PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 3, 1968

QUASI-ISOMORPHISM AND *TFM* RINGS

WILLIAM J. WICKLESS

Two rings A and B are quasi-isomorphic if and only if there exist ideals A' and B' contained in A and B respectively such that $A' \cong B'$ as rings and A/A' and B/B' are of bounded order as abelian groups. A ring is TFM if and only if it only admits torsion free abelian groups as irreducible modules. It is shown that quasi-isomorphic TFM rings have exactly the same abelian groups as irreducible modules. Several examples of TFM rings are given.

A classification of TFM rings is given. The following results are obtained:

(1) A is TFM if and only if A/pA is radical for all primes p.

(2) A is TFM if and only if A/N is torsion free and no maximal modular right ideal is dense in the subgroup lattice of A/N, where N is the Jacobson Radical of A.

(3) If A/N divisible then A is TFM. The converse holds under the assumption of minimum condition.

(4) A/D radical $\Rightarrow A$ $TFM \Rightarrow A/D$ has no nonzero idempotents, where D is the maximal divisible subgroup of A. These conditions are equivalent under the assumption of minimum condition.

Finally, the questions of the existance of a TFM radical and the determination of the unique maximal TFM ideal of a ring are discussed.

In matters of abelian group theory our definitions and notations are consistent with [4]; in matters of ring theory our definitions and notations are consistent with [3].

2. Quasi-isomorphism and *TFM* rings.

In [1] Beaumont and Pierce introduced the notion of quasi-isomorphism for abelian groups and rings. Our definition is a slight modification of their original definition.

DEFINITION 2.1. Two rings A and B are quasi-isomorphic if and only if there exist ideals A' and B', contained in A and B respectively, such that A' and B' are ring isomorphic and the quotient rings A/A' and B/B' are of bounded order as abelian groups.

In this paper we study quasi-isomorphic rings of the following type.

DEFINITION 2.2. A ring A is a TFM ring if and only if every irreducible right A module is torsion free as an abelian group.

The fields Q, R and C, the rational, real and complex numbers, are immediate examples of *TFM* rings. Any radical ring is *TFM*, as it satisfies the above definition vacuously. Z, the ring of integers, and P, the ring of p-adic integers, are simple examples of non-*TFM* rings.

The following are less trivial examples of *TFM* rings.

EXAMPLE 2.1. Let $A = \sum_{i=1}^{\infty} \oplus A_i$, where $A_1 \cong Z$, $A_i \cong Q$, $i \geqq 2$. (The isomorphisms here are abelian group isomorphisms only.) Give A the ordinary direct sum abelian group structure. Thinking of A as a set of sequences $\langle x_i \rangle_{i=1}^{\infty}$, where only finitely many of the x_i's are nonzero, define, for any positive integer j,

$$e_j = \langle x_i \rangle, x_i = 0 \quad \text{for} \quad i \neq j, x_j = 1 \, .$$

Define $e_i \cdot e_j = e_{i+j}$, where i and j are arbitrary positive integers. This definition clearly can be extended to arbitrary elements of A, thereby determining a product for A, making A a commutative ring.

EXAMPLE 2.2. Let $\{p_1, p_2, \cdots p_i \cdots\}$ be an arbitrary ordering of the set π of all primes. For any positive integer i, let $\pi_i = \{p_1 \cdots p_i\}$, and let $A_i = \{r/s \in Q \mid (r, s) = 1; p \in \pi, p \mid s \Rightarrow p \in \pi_i\}$. Each A_i is an abelian group under ordinary rational addition. Let $A = \sum_{i=1}^{\infty} \oplus A_i$ with the ordinary direct sum addition. Define a ring multiplication on A exactly as in Example 2.1.

EXAMPLE 2.3. Let p be a fixed prime. For any positive integer k, let $C(p^k)$ be the cyclic group of order p^k. Let $B = \prod_{k=1}^{\infty} C(p^k)$; give B the ordinary direct product abelian group structure. Define a ring multiplication on B by specifying the i-th co-ordinate of the product $\langle x_i \rangle \cdot \langle y_i \rangle$ to be px_iy_i, where $\langle x_i \rangle$ and $\langle y_i \rangle$ are arbitrary elements of B. This makes B a commutative radical ring. (Every element of B is quasi-regular, see [3], p. 9.) Let $A = B[\lambda]$, the ring of all polynomials in a commuting indeterminate λ over B.

