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THIN ABELIAN p-GROUPS

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An abelian p-group G is thin if every map from a torsion complete group to G is small. The class of thin groups is shown to be closed under arbitrary direct sums and extensions and hence any direct sum of countable reduced p-groups is thin. An example shows that, unlike for groups with no elements of infinite height, a reduced group may contain no unbounded torsion complete subgroups and still fail to be thin. Finally, these groups are used to settle questions in a certain relative homological algebra.

This investigation was motivated by the study of the relative homological algebra induced in the category of abelian p-groups (or abelian groups) by declaring the torsion complete p-groups to be projective. An exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is proper in this setting if every map from a torsion complete p-group G to C lifts to a map from G to B. In particular, since cyclic p-groups are torsion complete, the sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ must be p-pure, i.e., $p^n B \cap A = p^n A$ for every positive integer n. This condition alone enables us to lift any map from G to C which factors through a direct sum of cyclic p-groups. Thus if every map from a torsion complete p-group to C factors through a direct sum of cyclic groups the notions of p-pure and proper coincide.

Pierce [5] called a homomorphism ϕ from a p-group G *small* if the kernel of ϕ contained a large subgroup. A *large subgroup* is a subgroup of the form $\sum p^{n_k} G[p^k]$ where $\{n_k\}$ is an increasing sequence of nonnegative integers. If L is a large subgroup then G/L is a direct sum of cyclics. (This is clear for G a direct sum of cyclics. If B is a basic subgroup of G then $L \cap B$ is large in B, since B is pure, and $B + L = G$ by the Baer decomposition. Hence $G/L = (B + L)/L \cong B/L \cap B$ is a direct sum of cyclics[1].) Conversely, Megibben [3] has shown that any map from a torsion complete p-group to a direct sum of cyclics is small. It is therefore of some interest to study p-groups to which any homomorphism from a torsion complete group is small.

There is an analogy here with the notion of a slender torsion free group as defined by Łoś. A group G is *slender* if any map from the direct product of countably many infinite cyclic groups annihilates cofinitely many of these cyclic groups. The idea in both cases is to examine groups for which any homomorphism from a fixed, rather

[1] I am indebted to Doyle Cutler for this important but little publicized observation.

tightly held together group must have a rather large kernel (a fixed unbounded torsion complete group may be selected, instead of considering the whole class). In either case the point of the study is perhaps more to understand the structure of the fixed group than the class to which it gives rise.

All groups considered in this paper will be abelian.

2. Thin groups.

We formally introduce the main characters:

DEFINITION. A p-group G is *then* if every map from a torsion complete group to G is small.

Our class is clearly closed under subgroups. It is also closed under extensions and direct sums.

THEOREM 1. *If H and G/H are thin then so is G.*

Proof. Let T be torsion complete and ϕ: $T \rightarrow G$. Then $T \rightarrow G \rightarrow G/H$ is small so for every k there exists an n_k such that $\phi(L) \subseteqq H$ where $L = \sum p^{n_k} T[p^k]$. Now L is torsion complete. Indeed, since T/L is a direct sum of cyclics we have the exact sequence

$$0 \rightarrow \operatorname{Ext}(Z_{p^\infty}, L) \rightarrow \operatorname{Ext}(Z_{p^\infty}, T)$$

and since $p^\omega \operatorname{Ext}(Z_{p^\infty}, T) = \operatorname{Pext}(Z_{p^\infty}, T) = 0$ we have

$$\operatorname{Pext}(Z_{p^\infty}, L) = p^\omega \operatorname{Ext}(Z_{p^\infty}, L) = 0$$

and so L is torsion complete. Thus ϕ restricted to L is small. Hence ϕ is small.

