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SEQUENCES OF CONTRACTIONS AND FIXED POINTS

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Given a convergent sequence of contraction mappings, the convergence of the sequence of their fixed points is investigated in § 1 of this paper. The results obtained lead to a necessary and sufficient condition in order that a separable or a reflexive Banach space be finite dimensional. An application to differential equations is also included.

In § 2 we consider mappings defined on the cartesian product of two metric spaces which are contraction mappings in one variable or in each variable separately. Using some of the results of § 1 we prove that, with certain restrictions, such mappings have fixed point.

Let (X, d) be a metric space. A function $A: X \rightarrow X$ is said to be a *contraction mapping* if and only if there is a real number α, $0 \leqq \alpha < 1$, such that $d(A(x), A(y)) \leqq \alpha d(x, y)$ for all $x, y \in X$. The contraction mapping principle of Banach guarantees a unique fixed point of each contraction mapping of a complete metric space into itself. A natural question to ask is the following: In a complete metric space does the convergence of a sequence of contraction mappings to a contraction mapping A_0 imply the convergence of the sequence of their fixed points to the fixed point of A_0? A partial answer to this question appears as Theorem 1.2 of [1, p. 6] ("Let E be a complete metric space, and let T and $T_n (n = 1, 2, \cdots)$ be contraction mappings of E into itself with the same Lipschitz constant $K < 1$, and with fixed points u and u_n respectively. Suppose that $\lim_{n\to\infty} T_n(x) = T(x)$ for every $x \in E$. Then $\lim_{n\to\infty} u_n = u$.") The restriction in this theorem, that all the contraction mappings have the "same Lipschitz constant $K < 1$", is very strong; for one can easily construct a sequence of contraction mappings from the reals into the reals which converges uniformly to the zero mapping but whose Lipschitz constants tend to one.

In the next section two types of convergence of contraction mappings are considered uniform convergence and pointwise convergence. The question posed above is answered affirmatively in the case of uniform convergence and in the case of pointwise convergence on locally compact spaces.

1. Continuity of fixed points.

We prove the following two main theorems:

THEOREM 1. *Let (X, d) be a metric space, let $A_i: X \rightarrow X$ be a* *function with at least one fixed point a_i for each $i = 1, 2, \cdots$, and let $A_0\colon X \to X$ be a contraction mapping with fixed point a_0. If the sequence $\{A_i\}_{i=1}^\infty$ converges uniformly to A_0, then the sequence $\{a_i\}_{i=1}^\infty$ converges to a_0.*

THEOREM 2. *Let (X, d) be a locally compact metric space, let $A_i\colon X \to X$ be a contraction mapping with fixed point a_i for each $i = 1, 2, \cdots$, and let $A_0\colon X \to X$ be a contraction mapping with fixed point a_0. If the sequence $\{A_i\}_{i=1}^\infty$ converges pointwise to A_0, then the sequence $\{a_i\}_{i=1}^\infty$ converges to a_0.*

Proof of Theorem 1. Let $\varepsilon > 0$ and choose a natural number N such that $i \geqq N$ implies $d(A_i(x), A_0(x)) < \varepsilon(1 - \alpha_0)$ for all $x \in X$, where $\alpha_0 < 1$ is a Lipschitz constant for A_0. Then, if

$$\begin{aligned} i \geqq N,\ d(a_i, a_0) = d(A_i(a_i), A_0(a_0)) &\leqq d(A_i(a_i), A_0(a_i)) \\ &+ d(A_0(a_i), A_0(a_0)) < \varepsilon(1 - \alpha_0) + \alpha_0 d(a_i, a_0)\ . \end{aligned}$$

Hence, $d(a_i, a_0) < \varepsilon$ for all $i \geqq N$. This proves that $\{a_i\}_{i=1}^\infty$ converges to a_0 and completes the proof of Theorem 1.