These examples will be discussed in greater detail in § 3.

The following theorem is immediate.

THEOREM 2.1. *The class of TFM rings is closed under the taking of direct sums, homomorphisms, and extensions.*

The motivation for considering quasi-isomorphic *TFM* rings is given by Theorem 2.2. First we prove a lemma.

LEMMA 2.1. *Let A be any ring and M be any irreducible A module. Then either M is torsion free and divisible as an abelian group, or $pM = 0$ for some prime p.*

Proof. Since M is an irreducible A module, M may be regarded as a vector space over a division ring D. ([3], p. 26.) If characteristic $D = 0$, M is torsion free divisible. If characteristic $D = p$, p a prime, then $pM = 0$.

THEOREM 2.2. *Let A and B be quasi-isomorphic TFM rings. Let φ be the ring isomorphism mapping A' onto B', A' and B' ideals in A and B with A/A' and B/B' of bounded order. Let M be any irreducible right A module. Then M can be assigned a unique right B module structure via φ. Under this assignment M becomes an irreducible B module. Every irreducible B module is obtained in this manner.*

Proof. Let M be as above. Let $x \in M, b \in B$. Let k be any positive integer such that $k(B/B') = 0$. Define $xb = y\varphi^{-1}(kb)$, where y is the unique solution in M to the equation $ky = x$. It is easy to check that M becomes an irreducible B module under this definition. We have given M the unique B module structure such that if $x \in M$, $a' \in A'$, then $xa' = x\varphi(a')$. It is clear that every irreducible B module can be obtained in this way.

3. Classfication of *TFM* rings.

THEOREM 3.1. *Let A be any ring. Then A is TFM if and only if A/pA is a radical ring for all primes p.*

Proof. If A is not *TFM*, then A has an irreducible module M with $pM = 0$ for some prime p. M can be regarded as an A/pA module in the obvious fashion; as such M is still irreducible. Hence, A/pA is not a radical ring.

Conversely, if A/pA is not radical for some prime p, then A/pA has an irreducible M. M can be regarded as an irreducible A module is the obvious way. M is p-bounded, being a group homomorphic image of A/pA. Hence, A is not *TFM*.

REMARK. Using the above theorem it is easy to see that the ring A of Example 2.2 is *TFM*. We note that A/pA is nilpotent for any prime p.

COROLLARY. *Let A be a ring with identity. Then A is TFM* *if and only if A is divisible as an abelian group.*

Proof. If $1 \in A$, then A/pA is radical if and only if $A = pA$. ([2], p. 58.) Hence, by 3.1, if $1 \in A$, then A is *TFM* if and only if $A = pA$ for all primes p. But $A = pA$ for all primes p if and only if A is divisible.

THEOREM 3.2. *Let A be any ring. Then A is TFM if and only if the following conditions hold:*

(1) *A/N is torsion free*

(2) *If I is any maximal modular right ideal of A, then I/N is not dense in the subgroup lattice of A/N.*

Proof. Let A be *TFM*. Assume $(x + N)$ is a nonzero element of A/N of finite order. Then $x \notin N$, but $kx \in N$ for some positive integer k. Since $x \notin N$, $x \notin P$ for some primitive ideal P. Since $kx \in N$, $kx \in P$. Hence, A/P is not torsion free. But A/P must be torsion free, being a subring of the complete endomorphism ring of some torsion free irreducible module M. Contradiction. Thus, A/N is torsion free.

Now let I be any maximal modular right ideal of A. Since A is *TFM*, $A - I$ is torsion free. Let $x \in A$, $x \notin I$. Let G be the subgroup of A/N generated by the nonzero element $(x + N)$. Clearly, $G \cap I/N = \{0\}$—otherwise we have $kx \in I$ for some positive integer k. Hence, I/N is not dense in the subgroup lattice of A/N.