THEOREM 2. *If $G = \sum G_\alpha$ and G_α is thin for all α then G is thin.*

Proof. Let T be torsion complete and $\phi : T \rightarrow G$. If ϕ is not small then there exist $x_1, x_2, x_3 \cdots \in T[p^n]$, $ht(x_i) \geqq i$, such that $\phi(x_i) \neq 0$. Since $\pi_\alpha \circ \phi$ is small, where π_α is the projection $G \rightarrow G_\alpha$, the x_j may be chosen such that the $\phi(x_j)$ have disjoint supports. Relabeling (and applying Theorem 1) we may assume that $\phi(x_i) \in G_i$. Let π_i be the projection on G_i. Since $\pi_i \varphi$ is small there is an m such that $\pi_i \varphi(x_{m+1} + x_{m+2} + \cdots) = 0$. Hence

$$\pi_i \varphi(x) = \pi_i \varphi(x_1 + \cdots + x_m) = \varphi(x_i) \neq 0 \ .$$

This is a contradiction since $\varphi(x)$ can have only finitely many nonzero coordinates.

The following is the analogue of the theorem that a direct sum of countable reduced torsion free groups is slender.

COROLLARY 1. *Any direct sum of countable reduced p-groups is thin.*

Proof. Clearly Z_p is thin. Thus any group constructable from Z_p by a sequence of taking direct sums, subgroups and extensions is also thin. In particular, direct sums of countable reduced groups are thin.

A p-group is *fully starred* if no subgroup admits a dense subset of smaller cardinality (in the p-adic topology on the subgroup). Fully starred groups are closed under subgroups, direct sums [4] and extensions [1]. Since Z_p is fully starred so is any direct sum of countable reduced p-groups. We may extend Corollary 1 to:

COROLLARY 2. *Any fully starred p-group is thin.*

Proof. Let G be fully starred and ϕ a map from a torsion complete group $\bar{B}$ to G. If ϕ is not small there exists a sequence $x_1, x_2, x_3, \cdots \in \bar{B}[p^n]$ such that $ht(x_i) \to \infty$ and $\phi(x_i) \neq 0$, $i = 1, 2, 3, \cdots$. Let H be a countable pure subgroup of $\bar{B}$ containing each x_i. Then the closure, $\bar{H}$, of H in $\bar{B}$ is a torsion complete group and ϕ restricted to $\bar{H}$ is not small. Now $\phi(\bar{H})$ is not countable by Corollary 1 (a fully starred group must be reduced since 0 is dense in Z_{p^∞}). But $\phi(H)$ is countable and dense in $\phi(\bar{H})$, which is impossible since G is fully starred.

3. A counterexample. If G contains an unbounded torsion complete subgroup then G is certainly not thin. Megibben showed that the converse is true for groups without elements of infinite height. In this section we show that this restriction cannot be dropped.

We shall be looking at the socle of a group G with no elements of infinite height. This socle inherits the p-adic topology from G. All topological terminology refers to this topology.

LEMMA 1. *Let G be a p-group with no elements of infinite height and H a torsion complete subgroup of G. Then H contains a nondiscrete complete subgroup of $G[p]$.*

Proof. Choose $x_1, x_2, x_3, \cdots \in H[p]$ such that $ht_H(x_{i+1}) > ht_G(x_i)$. Then $\sum n_i x_i$ converges (in H or G) and $ht(\sum n_i x_i) = ht(x_j)$ where $j = \min\{i \mid p \nmid n_i\}$. Then $S = \{\sum n_i x_i \mid n_i \in Z\}$ is a nondiscrete complete subgroup of $G[p]$.

LEMMA 2. *If H is an extension of a bounded p-group by a torsion complete p-group and $p^{\omega}H = 0$ then H is torsion complete.*

Proof. Let $0 \to S \to H \to \bar{B} \to 0$ be exact with S bounded and $\bar{B}$ torsion complete. Then

$$0 \to \operatorname{Ext}(Z_{p^\infty}, S) \to \operatorname{Ext}(Z_{p^\infty}, H) \to \operatorname{Ext}(Z_{p^\infty}, \bar{B}) \to 0$$

is exact. Now $\operatorname{Pext}(Z_{p^\infty}, \bar{B}) = 0$ so $\operatorname{Ext}(Z_{p^\infty}, \bar{B})$ has no elements of infinite height. Hence $\operatorname{Pext}(Z_{p^\infty}, H) \subseteqq \operatorname{Ext}(Z_{p^\infty}, S)$ is bounded. But H is the torsion subgroup of $\operatorname{Ext}(Z_{p^\infty}, H)$ and H has no elements of infinite height. Hence $\operatorname{Pext}(Z_{p^\infty}, H) = 0$ and so H is torsion complete.

