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SIMULTANEOUS INTERPOLATION IN H_2, II

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Let $\{z_n\}$ denote a fixed sequence of complex numbers in the unit disc satisfying $(1 - |z_{n+1}|^2)/(1 - |z^n|^2) \leqq \delta < 1$ for some δ. Let M be a nonnegative integer, and let m be generic for integers between 0 and M inclusive. We define the linear functionals $L_n^{[m]}$ on H_2 by $L_n^{[m]}f = f^{(m)}(z_n)$. Given $M+1$ sequences $w^{[0]}, \cdots, w^{[M]}$ in l_2, can there be found a function f in H_2 which solves the simultaneous weighted interpolation problem

$$f^{(m)}(z_n) = (w^{[m]})_n \,||\, L_n^{[m]} \,|| \ ?$$

Shapiro and Shields considered this problem for $M = 0$. Their results were generalized by the author to the case $M = 1$. The purpose of this paper is to extend this generalization to arbitrary M.

The technique which we used for $M = 1$ would suggest that to proceed to arbitrary M, we should let $w^{[0]}, \cdots, w^{[M]}$ be prescribed in l_2 and then try to find $f_0, \cdots, f_M$ in H_2 satisfying

$$\text{(A)} \qquad \begin{cases} f_m^{(m)}(z_n) = (w^{[m]})_n \,||\, L_n{}^{[m]} \,|| \\ f_m^{(i)}(z_n) = 0 \qquad (0 \leqq i \leqq M,\ i \neq m) \end{cases}$$

Then, $f_0 + \cdots + f_M$ could serve as the desired interpolating function. However, the computational difficulties which would be involved in such a program can be glimpsed even in the case $M = 1$. We found the following modification to be effective.

The work of Shapiro and Shields assures us that we can interpolate when $M = 0$. Fixing M and assuming the result for lesser values, let $w^{[0]}, \cdots, w^{[M]}$ be chosen from l_2. The induction hypothesis furnishes us with a function f_{M-1} corresponding to $w^{[0]}, \cdots, w^{[M-1]}$. We would like to alter f_{M-1} by finding a function g_{M-1} in H_2 for which the sum $f_M \equiv f_{M-1} + g_{M-1}$, together with its first M derivatives, assumes appropriate values on $\{z_n\}$. This is equivalent to demanding that

$$\begin{cases} g_{M-1}^{(M)}(z_n) = [(w^{[M]})_n - ||\, L_n^{[M]} \,||^{-1} f_{M-1}^{(M)}(z_n)] \,||\, L_n^{[M]} \,|| \\ g_{M-1}^{(m)}(z_n) = 0 \qquad (m < M) \ . \end{cases}$$

By proving that the quantity in brackets is in l_2, we reduce the problem to that of finding a function g, once m and $w^{[m]}$ have been prescribed, which satisfies

$$\text{(B)} \qquad \begin{cases} g^{(m)}(z_n) = (w^{[m]})_n \,||\, L_n^{[m]} \,|| \\ g^{(i)}(z_n) = 0 \qquad (i < m) \ . \end{cases}$$

(B) is simpler to solve than (A) because the restriction $i \neq m$ has been changed to $i < m$. This accounts for why, although we now deal with abitrary M, our work is even less computational than when we only treated the case $M = 1$.

2. Preliminary results.

2.1 In [1], Bari proved the following: Let $\{x_n\}$ be a sequence of elements in a separable Hilbert space H. Then $\{(x, x_n)\}$ belongs to l_2 for all x in H if and only if the infinite matrix with elements (x_i, x_j) determines a bounded operator on l_2.

2.2 In [3], Schur showed that for any infinite matrix (a_{ij}), if $\sum_i |a_{ij}| \leqq N_1$ for all j, and $\sum_j |a_{ij}| \leqq N_2$ for all i, then

$$|\Sigma_{ij} a_{ij} x_i \bar{x}_j| \leqq (N_1 N_2)^{1/2} \Sigma_i |x_i|^2 .$$

2.3 Let δ_n denote $(1 - |z_n|^2)^{-1/2}$. We say that $\{z_n\}$ *approaches the boundary exponentially*, provided that

$$\delta_n / \delta_{n+1} \leqq \delta < 1 \qquad (n = 1, 2, \cdots)$$

for some δ.

We say that $\{z_n\}$ is a *Carleson sequence* if

$$\prod_{k \neq n} \left| \frac{z_k - z_n}{1 - z_n \bar{z}_k} \right| > \sigma > 0 \qquad (n = 1, 2, \cdots)$$

for some σ.

If a sequence approaches the boundary exponentially then it is a Carleson sequence (see [4]).