Conversely, assume A is a ring which satisfies conditions (1) and (2) above. To show A is *TFM*, we show $A - I$ is torsion free for any maximal modular right ideal I. Let I be such an ideal. I/N is not dense in the subgroup lattice of A/N. Thus, we can find a nonzero subgroup $S/N \subseteq A/N$ such that $S/N \cap I/N = \{0\}$. The mapping

$$x + N \longrightarrow (x + I, x + S)$$

is an abelian group injection of A/N into $(A - I) \oplus (A - S)$. Let $y \in S$, $y \notin I$. As $(y + N)$ has infinite order, so does its image $(y + I, 0 + S)$. Hence, $y + I$ is a nonzero element of infinite order in $A - I$. But $A - I$ is torsion free or p-bounded. Therefore, we must have $A - I$ torsion free. This finishes the proof of the theorem.

THEOREM 3.3. *Let A be any ring. If A/N is divisible, then A is TFM. The converse holds if A has minimum condition.*

Proof. If A/N is divisible, then $A - I$ is divisible for any maximal modular right ideal I. Hence, by Lemma 2.1, A is *TFM*.

Now assume A has minimum condition and A is *TFM*. By 3.2, A/N is torsion free. It is a simple consequence of Wedderburn's Theorem that if A has minimum condition then A/N is torsion free if and only if A/N is divisible. This completes the proof.

REMARKS. The converse to Theorem 3.3 is false in general. The ring constructed in Example 2.2 is *TFM*, semisimple, and reduced as an abelian group.

Using Theorem 3.3 we see that the ring A of Example 2.3 is *TFM*. The radical of A is $T[\lambda]$, where T is the maximal nil ideal in B. ([3], p. 13.) Here $A/N = B[\lambda]/T[\lambda] \cong B/T[\lambda]$. B/T is divisible, since $T \supseteqq \sum_{k=1}^{\infty} \oplus C(p^k)$. Thus A/N is divisible, and A is *TFM*.

THEOREM 3.4. *Let A be any ring. Let D be the maximal divisible subgroup of A. (Note that D is actually an ideal.) Then:*

(1) *If A/D is radical, then A is TFM.*

(2) *If A is TFM, then A/D has no nonzero idempotents. These three conditions are equivalent if A has minimum condition.*

Proof. (1) If A is not *TFM*, then there exists an irreducible A module M with $pM = 0$. Clearly, $MD = 0$. Hence, M can be regarded as an irreducible A/D module, and A/D is not radical.

(2) Let A be *TFM*. Since every irreducible A/D module may be regarded as an irreducible A module, A/D is also *TFM*. If A/D is radical, clearly A/D can have no nonzero idempotents. Hence, for the remainder of the proof, we may assume A/D is not radical.

Let $N[A/D]$ be the radical of A/D. Now assume $(e + D)$ is a nonzero idempotent in A/D. Then $(e + D) + N[A/D]$ is a nonzero element of $A/D/N[A/D]$. Since A/D is *TFM*, $A/D/N[A/D]$ is torsion free. Hence, $(e + D) + N[A/D]$ has infinite order. Hence, e must be an element of infinite order in A.

By assumption, $(e + D)^2 = (e + D)$. We may write $e^2 + d = e$ where $d \in D$. Now let p be any prime. Let $h_p(e)$ denote the p-height of e in A. (See [4].) We must have $h_p(e) = 0$ or $h_p(e) = \infty$, for if $h_p(e) = k, 0 < k < \infty$, then $k = h_p(e) = h_p(e^2 + d) \geqq 2k$.

We finally claim $h_p(e) = 0$ for some p. Otherwise, we have $h_p(e) = \infty$ for all p. But then, since e has infinite order, it is easy to see that $e \in D$. Contradiction. So $h_p(e) = 0$ for at least one prime p.

But now we have $(e + pA)$ is a nonzero idempotent in A/pA. Hence, A/pA is not radical. Since A was assumed to be *TFM*, this yields a contradiction.