THEOREM 3. *Let G be torsion complete, ϕ a map from G onto K such that $\ker \phi$ is bounded. If K contains an unbounded torsion complete subgroup $\bar{B}$ then* $\ker \phi$ *either contains or misses a nondiscrete complete subgroup of $G[p]$.*

Proof. Let $H = \phi^{-1}(\bar{B})$. Then H is torsion complete by Lemma 2. Let $S = \ker \phi \cap H[p]$. Since $\ker \phi$ and $H[p]$ are closed in H, so is S. Let H_1 be a pure subgroup of H such that $H_1[p] = S$ (see [2], Lemma 1). Then H_1 is torsion complete ([2], Lemma 2) and so $H = H_1 \oplus H_2$. Lemma 1 says that if H_1 is unbounded then S, and hence $\ker \phi$, contains a nondiscrete complete subgroup of $G[p]$ whereas if H_2 is unbounded then S, and hence $\ker \phi$, misses such a subgroup of $G[p]$.

To find our example we need only find a subgroup of the socle of an unbounded torsion complete group which neither misses nor contains a nondiscrete complete subsocle.

THEOREM 4. *Let G be a torsion complete group with a countable unbounded basic subgroup. Then there exists a subgroup $S \subseteqq G[p]$ such that if $K \subseteqq G[p]$ is nondiscrete complete then $0 \neq S \cap K \neq K$.*

Proof. Since G has a countable basic there are $2^{\aleph_0}$ complete nondiscrete subgroups of $G[p]$. Well order these subgroups, K_α, by the ordinals $\alpha < 2^{\aleph_0}$. Construct subgroups S_α and T_α of $G[p]$ inductively by:

(1) $S_0 = 0 = T_0$.

(2) $S_{\alpha+1} = S_\alpha + \langle s_\alpha \rangle$, $T_{\alpha+1} = T_\alpha + \langle t_{\alpha+1} \rangle$

where $s_\alpha \in K_\alpha \backslash (S_\alpha + T_\alpha)$ and $t_\alpha \in K_\alpha \backslash (S_{\alpha+1} + T_\alpha)$.

(3) $S_\beta = \bigcup_{\alpha<\beta} S_\alpha$, $T_\beta = \bigcup_{\alpha<\beta} T_\alpha$ at limit ordinals β.

The existence of s_α and t_α as required in the second step is guaranteed since S_α and T_α have cardinality less than K_α. Set $S = \cup S_\alpha$ and $T = \cup T_\alpha$. Then $S \cap T = 0$ but every nondiscrete complete subsocle K_α has nontrivial intersection with both S and T.

COROLLARY. *There exists a reduced p-group A such that A contains no unbounded torsion complete subgroup but A is not thin.*

Proof. Let $A = G/S$ where G and S are as in Theorem 4. Then Theorem 3 implies that A contains no unbounded torsion complete subgroup whereas the map $G \to A$ is certainly not small.

4. An application. We return to the relative homological algebra induced on the category of p-groups (or all groups) by declaring torsion complete p-groups to be projective. It turns out that the thin p-groups enable us to characterize the injectives in this algebra without requiring a deep understanding of what "proper" means.

