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ON A RADON-NIKODYM THEOREM FOR FINITELY ADDITIVE SET FUNCTIONS

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The purpose of this note is to comment on and extend recent results of C. Fefferman. A proof of his Radon-Nikodym theorem that is, perhaps, more amenable to generalization is given. A Lebesgue decomposition is also obtained.

Since the notations in [3] and [7] conflict, we shall make the following compromises in notation and terminology, and beg the reader's indulgence.

Let S be a set, Σ be an algebra of subsets of S, C be the complex numbers, and R be the real numbers. Let $H(C)=H(S, \Sigma: C)$ denote the set of all bounded, complex valued, finitely additive set functions on Σ. Then $H(R)$ will denote the real valued elements of $H(C)$. If $\alpha \in H(C)$ and $E \in \Sigma$, we denote the total variation of α over E by $v(\alpha, E)$. If $\alpha, \beta \in H(C)$ then

(i) α is absolutely continuous with respect to $\beta(\alpha \ll \beta)$ means: given $\varepsilon > 0$, there exists $\delta > 0$ such that $v(\beta, E) < \delta$ $(E \in \Sigma)$ implies $v(\alpha, E) < \varepsilon$.

(ii) α is singular with respect to $\beta(\alpha \perp \beta)$ means: given $\varepsilon > 0$, there exists $E \in \Sigma$ such that $v(\alpha, E) < \varepsilon$ and $v(\beta, S\text{-}E) < \varepsilon$.

The classical Radon-Nikodym theorem (eg., [6, Th. III. 10.2]) asserts that if Σ is a sigma algebra and λ is a countably additive element of $H(C)$, then λ can be given an integral representation with respect to a nonnegative, countably additive element μ of $H(R)$ if, and only if, λ is absolutely continuous with respect to μ.

In 1939, S. Bochner published a generalization ([1]) which removed the restrictions that Σ be a sigma algebra and that the set functions be countably additive. Then S. Bochner and R. S. Phillips [2] used a vector lattice approach to give a new proof of Bochner's Theorem and, also, to obtain a Lebesgue decomposition. S. Leader [8] studied the L^p-spaces associated with finitely additive measures. A representation for the case where $\mu \in H(R)$ appeared ([3]) in 1962. Theorem III. 10.7 of [6] supplements the classical theorem by allowing μ to be complex valued, and recently C. Fefferman ([7]) extended the latter result to the case of a general algebra of subsets of a set.

Let us turn to some comments on the paper of Fefferman.

(i) The definition of absolute continuity given in [7] seems to contain a misprint: Suppose that $S=[-1, 1]$, Σ is the sigma algebra of Lebesgue measurable subsets of S, α is Lebesgue measure m restricted to $[-1, 0]$ (i.e., $\alpha(E)=m(E\cap[-1, 0])$), β is Lebesgue measure restricted to $[0, 1]$, $\mu=\alpha-\beta$, $\gamma=\alpha+\beta$ and $E=S$. Then $\mu(E)=0$ while $\gamma(E)=2$ and, hence, γ is not absolutely continuous with respect to μ, according to [7]. Replacing $|\mu(E)|$ by $v(\mu, E)$ or by $\sup\{|\mu(F)|; F\in\Sigma, F\subset E\}$ in the definition of absolute continuity given in [7] prevents the pathology illustrated by the preceding example.

(ii) After correcting the misprint, one should replace the last statement in [7, p. 35, paragraph 1] by "Unless γ is bounded and countably additive the last two definitions need not be equivalent." (See [4, Th. 3.4]. For a counterexample to the original statement, let S be the positive integers, let Σ be the σ-algebra of all subsets of S, let $\mu(E) = \Sigma_{n\in E}2^{-n}$, and let γ be a Banach measure).

(iii) Existence of a Lebesgue decomposition in the setting of [7] follows immediately from [5] upon letting $T_\varepsilon = \{E\in\Sigma;\ v(\mu, E) < \varepsilon\}$.

(iv) The first four lemmas in [7] are obtained in [3] by a technical modification of a standard proof of the classical Radon-Nikodym theorem.

(v) While the last two lemmas in [7] are neat, our argument seems to make certain generalizations more transparent. Consider, for example, finitely additive set functions on Σ to a Banach space over R with basis. The coordinate functionals are finitely additive, and if enough conditions are imposed, then our procedure can be applied.

For the sake of completeness and clarity, we state Theorem 2.2 of [3].

If f, $g\in H(R)$, then there exist uniquely functions h and s in $H(R)$ such that

(i) $h\ll g$,

(ii) $s\perp g$,

(iii) $f=h+s$

(iv) $E\in\Sigma$ implies $v(f, E) = v(h, E) + v(s, E)$.

Moreover, there exists a sequence $\{y_n\}$ of $(S, \Sigma: R)$-simple functions which converges in g-measure and such that if $h_n(E) = \int_E y_n dg$ for each $E\in\Sigma$, then $\lim_n v(h-h_n, S) = 0$.

The following is the desired extension.