If A has minimum condition, then A/D has minimum condition. Let $N[A/D]$ be the radical of A/D. If A/D is not radical, then there exists $\bar{e} \in A/D/N[A/D]$ with $\bar{e} \neq 0, \bar{e}^2 = \bar{e}$. (This is simple consequence of Wedderburn's Theorem.) But then $\bar{e}$ can be "lifted" to an idempotent $e \in A/D$. ([3], p. 54.) Hence, the three conditions of the theorem coincide if A has minimum condition.

REMARKS. The ring A in Example 2.1 is such that A/D is radical. Hence, by Theorem 3.4, A is *TFM*.

The converse to each implication in Theorem 3.4 is false in general. The ring of even integers in an easy counterexample to the converse of 2; the ring in Example 2.2 is a counterexample to the converse of 1.

4. *TFM* Radical and Maximal *TFM* Ideal. Given an arbitrary ring A, we first wish to determine the unique maximal ideal I of A such that I is a *TFM* ring. This is accomplished in the following simple theorem.

THEOREM 4.1. *Let A be any ring. Let $P_B = \bigcap_{\alpha \in B} P_\alpha$, where $\{P_\alpha \mid \alpha \in B\}$ is the set of all primitive ideals associated with the p-bounded irreducible modules of A. ($P_B = A$ if $B = \emptyset$.) Then P_B is the unique maximal TFM ideal of A.*

Proof. Let M be an irreducible P_B module. As P_B is an ideal in A, M can be regarded as an irreducible A module. (See [2], p. 51-53.) M must be torsion free, otherwise we have $MP_B = 0$. Hence, P_B is *TFM*.

Now if I is any ideal in A such that I is a *TFM* ring, we must have $I \subseteqq P_B$—otherwise $I \nsubseteq P_\alpha$ for some $\alpha \in B$, and M_α, a bounded irreducible module associated with P_α, would be an irreducible I module. This proves the theorem.

Finally, we consider the question of the existance of a radical for the class of *TFM* rings. It is clear that a ring A is *TFM* if and only if A/N is *TFM*. We pose the the problem as follows: Given an arbitrary non-*TFM* ring A, find an ideal I of A containing the radical N such that:

(1) A/I is *TFM*.

(2) If J is an ideal of A with $N \subseteqq J \subset I$, A/J is not *TFM*.

The following theorem shows that, under the assumption of minimum condition, such a *TFM* radical exists.

THEOREM 4.2. *Let A be a non-TFM ring with minimum condition. Let $P_D = \bigcap_{\alpha \in D} P_\alpha$, where $\{P_\alpha \mid \alpha \in D\}$ is the set of all primitive ideals of A associated with the torsion free irreducible A modules. Then P_D is a TFM radical for A.*

Proof. By examining the Wedderburn decomposition for A/N, it is clear that P_D/N is the unique minimal ideal I in A/N such that $A/N/I$ is *TFM*. The theorem follows.

REMARK. It is easy to give an example to show that no reasonably defined *TFM* radical exists in the general case. For instance, let $A = \sum_{i=1}^{\infty} \oplus Z_i$, with the ordinary addition and the shift multiplication used in Examples 2.1 and 2.2.

The author gratefully acknowledges the help and encouragement of his thesis advisor, Professor Ross A. Beaumount. The author also wishes to thank Professor Richard S. Pierce for several helpful suggestions.

REFERENCES

1. R. A. Beaumount and R. S. Pierce, *Torsion free rings*, Illinois J. Math. **5** (1961), 61-98.
2. R. Brauer and E. Weiss, *Non Commutative Rings*, Department of Math., Harvard University.
3. N. Jacobson, *Structure of Rings*, A.M.S. Coll. Publ. 37, Amer. Math. Soc., Providence, 1964.
4. I. Kaplansky, *Infinite Abelian Groups*, Univ. of Mich. Press, Ann Arbor, 1954.

Received January 4, 1968. The preparation of this paper was sponsored in part by NSF Grant GP-5497. The results in this paper comprise a partion of the authors doctoral thesis, written while the author was an NDEA fellow at the University of Washington.

UNIVERSITY OF CALIFORNIA, LOS ANGELES