LEMMA 3. *If $G[p]$ is complete then $p^\omega G$ is p-divisible.*

Proof. Let $x \in p^\omega G$ and $p^{i+1}y_i = x$ for $i = 0, 1, 2, \cdots$. Then $py_0 = p^{i+1}y_i$ and so $y_0 - p^i y_i \in G[p]$. Since $y_0 - p^i y_i$ is a Cauchy sequence in $G[p]$ it converges to some element $z \in G[p]$. Since $p^i y_i$ converges to 0 we have y_0 converges to z, i.e., $y_0 = z + g$ where $g \in p^\omega G$. Thus $x = py_0 = pg$ where $g \in p^\omega G$.

COROLLARY. *A p-group G is the direct sum of a torsion complete group and a divisible group if and only if $G[p]$ is complete.*

Proof. The "only if" is clear. The "if" part follows from the lemma, which implies that $G = R \oplus D$ where R has no elements of infinite height and D is divisible, and Lemma 2 of [2] which says that R is torsion complete.

THEOREM 5. *A reduced p-group G is injective in the relative homological algebra induced in the category of p-groups by declaring the torsion complete groups projective $\Leftrightarrow$ G is torsion complete.*

Proof. $\Leftarrow$ is well known. The torsion complete groups are pure injective in this category.

$\Rightarrow$ We need only show that $G[p]$ is complete. Let $B \xrightarrow{\phi} C$ be a pure map of a direct sum of cyclic p-groups onto a reduced countable group C such that $C^1 \neq 0$. The kernel, K, of ϕ is a pure direct sum of cyclics. Now $K[p]$ is not closed in B since $C^1 \neq 0$. Let $z \in B[p]\backslash K$ be in the closure of $K[p]$ in B. Let $K = K_0 \oplus K_1 \oplus \cdots$ where K_1 is a direct sum of cyclic groups of order p^{i+1}. Then there exist integers

$0 \leqq n_1 < n_2 < \cdots$ and elements $0 \neq z_i \in K_{n_i}[p]$ such that

$$ht(z - z_1 - z_2 - \cdots - z_i) > p^{n_i} .$$

Choose $x_i \in K_{n_i}$ such that $p^{n_i} x_i = z_i$. Then $\sum \langle x_i \rangle$ is direct and is a summand of K, $o(x_i) = p^{n_i + 1}$ and $\sum p^{n_i} x_i$ converges to z.

Let $\sum g_i$ be a Cauchy series in $G[p]$, i.e., $ht(g_i) \rightarrow \infty$. We must show that $\sum g_i$ converges to an element of $G[p]$. We may assume that $ht(g_i) \geqq n_i$. Thus we can map $K \rightarrow G$ taking $p^{n_i} x_i$ to g_i. This map extends to B since every map of a torsion complete group into C is small and hence lifts to B. The image of z is then a limit of the Cauchy series $\sum g_i$.

Similarly we may characterize the relative injectives in the category of all groups.

THEOREM 6. *A reduced group G is injective in the relative homological algebra induced on the category of groups by declaring the torsion complete p-groups projective $\Leftrightarrow$ G is a complete Q_p-module, where Q_p is the ring of integers localized at p.*

Proof. $\Leftarrow$ is well known. These are the reduced p-pure injectives.

$\Rightarrow$ That G is a Q_p module follows from the fact that $Z \subseteqq Q_p$ is proper in this setting (since the quotient is p-torsion free) and the only reduced images of Q_p are cyclic Q_p-modules. By Theorem 5 the torsion subgroup of G is torsion complete and hence $p^\omega G$ is divisible by Lemma 3. Hence G has no elements of infinite height. Finally, to show that G is complete consider $\sum_{i=1}^{\infty} Q_p$ and the pure subgroup generated by $(1, -p, 0, 0, \cdots)$, $(0, 1, -p, 0, \cdots)$, $(0, 0, 1, -p, 0, \cdots)$ etc. If $\sum g_i$ is a Cauchy series in G, $ht(g_i) \geqq i$, then $g_i = p^i h_i$ and we can map $(0, 0, \cdots, 1, -p, \cdots)$ onto h_i and extend to $\sum_{i=1}^{\infty} Q_p$. Then $\sum g_i$ converges to the image of $(1, 0, 0, \cdots)$.

