in the same manner onto the group of matrices of its generalized row sums is a homomorphism.*

3. Powers of quasi-block-stochastic matrices. We denote the ith generalized row sum of the quasi-block-stochastic matrix A^n by $a_i^{(n)}$:

$$a_i^{(n)} = \sum_{j=1}^{l} a_{ij}^{(n)} d_j \qquad (i = 1, \cdots, l) \tag{3.1}$$

with

(3.2) $$d_1 = d_{n_1+1} = d_{n_2+1} = \cdots = d_{n_k+1} = 1 ,$$

the i^{th} (usual) row sum of the matrix $S_A^{(n)}$ by $s_i^{(n)}$:

(3.3) $$s_i^{(n)} = \sum_{j=1}^{k} s_{ij}^{(n)} d_{n_j+1} = \sum_{j=1}^{k} s_{ij}^{(n)} .$$

We define two series of vectors:

(3.4) $$\left.\begin{aligned} a^{(0)} &= \sum_{i=1}^{l} d_i u_i , \qquad & a^{(n+1)} &= A a^{(n)} \\ s^{(0)} &= \sum_{i=1}^{k} v_i , \qquad & s^{(n+1)} &= S_A s^{(n)} \end{aligned}\right\}$$

where u_i and v_i are the i^{th} unit vectors of dimension l and k, respectively.

LEMMA 1. *The i^{th} component of the vector $a^{(n)}$ is $a_i^{(n)}$, i.e., $a^{(n)} = A^n \cdot a^{(0)}$, the i^{th} component of the vector $s^{(n)}$ is $s_i^{(n)}$, i.e., $s^{(n)} = S_A^n s^{(0)}$ $(n \geqq 1)$.*

Proof. By induction. The lemma is valid for $n = 1$.
Assume

$$a^{(n)} = A^n a^{(0)} .$$

then

$$a^{(n+1)} = A^{n+1} a^{(0)} .$$

Similarly holds

$$s^{(n)} = S_A^n s^{(0)} .$$

With Theorem 1

(3.5) $$A^n F = F S_A^n$$

holds.
Because of

$$A^n F s^{(0)} = F S_A^n s^{(0)}$$

we get the following.

COROLLARY.

(3.6) $$a^{(n)} = \sum_{j=1}^{k} s_j^{(n)} f_j$$

for all n.

The corollary admits no immediate converse, since the property (3.6) does not imply the quasi-block-stochastic structure. We are interested in matrix properties which, combined with the property (3.6) assure the quasi-block-stochastic structure.

We now state the following:

LEMMA 2. *Linear relations which hold among the vectors*

$$v_1, \cdots, v_k, s^{(0)}, s^{(1)}, \cdots, s^{(k)}$$

also hold among the vectors

$$f_1, \cdots, f_k, a^{(0)}, a^{(1)}, \cdots, a^{(k)} .$$

Proof. We consider the vector equation

$$\sum_{i=1}^{k} \alpha_i v_i + \sum_{i=0}^{k} \beta_i s^{(i)} = 0$$

which implies that

$$\sum_{i=1}^{k} \alpha_i \delta_{ij} + \sum_{i=0}^{k} \beta_i s_j^{(i)} = 0 \qquad \text{for } j = 1, \cdots, k$$

and

$$\sum_{i=1}^{k} \alpha_i \delta_{ij} f_j + \sum_{i=0}^{k} \beta_i s_j^{(i)} f_j = 0 \qquad \text{for } j = 1, \cdots, k$$

$$\sum_{j=1}^{k} \left(\sum_{i=1}^{k} \alpha_i \delta_{ij} f_j + \sum_{i=0}^{k} \beta_i s_j^{(i)} f_j \right) = 0$$

$$\sum_{i=1}^{k} \alpha_i \sum_{j=1}^{k} \delta_{ij} f_j + \sum_{i=0}^{k} \beta_i \sum_{j=1}^{k} s_j^{(i)} f_j = 0$$

$$\sum_{i=1}^{k} \alpha_i f_i + \sum_{i=0}^{k} \beta_i a^{(i)} = 0 .$$

THEOREM 3. *If the generalized row sums of a matrix A satisfy the condition* (3.6)

$$a^{(n)} = \sum_{j=1}^{k} s_j^{(n)} f_j$$

for all n, and if in addition the k vectors

$$s^{(0)}, s^{(1)}, \cdots, s^{(k-1)}$$

are linearly independent, then the matrix A is quasi-block-stochastic.