Proof of Theorem 2. Let $\varepsilon > 0$ and assume ε is sufficiently small so that $K(a_0, \varepsilon) = \{x \in X \mid d(a_0, x) \leqq \varepsilon\}$ is a compact subset of X. Then, since $\{A_i\}_{i=1}^\infty$ is an equicontinuous sequence of functions converging pointwise to A_0 and since $K(a_0, \varepsilon)$ is compact, the sequence $\{A_i\}_{i=1}^\infty$ converges uniformly on $K(a_0, \varepsilon)$ to A_0. Choose N such that if $i \geqq N$, then $d(A_i(x), A_0(x)) < (1 - \alpha_0)\varepsilon$ for all $x \in K(a_0, \varepsilon)$, where $\alpha_0 < 1$ is a Lipschitz constant for A_0. Then, if $i \geqq N$ and $x \in K(a_0, \varepsilon)$, $d(A_i(x), a_0) \leqq d(A_i(x), A_0(x)) + d(A_0(x), A_0(a_0)) < (1 - \alpha_0)\varepsilon + \alpha_0 d(x, a_0) \leqq (1 - \alpha_0)\varepsilon + \alpha_0\varepsilon = \varepsilon$. This proves that if $i \geqq N$, then A_i maps $K(a_0, \varepsilon)$ into itself. Letting B_i be the restriction of A_i to $K(a_0, \varepsilon)$ for each $i \geqq N$ we see that each B_i is a contraction mapping of $K(a_0, \varepsilon)$ into itself. Since $K(a_0, \varepsilon)$ is a complete metric space, B_i has a fixed point for each $i \geqq N$ which must, from the definition of B_i and the fact that A_i has only one fixed point, be a_i. Hence, $a_i \in K(a_0, \varepsilon)$ for each $i \geqq N$. It follows that the sequence $\{a_i\}_{i=1}^\infty$ of fixed points converges to a_0. This completes the proof of Theorem 2.

We now give an example which shows that, in non-locally compact spaces, a sequence of contraction mappings may converge *pointwise* to a contraction mapping without the sequence of their fixed points converging. In fact, the example is a construction in any infinite dimensional separable or reflexive Banach space of a sequence of contraction mappings which converges pointwise to the zero mapping but such that the sequence of their fixed points has no convergent subsequence. The following lemma will be useful in this example.

LEMMA. *Let (X, d) be a metric space, let A_i: $X \to X$ be a contraction mapping with fixed point a_i for each $i = 1, 2, \cdots$, and let A_0: $X \to X$ be a contraction mapping with fixed point a_0. If the sequence $\{A_i\}_{i=1}^{\infty}$ converges pointwise to A_0 and if a subsequence $\{a_{i_j}\}_{j=1}^{\infty}$ of $\{a_i\}_{i=1}^{\infty}$ converges to a point $x_0 \in X$, then $x_0 = a_0$.*

Proof. Let $\varepsilon > 0$. Then there is a positive integer N such that $j \geqq N$ implies $d(a_{i_j}, x_0) < (\varepsilon/2)$ and $d(A_{i_j}(x_0), A_0(x_0)) < (\varepsilon/2)$. Therefore, $d(a_{i_j}, A_0(x_0)) = d(A_{i_j}(a_{i_j}), A_0(x_0)) \leqq d(A_{i_j}(a_{i_j}), A_{i_j}(x_0)) + d(A_{i_j}(x_0), A_0(x_0)) < d(a_{i_j}, x_0) + d(A_{i_j}(x_0), A_0(x_0)) < \varepsilon$ for all $j \geqq N$. This proves that the sequence $\{a_{i_j}\}_{j=1}^{\infty}$ converges to $A_0(x_0)$. Hence, $A_0(x_0) = x_0$ and it follows that $x_0 = a_0$.