5. An aside. One of the questions which arises in the study of any relative homological algebra is whether a proper (i.e., "pure") subgroup of a projective is again projective. For the relative homological algebra induced in the category of p-groups by declaring torsion complete groups to be projective, the projectives are direct sums of torsion complete and divisible groups (see [6]). Just what a proper subgroup "is" in this setting is still a little mysterious. The only nontrivial ones which have been exhibited thus far are (ordinary) pure subgroups with thin cokernels. These kinds of subgroups of projectives are again projective. It suffices to consider only direct sums of torsion complete groups. We first prove a pair of very general lemmas.

LEMMA 4. *If K is a p-group, the map $\phi\colon G \rightarrow K$ is pure and onto* $\Leftrightarrow \phi(p^nG[p]) = p^nK[p]$ *for every nonnegative integer n.*

Proof. $\Rightarrow$ If $x \in p^nK[p]$ then $x = p^ny$. We can find $g \in G$ of the same order as y such that $\phi(g) = y$. Then $p^ng \in p^nG[p]$ and $\phi(p^ng) = x$.
$\Leftarrow$ Suppose $y \in K$ and $p^{n+1}y = 0$. Then $p^ny \in p^nK[p]$ and so $p^ny = \phi(p^ng)$ where $p^ng \in G[p]$. Now $p^n(\phi(g) - y) = 0$ and so, by induction, $\phi(g) - y = \phi(g_0)$ where $g_0 \in G$ and $p^ng_0 = 0$. Thus $y = \phi(g - g_0)$ and $p^{n+1}(g - g_0) = 0$.

LEMMA 5. *If f_1 and f_2 are maps from G_1 and G_2 to H, and ϕ is a map from G_1 to G_2 such that $f_2\phi = f_1$ then the kernel of $f_1 \oplus f_2 : G_1 \oplus G_2 \rightarrow H$ is $A \oplus \ker f_2$ where $A \cong G_1$.*

Proof. Let $A = \{g_1 - \phi(g_1) \mid g_1 \in G_1\}$. Then $A \subseteqq \ker(f_1 \oplus f_2)$. The map taking g_1 to $g_1 - \phi(g_1)$ is an isomorphism from G_1 to A. Since $A \cap G_2 = 0$ we have $A \cap \ker f_2 = 0$. It remains to show that $\ker(f_1 \oplus f_2) \subseteqq A + \ker f_2$. Suppose $f_1(g_1) + f_2(g_2) = 0$. Then $f_2(\phi(g_1) + g_2) = 0$ so $\phi(g_1) + g_2 \in \ker f_2$. But $g_1 + g_2 = g_1 - \phi(g_1) + \phi(g_1) + g_2$.

THEOREM 7. *Let G be a direct sum of torsion complete p-groups and K a thin p-group. If ϕ is a pure map from G onto K then the kernel of ϕ is a direct sum of torsion complete groups.*

Proof. Write $G = \sum T_i$ where T_i is torsion complete. Since ϕ induces a small map on T_i there exists an n_i such that $\phi(p^{n_i}T_i[p]) = 0$. Letting B_i be a maximal p^{n_i} bounded summand of T_i we have $T_i = A_i \oplus B_i$ where B_i is bounded and $\phi(A_i[p]) = 0$. Since $\phi(\sum A_i[p]) = 0$ and ϕ is pure onto K we have, by Lemma 4, $\phi(p^n \sum B_i[p]) = p^nK[p]$ for every $n \geqq 0$. Hence, by Lemma 4 again, ϕ restricted to $\sum B_i$ is pure onto K. Since ϕ restricted to $\sum A_i$ factors through a direct sum of cyclics (ϕ is small on A_i), and $\phi \mid \sum B_i$ is pure onto K, we can factor $\phi \mid \sum A_i$ through $\phi \mid \sum B_i$. Lemma 5 then implies that $\ker \phi \cong \sum A_i \oplus C$ where C is a subgroup of $\sum B_i$ and is therefore a direct sum of cyclic groups.

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