Proof. According to the assumption, we may introduce the following representations:

$$\left.\begin{aligned} v_i &= \sum_{j=0}^{k-1} \alpha_{ji} s^{(j)} \qquad (i = 1, \cdots, k) \\ s^{(k)} &= \sum_{j=0}^{k-1} \beta_j s^{(j)} \end{aligned}\right\} \tag{3.7}$$

and therefore, due to Lemma 2:

$$\left.\begin{aligned} f_i &= \sum_{j=0}^{k-1} \alpha_{ji} a^{(j)} \qquad (i = 1, \cdots, k) \\ a^{(k)} &= \sum_{j=0}^{k-1} \beta_j a^{(j)} \end{aligned}\right\} \tag{3.8}$$

With (3.4)

$$\begin{aligned} Af_i &= \sum_{j=0}^{k-1} \alpha_{ji} A a^{(j)} = \sum_{j=0}^{k-1} \alpha_{ji} a^{(j+1)} \\ &= \sum_{j=0}^{k-2} \alpha_{ji} a^{(j+1)} + \alpha_{k-1,i} \sum_{j=0}^{k-1} \beta_j a^{(j)} \\ &= \alpha_{k-1,i}\beta_0 a^{(0)} + \sum_{j=1}^{k-1} (\alpha_{j-1,i} + \alpha_{k-1,i}\beta_j) a^{(j)} \\ &= \sum_{j=0}^{k-1} \gamma_{ji} a^{(j)} = \sum_{j=0}^{k-1} \gamma_{ji} \sum_{m=1}^{k} s_m^{(j)} f_m \\ &= \sum_{m=1}^{k} f_m \sum_{j=0}^{k-1} \gamma_{ji} s_m^{(j)} = \sum_{m=1}^{k} s_{mi} f_m \,. \end{aligned}$$

The representation $Af_i = s_{1i}f_1 + \cdots + s_{ki}f_k$ for $i = 1, \cdots, k$ indicates that A is quasi-block-stochastic:

$$\begin{aligned} Af_i &= \begin{pmatrix} A_{11} & \cdots & A_{1i} & \cdots & A_{1k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{k1} & \cdots & A_{ki} & \cdots & A_{kk} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ e_i \\ \vdots \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} A_{1i} \\ \vdots \\ A_{ki} \end{pmatrix} e_i = \begin{pmatrix} s_{1i}e_1 \\ \vdots \\ s_{ki}e_k \end{pmatrix} \qquad (i = 1, \cdots, k) \end{aligned}$$

This condition is equivalent to condition (1.2).

4. A reduction formula for quasi-block-stochastic matrices. We refer to the following theorem of Haynsworth [2]: Suppose the $(n_i \times n_j)$ blocks A_{ij} $(i, j = 1, \cdots, t)$ of the partitioned $(N \times N)$ matrix A satisfy

$$A_{ij}X_j = X_iB_{ij} \tag{*}$$

where B_{ij} is a square matrix of order r, $0 < r \leqq n_i$, with strict inequality for at least one value of i, and X_i is an $(n_i \times r)$ matrix with a nonsingular matrix of order r, $X_1^{(i)}$, in the first r rows. Let the last $n_i - r$ rows of X_i be $X_2^{(i)}$, and let

$$A_{ij} = \begin{pmatrix} A_{11}^{(ij)} & A_{12}^{(ij)} \\ A_{21}^{(ij)} & A_{22}^{(ij)} \end{pmatrix}$$

where $A_{11}^{(ij)}$ is square, of order r. Then A is similar to the matrix

$$R = \begin{pmatrix} B & D \\ \varnothing & C \end{pmatrix}$$

where B is a partitioned matrix of order tr with blocks B_{ij}, as defined in (*), and C has blocks

$$C_{ij} = (A_{22}^{(ij)} - X_2^{(i)}(X_1^{(i)})^{-1}A_{12}^{(ij)})$$

with dimensions $(n_i - r) \times (n_j - r)$. If either n_i or $n_j = r$, the corresponding block C_{ij} does not appear.