EXAMPLE 1. Let B be an infinite dimensional separable or reflexive Banach space. Let B^* be the first conjugate of B and let $T = \{f \in B^* \mid \| f \| \leqq 1\}$. Then T is weak* sequentially compact. In the separable case this follows from the metrizability of the weak* topology for T[2, p. 426]; in the reflexive case the weak compactness of T implies, by the Eberlein-Smulian Theorem [2, p. 430], that T is weakly sequentially compact and, therefore, weak* sequentially compact. Since B is infinite dimensional, there is a sequence $\{g_k\}_{k=1}^{\infty}$ of linear functionals in T which has no norm convergent subsequence. Let $\{g_{k_i}\}_{i=1}^{\infty}$ be a weak* convergent subsequence of $\{g_k\}_{k=1}^{\infty}$ and let g be the weak* limit of $\{g_{k_i}\}_{i=1}^{\infty}$. For each $i = 1, 2, \cdots$ let

$$f_i = \frac{g_{k_i} - g}{\| g_{k_i} - g \|} .$$

The sequence $\{f_i\}_{i=1}^{\infty}$ is weak* convergent to the zero linear functional and $\| f_i \| = 1$ for all $i = 1, 2, \cdots$.

For each $i = 1, 2, \cdots$, let $a_i \in B$ such that $\| a_i \| = 1$ and $| f_i(a_i) | > 1 - (1/i^2)$. For each $i = 1, 2, \cdots$, define A_i: $B \to B$ by

$$A_i(x) = \left(1 - \frac{1}{i}\right) \frac{f_i(x)}{f_i(a_i)} a_i + \frac{1}{i} a_i$$

for all $x \in B$. Since

$$\| A_i(x) - A_i(y) \| = \left\| \left(1 - \frac{1}{i}\right) \frac{f_i(x)}{f_i(a_i)} a_i - \left(1 - \frac{1}{i}\right) \frac{f_i(y)}{f_i(a_i)} a_i \right\| = \left(1 - \frac{1}{i}\right) \frac{| f_i(x) - f_i(y) |}{| f_i(a_i) |} \| a_i \|$$

$$\leqq \left(1 - \frac{1}{i}\right) \frac{\| x - y \|}{| f_i(a_i) |} \leqq \frac{i}{1 + i} \| x - y \|$$

for all x and y in B, A_i is a contraction mapping for each $i = 1, 2, \cdots$. Since the sequence $\{f_i\}_{i=1}^{\infty}$ is weak* convergent to the zero linear functional and the sequence $\{f_i(a_i)\}_{i=1}^{\infty}$ is bounded away from zero, it follows that the sequence $\{A_i\}_{i=1}^{\infty}$ converges pointwise to the zero mapping. It is easy to verify that, for each $i = 1, 2, \cdots$, a_i is the unique fixed point of A_i. Since $\| a_i \| = 1$ for all $i = 1, 2, \cdots$, it follows from the lemma that the sequence $\{a_i\}_{i=1}^{\infty}$ of fixed points has no convergent subsequence. Hence, $\{A_i\}_{i=1}^{\infty}$ is a sequence of contraction mappings which converges pointwise to the zero mapping and such that the sequence $\{a_i\}_{i=1}^{\infty}$ of fixed points has no convergent subsequence. This example may be slightly modified so that the sequence of fixed points is unbounded [5]. Also it is clear that this type of construction may be done in any infinite dimensional Banach space in which the unit ball of the first conjugate is weak* sequentially compact.

The next theorem is a characterization of those separable Banach spaces which are finite dimensional in terms of pointwise convergent sequences of contraction mappings and the convergence of their fixed points.

THEOREM 3. *A separable or reflexive Banach space B is finite dimensional if and only if whenever a sequence of contraction mappings of B into B converges pointwise to a contraction mapping A_0, then the sequence of their fixed points converges to the fixed point of A_0.*

Proof. Since a finite dimensional Banach space is locally compact, half of the theorem follows from Theorem 2. The proof of the other half of the theorem is obtained by supposing B is not finite dimensional and applying Example 1.

It is not known by the author whether or not the statement of Theorem 3 remains valid if the condition, "separable or reflexive," is removed.

As an application of Theorem 2 we give the following proposition due to Professor J. R. Dorroh.