2.4 The functionals $L_n^{[m]}$ are continuous with Riesz representatives

$$K_n^{[m]}(z) = \frac{m! z^m}{(1 - \bar{z}_n z)^{m+1}} .$$

Their norms satisfy $\delta_n^{2m+1} \leqq \| L_n^{[m]} \| = 0(\delta_n^{2m+1})$ (for M fixed).

This is suggested by applying $\partial^m / \partial z_n^m$ to both sides of

$$f(z_n) = \frac{1}{2\pi i} \lim_{r \uparrow 1} \oint \frac{f(z)}{z} \frac{dz}{1 - z_n \bar{z}} \qquad (|z| = r)$$

and then formally bringing the operator past the limit and the integral sign. The result is more readily established by hindsight by finding the Taylor expansion of $m!(1 - \bar{z}_n z)^{-m-1}$ and then raising the exponents by m to get the expansion of $K_n^{[m]}$. The identity

$$(\Sigma a_n z^n, \Sigma b_n z^n) = \Sigma a_n \bar{b}_n$$

(for functions in H_2) then yields

$$(f, K_n^{[m]}) = f^{(m)}(z_n) .$$

The norm can be computed easily by noting that

$$\| K_n^{[m]} \|^2 = (K_n^{[m]}, K_n^{[m]}) = \left[\frac{d^m}{dz^m} K_n^{[m]}(z)\right]_{z=z_n} .$$

3. **Simultaneous interpolation.** We will prove that if $\{z_n\}$ approaches the boundary exponentially, then simultaneous weighted interpolation can be done with an H_2 function and its first M derivatives for M arbitrary.

THEOREM 1. *If $\{z_n\}$ approaches the boundary exponentially and if f is in H_2 then*

$$f^{(m)}(z_n)/\| K_n^{[m]} \|\}$$

is in l_2 for arbitrary m.

Proof. By a method similar to that used for the computation of $\| K_n^{[m]} \|$, we find that $|(K_n^{[m]}, K_p^{[m]})| = 0(|1 - \bar{z}_n z_p|^{-2m-1})$. Let $k_n^{[m]}$ denote the normalization of $K_n^{[m]}$. Since $1/|1 - \bar{z}_n z_p|$ is less than both $2\delta_n^2$ and $2\delta_p^2$ thus $|(k_n^{[m]}, k_p^{[m]})|$ is dominated by both $(\delta_n/\delta_p)^{2m+1}$ and $(\delta_p/\delta_n)^{2m+1}$ and thus by $(\delta^{2m+1})^{|n-p|}$. This, together with Schur's result, allows us to conclude that the matrix whose elements are $(k_n^{[m]}, k_p^{[m]})$ determines a bounded operator in l_2. Bari's theorem then applies to complete the proof.

THEOREM 2. *If $\{z_n\}$ approaches the boundary exponentially and if M is any nonnegative integer then, corresponding to any choice of $M + 1$ sequences $w^{[0]}, \cdots, w^{[M]}$ in l_2, there can be found an f in H_2 for which*

$$f^{(m)}(z_n) = (w^{[m]})_n \|L_n^{[m]}\| \qquad (0 \leqq m \leqq M; n = 1, 2, \cdots) .$$

Proof. The proof is by induction on M. As we've noted, the case $M = 0$ has been treated by Shapiro anc Shields. Let $M > 0$ and assume the result for lesser values. If $w^{[0]}, \cdots, w^{[M]}$ are in l_2, let f_{M-1} be a function in H_2 corresponding to $w^{[0]}, \cdots, w^{[M-1]}$. We let $B(z)$ denote the Blaschke product for $\{z_n\}$ and let $B_n(z)$ denote $B(z)$ with the factor $\bar{z}_n(z - z_n)/z_n(1 - \bar{z}_n z)$ deleted. By Theorem 1,

$$(w')_n \equiv (w^{[M]})_n - \| L_n^{[M]} \|^{-1} f_{M-1}^{(M)}(z_n)$$

determines a sequence in l_2. Then, since $\{z_n\}$ is a Carleson sequence,

$$(w'')_n \equiv \frac{(w')_n \,\|\, L_n^{[M]} \,\|\, |\, z_n\,|^M}{B_n^M(z_n)\delta_n^{2M+1}M!}$$

also determines a sequence in l_2. Again using the results of Shapiro and Shields, we can find a function φ in H_2 for which $\varphi(z_n) = (w'')_n\delta_n$. We define f_M to be $f_{M-1} + B^M\varphi$. Clearly, f_M is in H_2 and a simple computation shows that it solves our interpolation problem.

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