THEOREM 1. *Suppose* α, $\beta\in H(C)$ *and* $\alpha\ll\beta$. *Then there exists a sequence* $\{z_n\}$ *of* $(S, \Sigma: C)$*-simple functions such that if* $\alpha_n(E) = \int_E z_n d\beta$ *for each* $E\in\Sigma$, *then* $\lim_n v(\alpha-\alpha_n, S) = 0$.

In order to apply Theorem 2.2, we need the following

LEMMA 1. *Suppose*

(i) $\alpha, \beta, \gamma \in H(C)$,

(ii) $\{g_n\}$ *is a sequence of* $(S, \Sigma: C)$*-simple functions such that, if* $\gamma_n(E) = \int_E g_n d\alpha$ *for each* $E \in \Sigma$, *then* $\lim_n v(\gamma - \gamma_n, S) = 0$,

(iii) $\beta \perp \gamma$.

Then there exists a sequence $\{h_n\}$ *of* $(S, \Sigma: C)$*-simple functions such that if* $\gamma'_n(E) = \int_E h_n d\alpha$ and $\gamma''_n(E) = \int_E h_n \, d(\alpha + \beta)$ *for each* $E \in \Sigma$, *then* $\lim_n v(\gamma - \gamma'_n, S) = \lim_n v(\gamma - \gamma''_n, S) = 0$.

Proof. Let $M_n = \max\{| g_n(x) | \mid x \in S\}$. Since $\gamma \perp \beta$, we choose a sequence $\{A_n\}$ of elements of Σ such that $v(\beta, A_n) < a_n$ and $v(\gamma, S - A_n) < a_n$ where $a_n = \min\{1/nM_n, 1/n\}$. Define $\{h_n\}$ by $h_n(x) = g_n(x)$ if $x \in A_n$ and $h_n(x) = 0$ if $x \in S - A_n$. Let $\varepsilon > 0$ be given and choose N so large that $n \geqq N$ implies both $v(\gamma - \gamma_n, S) < \varepsilon/3$ and $1/n < \varepsilon/3$. Then if $E \in \Sigma$ and

$$\begin{aligned} n \geq N, | \gamma(E) - \gamma'_n(E) | &\leqq | \gamma(E) - \gamma_n(E) | + |\gamma_n(E) - \gamma'_n(E) \\ &- \gamma(E \cap (S - A_n)) | + | \gamma(E \cap (S - A_n) | < (\varepsilon/3) + | \gamma_n(E \cap (S - A_n)) \\ &- \gamma(E \cap (S - A_n)) | + v(\gamma, S - A_n) < (\varepsilon/3) + (\varepsilon/3) + (\varepsilon/3) = \varepsilon \,. \end{aligned}$$

Therefore, $\lim_n v(\gamma - \gamma'_n, S) = 0$. Finally, for $E \in \Sigma$ and

$$\begin{aligned} n \geq N, | \gamma(E) - \gamma''_n(E) | &\leqq | \gamma(E) - \gamma'_n(E) | + | \int_E h_n d\beta | < \varepsilon \\ &+ | \int_{E \cap A_n} g_n d\beta | \leqq \varepsilon + M_n V(\beta, A_n) \\ &< \varepsilon + M_n \frac{1}{nM_n} < \varepsilon + \frac{\varepsilon}{3} < 2\varepsilon \,. \end{aligned}$$

Therefore $\lim_n v(\gamma - \gamma''_n, S) = 0$, and Lemma 1 is proved.

Proof of Theorem 1. Any $\alpha \in H(C)$ may be separated into its real and complex parts, so it suffices to prove the theorem when α is real valued. According to Theorem 2.2, we can express β as follows: $\beta = \mu + i\nu = (\mu_1 + \mu_2) + i(\nu_1 + \nu_2)$ where $\mu, \nu, \mu_1, \mu_2, \nu_1, \nu_2 \in H(R)$ and

(1) (a) $\mu_1 \perp \nu$ (b) $\mu_2 \ll \nu$

(c) $\nu_1 \perp \mu$ (d) $\nu_2 \ll \mu$.

We also write $\alpha = \alpha_1 + \alpha_2$ where

(2) (a) $\alpha_1 \perp \nu$ (b) $\alpha_2 \ll \nu$.

Notice now that $\alpha_1 \ll \mu_1$. The reason for this is:

$\alpha_1 \ll \alpha \ll \beta$ and (2) (a) imply $\alpha_1 \ll \mu = \mu_1 + \mu_2$. But (1) (b) and (2) (a) imply $\alpha_1 \perp \mu_2$, and hence $\alpha_1 \ll \mu_1$. Therefore, again by Theorem 2.2, there exists a sequence $\{f_n\}$ of $(S, \Sigma: R)$-simple functions such that $\lim_n \int_E f_n d\mu_1 = \alpha_1(E)$ uniformly for $E \in \Sigma$. Since $\alpha_1 \perp \nu$ and $\alpha_1 \perp \mu_2$, we conclude that $\alpha_1 \perp \mu_2 + i\nu$, and apply Lemma 1 to get a sequence $\{f'_n\}$ of $(S, \Sigma: C)$-simple functions such that

(A) $$\lim_n \int_E f'_n d(\mu_1 + (\mu_2 + iv)) = \lim_n \int_E f'_n d\beta = \alpha_1(E)$$

uniformly for $E \in \Sigma$.