PROPOSITION. Let D be an open subset of the plane, let $(a, b) \in D$, let $M > 0$ be a real number, and let $\{K_i\}_{i=0}^{\infty}$ be a bounded sequence of strictly positive real numbrs. For each $i = 0, 1, 2, \cdots$, let f_i be a real valued continuous function defined on D such that $| f_i(x, y) | \leqq M$ for all $(x, y) \in D$ and $| f_i(x, y) - f_i(x, z) | \leqq K_i | y - z |$ for all $(x, y), (x, z) \in D$.

Suppose also that the sequence $\{f_i\}_{i=1}^{\infty}$ converges pointwise on D to f_0. Let h be such that $0 < k_i \cdot h < 1$ for all $i = 0, 1, 2, \cdots$, and such that the set $W = \{(x, y) \mid |x - a| \leqq h \text{ and } |y - b| \leqq M|x - a|\}$ is a subset of D. Then the sequence $\{y_i\}_{i=1}^{\infty}$ converges uniformly on $I = [a - h, a + h]$ to y_0 where, for each $i = 0, 1, 2, \cdots$, y_i is the unique solution on I of the initial value problem

$$y(a) = b$$
$$y'(x) = f_i(x, y(x)) .$$

Proof. Let X be the set of all real valued functions defined on I with graph lying in W and with Lipschitz constant less than or equal to M. Then X, with the supremum metric ρ, is a compact metric space. For each $i = 0, 1, 2, \cdots$ and each $g \in X$, define $A_i(g)$ at each $x \in I$ by $[A_i(g)](x) = b + \int_a^x f_i(t, g(t))dt$. It is easy to verify that, for each $i = 0, 1, 2, \cdots$, A_i is a contraction mapping from X into X with Lipschitz constant less than or equal to $K_i \cdot h$. For each $g \in X$, $x \in I$, and $i = 1, 2, \cdots$,

$$[A_i(g)](x) - [A_0(g)](x) = \int_a^x [f_i(t, g(t)) - f_0(t, g(t))]dt .$$

Since the sequence of integrands converges pointwise to zero and is uniformly bounded by $2M$, the Lebesgue bounded convergence theorem guarantees that the sequence of integrals goes to zero as i tends to infinity. Therefore, the sequence $\{A_i(g)\}_{i=1}^{\infty}$ converges pointwise on I to $A_0(g)$. This implies, by the equicontinuity of $\{A_i(g)\}_{i=1}^{\infty}$ on the compact set I, that the sequence $\{A_i(g)\}_{i=1}^{\infty}$ converges uniformly on I to $A_0(g)$. Hence, the sequence $\{A_i\}_{i=1}^{\infty}$ converges pointwise on X to A_0. By Theorem 2 the sequence $\{y_i\}_{i=1}^{\infty}$, where y_i is the unique fixed point of A_i for each $i = 1, 2, \cdots$, converges to the fixed point y_0 of A_0. The result follows since these fixed points are the unique solutions of the initial value problem.

The restriction in this proposition that each of the mappings f_1, $f_2, \cdots$ satisfy the type of Lipschitz condition given above can be significantly weakened. This and related matters will be considered for a later article.

2. A fixed point theorem for product spaces.

A number of mathematicians have investigated the problem of determining what kinds of mappings defined on the cartesian product of two spaces have fixed points (for an historical survey see [6]). In 1930, K. Kuratowski asked [3] if the cartesian product of two Peano continua, each with the fixed point property, had the fixed point property. Recently, W.

Lopez [4] gave an example of a finite polyhedron with the fixed point property whose cartesian product with the unit interval failed to have the fixed point property.

Throughout this section (X, d_X) and (Y, d_Y) will be metric spaces and $(X \times Y, d)$ will denote their cartesian product with the product metric d given by $d((x_1, y_1), (x_2, y_2)) = [(d_X(x_1, x_2))^2 + (d_Y(y_1, y_2))^2]^{1/2}$ for all $(x_1, y_1), (x_2, y_2) \in X \times Y$. A function $f: X \times Y \to X \times Y$ is said to be a *contraction mapping in the first variable* if and only if for each $y \in Y$ there is a real number $\alpha(y)$, $0 \leqq \alpha(y) < 1$, such that $d(f(x_1, y), f(x_2, y)) \leqq \alpha(y) d((x_1, y), (x_2, y))$ for all $x_1, x_2 \in x$. We define a *contraction mapping in the second variable* in an analogous fashion and we say that a function is a *contraction mapping in each variable separately* provided it is a contraction mapping in the first variable and in the second variable.

