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MINIMAL URYSOHN SPACES

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A topological space X is (1) H(i), (2) H(ii), (3) R(i), (4) R(ii) if (1) Every open filter on X has nonvoid adherence, (2) Every open filter on X with one-point adherence is convergent, (3) Every regular filter on X has nonvoid adherence, (4) Every regular filter on X with one point adherence is convergent. These properties, which were investigated by Scarborough and Stone in a recent paper, arose naturally from the study of minimal Hausdorff, H-closed, minimal regular and R-closed spaces. This paper investigates similar properties for minimal Urysohn and Urysohn closed spaces.

Urysohn filters are introduced and another characterization of minimal Urysohn and Urysohn closed spaces is obtained. In connection with product spaces, it is shown that in some sense open filters and regular filters have more in common with each other than either has with Urysohn filters, dispite the fact that Urysohn filters "lie between" these first two types.

A topological space X is said to be Urysohn if every two distinct points of X can be separated by disjoint closed neighborhoods. A space $(X, \mathscr{T})$ is said to be minimal Urysohn if $\mathscr{T}$ is a Urysohn topology on X and there is no strictly smaller Urysohn topology contained in $\mathscr{T}$. A space $(X, \mathscr{T})$ is said to be Urysohn closed (U-closed) if $\mathscr{T}$ is a Urysohn topology and X is closed in every Urysohn space in which it is embedded. Herrlich [5] has shown the existence of noncompact minimal Urysohn spaces and also of U-closed spaces which are not minimal Urysohn.[1] In what follows, we generalize the notions of minimal Urysohn and U-closed and extend them to spaces which are not necessarily Urysohn (or even Hausdorff).

A filter base $\mathscr{F}$ on a space X is said to be an open filter if each member of $\mathscr{F}$ is an open set. An open filter $\mathscr{F}$ is said to be a Urysohn filter or a U-filter if given any point x not in the adherence of $\mathscr{F}$, there exists a neighborhood U of x and an $F \in \mathscr{F}$ such that $U' \cap F' = \varnothing$ (A' denotes the closure of the set A). Using the same techniques as in [1] or [3], we see that the following is true:

(1) A Urysohn space is U-closed if and only if every U-filter has nonvoid adherence.

(2) A Urysohn space is minimal Urysohn if and only if every U-filter with one-point adherence converges to this point.

We say that a space X is a U(i) space or is U(i) if every U-filter

[1] The existence and characterizations of these spaces were also known to the author.

on X has nonvoid adherence. We also say that a space X is a U(ii) space or is U(ii) if every U-filter on X with one-point adherence converges to this point. A space can be U(ii) is two ways; it can satisfy the condition U(ii) vacuously or nonvacuously. Thus the statement "X is U(ii) vacuously" will mean that no U-filter on X has a unique adherent point, and the statement "X is U(ii) nonvacuously" will mean that X is U(ii) and also there is at least one U-filter on X with one-point adherence. This last requirement is certainly met if X is a nonempty Urysohn space. Every minimal Urysohn space is U-closed, and by an argument analagous to that in [3, p. 98], it follows that a space which is U(ii) nonvacuously is also U(i).

For an example of a countable T_1 space which satisfies U(ii) vacuously but does not satisfy U(i) see [7, Ex 2.2], and for an example of a countable T_1 space which satisfies U(ii) nonvacuously, see [7, Ex 2.3].

If $\{X_a: a \in A\}$ is a family of spaces, we write πX_a for the Cartesian product of this family, and assume πX_a is equipped with the product topology. We also assume that no factor is empty. The notation and definitions will be that of [7].

THEOREM 1. *If* $X = \pi X_a$ *is* U(i), *then each factor is* U(i).

This follows from the observation that the continuous image of a U(i) space is U(i).

In particular, if πX_a is U-closed, then so is each X_a. It is unknown whether the converse of the above theorem is true or not, even when all the factors are U-closed. However, in certain special cases we are able to give an affirmative answer.

LEMMA 2. *Let X be an* H(i) *space and Y an arbitrary space. If $\mathscr{F}$ is a U-filter on $X \times Y$, then $\pi_2\mathscr{F}$ is a U-filter on Y.*

If $y \notin ad\pi_2\mathscr{F}$, then for each $x \in X$, there exist neighborhoods U_x and V_x of x and y respectively and an $F_x \in \mathscr{F}$ such that

$$(U'_x \times V'_x) \cap F'_x = \varnothing .$$

Since X is H(i), there exist $x_1, \cdots, x_n \in X$ such that

$$\cup \{U'_{x_k}: 1 \leqq k \leqq n\} \supset X ;$$

see [6, p. 132]. Thus

$$(X \times \cap \{V'_{x_k}: 1 \leqq k \leqq n\}) \cap (\cap \{F'_{x_k}: 1 \leqq k \leqq n\}) = \varnothing ,$$

and it follows from [7, 2.17] that

$$(\cap \{V'_{x_k}: 1 \leqq k \leqq n\}) \cap (\pi_2 \cap \{F_{x_k}: 1 \leqq k \leqq n\})' = \varnothing .$$

We have produced a closed neighborhood $\cap\{V'_{x_k}: 1 \leqq k \leqq n\}$ of y which fails to meet the closure of $\pi_2 \cap \{F_{x_k}: 1 \leqq k \leqq n\}$; whence $\pi_2\mathscr{F}$ is a U-filter.