THEOREM 4. *A quasi-block-stochastic matrix A is similar to*

$$R = \begin{pmatrix} S_A & D \\ \varnothing & C \end{pmatrix}. \tag{4.1}$$

Proof. Theorem 4 is a special case of the theorem of Haynsworth [2] cited above. For proof take

$$\begin{aligned} &N = l,\ t = k,\ r = 1 && \\ &n_i = l_i,\ X_i = e_i && (i = 1, \cdots, k) \\ &B_{ij} = (s_{ij}) && (i, j = 1, \cdots, k) \\ &B = S_A && \end{aligned}$$

and $X_1^{(i)}$, $X_2^{(i)}$, $A_{11}^{(ij)}$, $A_{12}^{(ij)}$, $A_{21}^{(ij)}$, $A_{22}^{(ij)}$ in an obvious manner.

The $(l - k) \times (l - k)$ matrix C of (4.1) has blocks

$$C_{ij} = (A_{22}^{(ij)} - X_2^{(i)}A_{12}^{(ij)}) \qquad (i, j = 1, \cdots, k)$$

with dimensions $(l_i - 1) \times (l_j - 1)$. If either l_i or $l_j = 1$, the corresponding block C_{ij} does not appear.

5. Eigenvectors of quasi-block-stochastic matrices. There is a simple way of finding an eigenvector of A for each eigenvector of S_A, as is stated in

THEOREM 5. *If*

$$x = \sum_{i=1}^{k} x_i v_i \tag{5.1}$$

is an eigenvector belonging to the eigenvalue λ, *with regard to the rows of* S_A, *then*

$$y = \sum_{i=1}^{k} x_i f_i \tag{5.2}$$

is an eigenvector belonging to the eigenvalue λ *with regard to the rows of* A.

Proof. From $S_A x = \lambda x$, there follows by (1.5)

$$A(Fx) = F(S_A x) = \lambda(Fx) .$$

Hence $y = Fx = \sum_{i=1}^{k} x_i f_i$ is an eigenvector belonging to λ with regard to the rows of A.

6. Eigenvalues of nonnegative, irreducible, primitive quasi-block-stochastic matrices. For the following we consider only quasi-block-stochastic matrices whose elements are nonnegative and for which there is no permutation matrix P such that

$$P^{-1}AP = \begin{pmatrix} A_{11} & A_{12} \\ \varnothing & A_{22} \end{pmatrix} \tag{6.1}$$

with square sub-matrices A_{11} and A_{22} or such that

$$P^{-1}AP = \begin{pmatrix} \varnothing & A_1 & \varnothing \cdots \varnothing \\ \varnothing & \varnothing & A_2 \cdots \varnothing \\ A_t & \varnothing & \varnothing \cdots \varnothing \end{pmatrix} . \tag{6.2}$$

It has been proved by Wielandt [7] that under these conditions, the irreducibility (6.1) and the primitiveness (6.2), the matrix A has a positive eigenvalue which is greater than the absolute values of all other eigenvalues of A and which is associated with a positive eigenvector which is the only positive eigenvector of A. We use this result to prove the following:

THEOREM 8. *If the quasi-block-stochastic matrix* A *and the matrix* S_A *are nonnegative, further* A *irreducible and primitive, the components* $d_j (j = 1, \cdots, l)$ *of* f_i $(i = 1, \cdots, k)$ *are positive, then the greatest eigenvalue of* A *is equal to the greatest eigenvalue of* S_A. *This means that the eigenvalues of the matrix* C (Theorem 4) *are smaller than the greatest eigenvalue of* A *and* S_A.