It is worthwhile noting that, even if (X, d_X) and (Y, d_Y) are compact, there may be mappings from $X \times Y$ into $X \times Y$ which are contraction mappings in each variable separately but which are not themselves contraction mappings. The function $f: [0,1] \times [0,1] \to [0,1] \times [0,1]$ given by $f(x, y) = \{(x + y)/2, (x + y)/2\}$ for all $(x, y) \in [0,1] \times [0,1]$ is an example of such a mapping.

THEOREM 4. *Let (X, d_X) be a complete metric space, let (Y, d_Y) be a metric space with the fixed point property, and let f be a function from $X \times Y$ into $X \times Y$.*

(1) *If f is uniformly continuous on $X \times Y$ and a contraction mapping in the first variable, then f has a fixed point.*

(2) *If (X, d_X) is locally compact and f is continuous on $X \times Y$ and a contraction mapping in the first variable, then f has a fixed point.*

Proof. We prove (1) and (2) simultaneously. If $y \in Y$, then let $f_y: X \to X$ be defined by $f_y(x) = \pi_1 \circ f(x, y)$ for all $x \in X$ where π_1 is the natural projection of $X \times Y$ onto X. For each $y \in Y$, f_y is a contraction mapping of X into X and, therefore, has one and only one fixed point. Let $F: Y \to X$ be given by $F(y)$ is the unique fixed point of f_y. Now let $y_0 \in Y$ and let $\{y_i\}_{i=1}^{\infty}$ be a sequence of points of Y which converges to y_0. Under the assumption of 1, the sequence $\{f_{y_i}\}_{i=1}^{\infty}$ converges uniformly to f_{y_0} and hence, by Theorem 1, the sequence $\{F(y_i)\}_{i=1}^{\infty}$ converges to $F(y_0)$. Under the assumptions of 2, we may apply Theorem 2 to conclude that the sequence $\{F(y_i)\}_{i=1}^{\infty}$ converges to $F(y_0)$. Hence, in either case, this proves that F is continuous on Y. Next let $G: Y \to Y$ be the continuous function defined by $G(y) = \pi_2 \circ f(F(y), y)$ for each $y \in Y$ where π_2 is the natural projection of $X \times Y$ onto Y. Since Y has the fixed point property, there is a point $p \in Y$ such that $G(p) = p$. It follows that $(F(p), p)$ is a fixed point of f, which proves Theorem 4.

A special class of functions satisfying the conditions of (1) are Lipschitz functions which are contraction mappings in the first variable. Theorem 4, therefore, gives a class of Lipschitz mappings of $X \times Y$ into $X \times Y$ which have fixed points. Along these lines we have the following:

COROLLARY. *Let* (X, d_X) *be a complete metric space and let* (Y, d_Y) *be a metric space with the fixed point property. If* $f: X \times Y \to X \times Y$ *is a contraction mapping in each variable separately, then* f *has a fixed point.*

It may seem that the type of restriction placed on f in the corollary above would enable us to replace the condition "(Y, d_Y) has the fixed point property" by the condition "(Y, d_Y) is complete." However, the function $f: R^1 \times R^1 \to R^1 \times R^1$ (where R^1 denotes the real numbers) defined by $f(x, y) = \{(x + y)/2 + 1, (x + y)/2 + 1)\}$ for all $(x, y) \in R^1 \times R^1$ shows that this is not the case. For it is easy to see that f is a contraction mapping in each variable separately and has no fixed point.

REMARK: It is clear from the proof of Theorem 4 that less restrictive topological conditions could have been assumed about Y (for example, in part 2, Y need only be first axiom with the fixed point property).

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