LEMMA 3. *Let X be an H(i) space and Y be an arbitrary space. If $\mathscr{F}$ is an open filter on $X \times Y$ and if $y \in ad\pi_2\mathscr{F}$, then there exists an $x \in X$ such that $(x, y) \in ad\mathscr{F}$.*

The proof is essentially that of [7, 3.13] and is omitted.

THEOREM 4. *If X is H(i) and Y is U(i), then $X \times Y$ is U(i).*

The proof follows immediately from Lemma 2 and 3.

COROLLARY 5. *The product of a compact Hausdorff space (or equivalently, a regular U-closed space) with a U-closed space is U-closed.*

We now consider the question as to whether theorems similar to 2.9 and 3.8 of [7] also hold for U(ii) spaces. In view of these theorems and the fact that every regular filter is a U-filter and every U-filter is an open filter, it might be conjectured that a corresponding theorem is also valid for U(ii) spaces. The following theorem and example completely resolve this question.

THEOREM 6. *If $X = \pi X_a$, and X satisfies U(ii) vacuously, then at least one X_a does so.*

If no X_a satisfies U(ii) vacuously, then there exists a U-filter $\mathscr{F}_a$ on each X_a with a unique adherent point x_a. The product filter $\pi\mathscr{F}_a$ is a U-filter on X with the unique adherent point πx_a. This is a contradiction.

Next we present an example of a product space $X_1 \times X_2$ in which X_2 is U(ii) vacuously while $X_1 \times X_2$ is not U(ii) vacuously (it is not U(ii)).

EXAMPLE 7. Let I be the positive integers. Let X_1 be the Urysohn subspace of X (described in [3, p. 98]) consisting of the points a_{ij}, a where $i, j \in I$.

We next describe X_2. Let S be the set of all functions from I into I. Let

$$X_2 = \{a'_{ij}, b'_{ij}, c'_{ij}, d'_{ij}, e'_i, f'_i: i, j \in I\} \cup \{a', b'\} ,$$
$$A_{in} = \{a'_{ij}, b'_{ij}: j \geqq n\} \text{ and } B_{in} = \{c'_{ij}, d'_{ij}: j \geqq n\}$$

where all the symbols a'_{ij}, b'_{ij}, c'_{ij}, d'_{ij}, e'_i, f'_i, a', b' are assumed to represent distinct points. If we let

$$\mathscr{B} = \{\{a'_{ij}, b'_{ij}\}, \{c'_{ij}, d'_{ij}\}: i, j \in I\} \cup \{\{e'_i, f'_i\} \cup A_{in} \cup B_{im}: i, n, m \in I\}$$
$$\cup \{\{a'\} \cup \cup \{A_{if(i)}: i \geqq n\}: n \in I, f \in S\}$$
$$\cup \{\{b'\} \cup \cup \{B_{if(i)}: i \geqq n\}: n \in I, f \in S\} ,$$

then $\mathscr{B}$ is a base for a topology on X_2 which is U(ii) vacuously.

We will show that $X_1 \times X_2$ is not U(ii) by exhibiting a U-filter on $X_1 \times X_2$ with a unique adherent point to which it does not converge.

For each $n \in I$, let

$$F_n = \cup \{\{a_{ij}\} \times \{a'_{ij}, b'_{ij}\}: j \in I, i \geqq n\} .$$

Clearly $\mathscr{F} = \{F_n: n \in I\}$ is an open filter on $X_1 \times X_2$ with (a, a') as its only point of adherence. Since $F'_n = F_n \cup \{(a, a')\}$, $\mathscr{F}$ is a U-filter on $X_1 \times X_2$ with a unique adherent point to which it does not converge. Thus $X_1 \times X_2$ is not U(ii), but is R(ii) vacuously by Theorem 3.8 of [7] in view of the fact that every regular filter base is a U-filter.

Because of the above example, it is not possible to use the same technique as was employed in Theorem 3.9 of [7] to prove the following conjecture: If $X = \pi X_a$ is U(ii) nonvacuously, then each X_a is U(ii) nonvacuously. Using the same idea as in Theorem 6, it is easy to see that some X_a must be U(ii), but the above conjecture as well as its converse remain unsolved problems. Indeed, it is not known whether or not the product of minimal Urysohn spaces is minimal Urysohn. However, in certain special cases we were able to obtain the following results.