Proof. The greatest eigenvalue μ of S_A corresponds to the only positive eigenvector x_μ

$$x_\mu = \sum_{i=1}^{k} x_{\mu i} v_i \qquad x_{\mu i} \geqq 0 .$$

According to Theorem 5 is

$$y_\mu = \sum_{i=1}^{k} x_{\mu i} f_i$$

eigenvector of A for the eigenvalue μ. μ is the greatest eigenvalue of A, since y_μ is positive. All other eigenvalues of A have to be smaller than μ, so that all eigenvalues of C are smaller than μ.

7. Example. We construct a quasi-block-stochastic matrix by help of Theorem 3.

Our assumptions are

$$s^{(0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad s^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad s^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad l_1 = 2, l_2 = 3$$

$$(7.1) \qquad e_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad e_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad f_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -2 \end{pmatrix} .$$

We get following representations:

$$(7.2) \qquad s^{(2)} = s^{(0)} , \qquad v_1 = \frac{2}{3} s^{(0)} - \frac{1}{3} s^{(1)} , \qquad v_2 = \frac{1}{3} s^{(0)} + \frac{1}{3} s^{(1)} ;$$

and due to Lemma 2:

$$(7.3) \qquad a^{(2)} = a^{(0)} ; \qquad f_1 = \frac{2}{3} a^{(0)} - \frac{1}{3} a^{(1)} , \qquad f_2 = \frac{1}{3} a^{(0)} + \frac{1}{3} a^{(1)}$$

$$(7.4) \qquad \left. \begin{aligned} Af_1 &= A\Big(\frac{2}{3} a^{(0)} - \frac{1}{3} a^{(1)}\Big) = \frac{2}{3} a^{(1)} - \frac{1}{3} a^{(2)} = -f_1 + f_2 \\ Af_2 &= A\Big(\frac{1}{3} a^{(0)} + \frac{1}{3} a^{(1)}\Big) = \frac{1}{3} a^{(1)} + \frac{1}{3} a^{(2)} = \qquad\quad f_2 \end{aligned} \right\}$$

which yields

$$(7.5) \qquad \begin{aligned} s_{11} &= -1 & s_{12} &= 0 \\ s_{21} &= 1 & s_{22} &= 1 \end{aligned} \qquad S_A = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} .$$

By solving the system (7.4) or equivalently $A_{ij} e_j = s_{ij} e_i$ $(i, j = 1, 2)$, we can get following matrix:

$$A=\left(\begin{array}{cc|ccc} 1 & 2 & 3 & 5 & 4 \\ 2 & 1 & 1 & 5 & 3 \\ \hline 1 & 0 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 4 & 0 & 3 \end{array}\right). \tag{7.6}$$

According to Theorem 4 we can transform A by a similarity transformation to:

$$G^{-1}AG=\begin{pmatrix} -1 & 0 & 2 & 5 & 4 \\ 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 3 & 10 & 7 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & 4 & 5 \end{pmatrix}. \tag{7.7}$$

with

$$G=\left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 1 \end{pmatrix}.$$

By the reduction formula (7.7) we get the characteristic equation

$$(\lambda^2-1)(\lambda^3-6\lambda^2-25\lambda+24)=0\,. \tag{7.8}$$

Eigenvalues which belong to both A and S_A are

$$\lambda_1=1\,,\qquad \lambda_2=-1 \tag{7.9}$$

and the eigenvectors with regard to the rows of S_A are

$$x_{\lambda_1}=\begin{pmatrix}0\\1\end{pmatrix}\qquad x_{\lambda_2}=\begin{pmatrix}2\\-1\end{pmatrix}. \tag{7.10}$$

According to Theorem 5 we get the eigenvectors with regard to the rows of A by (7.10);

$$y_{\lambda_1} = 0\cdot f_1 + 1\cdot f_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -2 \end{pmatrix}.$$

(7.11)

$$y_{\lambda_2} = 2\cdot f_1 - 1\cdot f_2 = \begin{pmatrix} 2 \\ -2 \\ -1 \\ -1 \\ 2 \end{pmatrix}.$$

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