Now we look at α_2, and write it as $\alpha_2 = \alpha'_2 + \alpha''_2$ where

(3) (a) $\alpha'_2 \perp \mu$ (b) $\alpha''_2 \ll \mu$.

As before, $\alpha'_2 \ll \alpha_2 \ll \nu$, thus (3) (a), (1) (d) imply $\alpha'_2 \perp \nu_2$, and hence $\alpha'_2 \ll \nu_1$. Also $\alpha'_2 \perp \nu_2$ and $\alpha'_2 \perp \mu$ imply $\alpha'_2 \perp \nu_2 - i\mu$. Therefore by Theorem 2.2 and Lemma 1, there exists a sequence $\{g'_n\}$ of $(S, \Sigma: C)$-simple functions such that

(B) $$\lim_n \int_E g'_n \, d(\nu_1 + (\nu_2 - i\mu)) = \lim_n \int_E (-i) g'_2 d\beta = \alpha'_2(E)$$

uniformly for $E \in \Sigma$.

We are still left with α''_2. But (3) (b), $\alpha''_2 \ll \alpha_2$, (2) (b), and (1) (a) imply $\alpha''_2 \perp \mu_1$, hence $\alpha''_2 \ll \mu_2$. Hence there exists a sequence $\{k_n\}$ of $(S, \Sigma: R)$-simple functions such that $\lim_n \int_E g_n d\mu_2 = \alpha''_2(E)$ uniformly for $E \in \Sigma$. But we cannot apply Lemma 1 here since we do not have $\alpha''_2 \perp \mu_1 + i\nu$. However, (1) (d), $\nu_2 \ll \nu$, and (1) (a) imply $\nu_2 \ll \mu_2$. Therefore there exists a sequence $\{l_n\}$ of $(S, \Sigma: R)$-simple functions such that $\lim_n \int_E l_n d\mu_2 = \nu_2(E)$ uniformly for $E \in \Sigma$. Let $K_n = \max \{|k_n(x)|, x \in X\}$, and, by taking a subsequence of $\{l_n\}$ if necessary, suppose $v\left(\int l_n d\mu_2 - \nu_2, E\right) < 1/nK_n$ for all $E \in \Sigma$. Consider the sequence $\{h_n\}$ where $h_n = k_n/(1 + il_n)$. Notice that h_n is a step function, is defined everwhere since $1 + il_n$ is never zero, and $|h_n(x)| \leqq |k_n(x)|$ for each $x \in S$. Let $\varepsilon > 0$, and choose N so large that $n \geqq N$ implies both

(i) $\left|\alpha''_2(E) - \int_E k_n d\mu_2\right| < \varepsilon/2$ for all $E \in \Sigma$ and

(ii) $1/n < \varepsilon/2$.

If $E \in \Sigma$, then $\left|\alpha''_2(E) - \int_E h_n \, d(\mu_2 + i\nu_2)\right.$

$$\leqq \left|\alpha''_2(E) - \int_E k_n d\mu_2\right| + \left|\int_E k_n d\mu_2 - \int_E h_n(1 + il_n) d\mu_2\right|$$

$$+\left|\int_E h_n(1 + il_n)d\mu_2 - \int_E h_n\, d(\mu_2 + i\nu_2)\right|$$
$$< \frac{\varepsilon}{2} + 0 + \left|\int_E ih_n l_n d\mu_2 - \int_E ih_n d\nu_2\right|$$
$$\leqq \frac{\varepsilon}{2} + K_n\cdot\nu\Big(\Big(\int l_n d\mu_2 - \nu_2, E\Big) < \frac{\varepsilon}{2} + K_n\cdot\frac{1}{nK_n}$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ if } n \geqq N.$$

Therefore

$$\lim_n \int_E h_n d(\mu_2 + i\nu_2) = \alpha_2''(E)$$

uniformly for $E \in \Sigma$.

Noting that $\alpha_2'' \perp \mu_1 + i\nu_1$ and applying Lemma 1 again yields the existence of a sequence $\{h_n'\}$ of $(S, \Sigma\colon C)$-simple functions such that

$$\text{(C)} \qquad \lim_n \int_E h_n' d(\mu_2 + i\nu_2 + (\mu_1 + i\nu_1)) = \lim_n \int_E h_n' d\beta = \alpha_2''(E)$$

uniformly for $E \in \Sigma$.

Considering (A), (B), and (C) together we have

$$\lim_n \int_E (f_n' + (-ig_n') + h_n')d\beta = (\alpha_1 + (\alpha_2' + \alpha_2''))(E) = \alpha(E)$$

uniformly for $E \in \Sigma$. Thus, let $z_n = (f_n' + i(-g_n') + h_n')$ and the theorem is proved.

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