THEOREM 8. *If $X \times Y$ is U(ii) and if Y has a Urysohn filter $\mathscr{G}$ with a unique adherent point y, then X is U(ii).*

Let $\mathscr{F}$ be a U-filter on X with a unique adherent point x. Then $\mathscr{F} \times \mathscr{G}$ is a U-filter on $X \times Y$ with a unique adherent point (x, y). Since $X \times Y$ is U(ii), $\mathscr{F} \times \mathscr{G}$ converges to (x, y); whence $\mathscr{F}$ converges to x. Thus X is U(ii).

COROLLARY 9. *If $X = \pi X_a$ is minimal Urysohn, then each factor is minimal Urysohn.*

Let X_b be a factor and write Y for the product of the rest. Then Y is a nonempty Urysohn space (whenever we write πX_a, we assume that each $X_a \neq \varnothing$). Let $y \in Y$ and $\mathscr{N}(y)$ be the open neighborhood system of y. By the preceding theorem, X_b is U(ii); thus X_b is minimal Urysohn.

THEOREM 10. *If X is H(ii) nonvacuously and Y is U(ii), then $X \times Y$ is U(ii).*

Let $\mathscr{F}$ be a U-filter on $X \times Y$ with a unique adherent point (x, y). By Lemma 2, $\pi_2\mathscr{F}$ is a U-filter on Y and by Lemma 3, y is the only adherent point of $\pi_2\mathscr{F}$. Thus $\pi_2\mathscr{F}$ converges to y. If $w \in ad\pi_1\mathscr{F}$, then $(w, y) \in ad\mathscr{F}$, so $w = x$. Hence $\pi_1\mathscr{F}$ is an open filter on an H(ii) space with a unique adherent point x to which it must converge. Thus $\mathscr{F}$ converges to (x, y), so $X \times Y$ is U(ii).

COROLLARY 11. *If X is compact Hausdorff and Y is minimal Urysohn, then $X \times Y$ is minimal Urysohn.*

The proof is immediate from the preceding theorem.

Next we show that there are large numbers of U-closed spaces which are not minimal Urysohn and also large numbers of minimal Urysohn spaces which are not compact. Corresponding theorems for the Hausdorff and regular cases are found in [7].

THEOREM 12. *There exist U-closed spaces of every infinite cardinality which are not minimal Urysohn.*

Let X_1 be the Urysohn subspace of X (described in [3, p. 98]) consisting of the points $a_{ij}, c_i, a, i, j \in I$. X_1 is a countable absolutely closed Urysohn space which is not minimal Urysohn. Thus X_1 is U-closed. Given an infinite cardinal K, let Z be a compact Hausdorff space of cardinality K—for instance let Z be the one point compactification of a discrete space of cardinal K. By Corollary 5, $X_1 \times Z$ is U-closed and obviously has cardinal K. By Corollary 9, $X_1 \times Z$ is not minimal Urysohn, although this is easy to verify directly.

THEOREM 13. *There exist minimal Urysohn spaces of every uncountably infinite cardinality which are not compact.*

By Corollary 11 and the proof of Theorem 12 it suffices to exhibit a noncompact minimal Urysohn space with cardinality equal to the first uncountable cardinal. Such a space is found in [5, p. 289].

Every countable R-closed space is compact [7, p. 137], so every countable minimal regular space is compact. There exist countable absolutely closed spaces which are not minimal Hausdorff (see the proof of 12) and also noncompact minimal Hausdorff spaces which are countable; see [3, p. 98]. Do there exist noncompact minimal Urysohn spaces which are countable?

Berri [4] has shown that every minimal Hausdorff space which is countable has an isolated point. The same reasoning also shows that every countable absolutely closed space has an isolated point. With trivial modifications of this proof the following theorem results.

THEOREM 14. *Every countable U-closed space has an isolated point.*

COROLLARY 15. *Every countable minimal Urysohn space has an isolated point.*

It is well known [6] that absolutely closed plus semiregular is equivalent to minimal Hausdorff. The relation of semiregularity to the corresponding properties in the Urysohn case is given in the following.

THEOREM 16. *A minimal Urysohn space X is semiregular.*

If X is not semiregular at some point $x \in X$, then

$$\mathscr{F} = \{(N')^0 : N \in \mathscr{N}(x)\}$$

is a U-filter on X having a unique adherent point to which it does not converge. This is a contradiction.

COROLLARY 17. *Every minimal Urysohn space can be embedded densely in a minimal Hausdorff space.*

The result follows immediately from the fact that a space is semiregular if and only if it can be densely embedded in a minimal Hausdorff space; see [2].

It is not true that semiregularity plus U-closure implies minimal Urysohn as the following example shows.

EXAMPLE 18. Let X be the subspace consisting of $\{a\} \cup R_1 \cup R_2$ in the notation of [5, Ex 4]. Then X is U-closed and semiregular but not minimal Urysohn.

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