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A NEW PROOF OF THE MAXIMUM PRINCIPLE FOR DOUBLY-HARMONIC FUNCTIONS

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Let f be a real-valued Lebesque integrable function on a domain Ω in Euclidean space E_{2m}, and let f be doubly-harmonic on Ω so that it satisfies

$$\frac{\partial^2 f}{\partial x_{2k-1}^2}+\frac{\partial^2 f}{\partial x_{2k}^2}=0 \qquad \text{for } k=1,2,\cdots,m\,.$$

In this paper, a new proof of the maximum principle is given for nonconstant functions f satisfying the preceding conditions.

The proof depends on the fact that the associated forms

$$\varphi_p(H;f)=\sum_{r_1+\cdots+r_n=p}\frac{h_1^{r_1}\cdots h_n^{r_n}}{r_1!\cdots r_n!}\left(\frac{\partial^p f}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}}\right)_{x=A},$$

where $A\in\Omega$, are either indefinite or identically 0 for each $p\geqq 1$. The authors previously proved this under weaker hypotheses on f, but the proof used the strong form of the maximum principle for solutions of linear elliptic partial differential equations of the second order with constant coefficients. By means of the theory of distributions, the authors now prove that the $\varphi_p(H;f)$ have the stated property without using the maximum principle. Consequently, they obtain a new proof of this principle.

1. Introduction.

We say that f is *doubly-harmonic* on a domain $\Omega\subset E_{2m}$ if it is a real-valued function defined on Ω such that the equations

$$\frac{\partial^2 f}{\partial x_{2k-1}^2}+\frac{\partial^2 f}{\partial x_{2k}^2}=0\,, \qquad k=1,2,\cdots,m \tag{1}$$

hold for all $(x_1,\cdots,x_{2m})\in\Omega$. Such a function f is necessarily harmonic on Ω since on adding the m equations (1) we see that f satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x_1^2}+\frac{\partial^2 f}{\partial x_2^2}+\cdots+\frac{\partial^2 f}{\partial x_{2m-1}^2}+\frac{\partial^2 f}{\partial x_{2m}^2}=0\,. \tag{2}$$

Moreover, the class of doubly-harmonic functions contains each function that is the real part of a function of m complex variables which is holomorphic on Ω; this can be seen from the Cauchy-Riemann equations applied to each complex variable $x_{2k-1}+ix_{2k}$ separately. Obviously, if $m=1$, the class of doubly-harmonic functions coincides with the class of harmonic functions.

2. Two lemmas.

Throughout, we use the notation of our earlier paper [3]. In particular, $\varphi_p(H; f)$, which depends also on a point $A \in \Omega$, is defined in (2) of [3], and $\theta = (0, \cdots, 0)$ denotes the origin.

LEMMA 1. *If f is a homogeneous polynomial of degree q defined on E_n and if $A = \theta$, then*

$$\varphi_p(H; f) = \begin{cases} f(H) & \text{if } p = q \\ 0 & \text{if } p \neq q\ . \end{cases}$$

Proof. First, consider the special case

$$f_0(X) = cx_1^{s_1} \cdots x_n^{s_n}$$

where $s_1 + \cdots + s_n = q$. On applying the definition of $D_X^{|R|}$ in (1) of [3], we get

$$D_X^{|R|} f_0 = r_1! \cdots r_n!\, c \binom{s_1}{r_1} \cdots \binom{s_n}{r_n} x_1^{s_1 - r_1} \cdots x_n^{s_n - r_n}$$

so that $D_X^{|R|} f_0 = 0$ if $r_j > s_j$ for some j. And if $r_j < s_j$ for some j, then $(D_X^{|R|} f_0)_{X=\theta} = 0$. In the remaining case in which $r_j = s_j$ for all j, i.e., $R = S$, we have $D_X^{|R|} f_0 = S!c$. Hence, from (2) of [3], we obtain $\varphi_p(H; f_0) = 0$ if $p \neq q$ and

$$\varphi_q(H; f_0) = \frac{1}{S!} h_1^{s_1} \cdots h_n^{s_n} S!c = f_0(H)\ .$$

Second, consider the case of a general homogeneous polynomial f of degree q. Then f is a linear combination of terms of the kind f_0. Since $\varphi_q(H; f)$ is linear in the last argument, the result for f follows from the result for each of the f_0.

LEMMA 2. *If $P(x, y)$ is a harmonic polynomial, then it is either a constant or is an indefinite function.*

Proof. Suppose P is not a constant so that it is of exact degree $p \geqq 1$. Then

$$P(x, y) = J_p(x, y) + J_{p-1}(x, y) + \cdots + J_0(x, y)\ ,$$

where $J_k(x, y)$ is a homogeneous polynomial of degree k and J_p is not identically 0. If $p = 1$, then P is clearly indefinite. We therefore assume that $p \geqq 2$. Then

$$0 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) P(x, y) = I_{p-2}(x, y) + I_{p-3}(x, y) + \cdots + I_0(x, y)$$

where, for $0 \leqq k \leqq p - 2$,

$$I_k(x, y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)J_{k+2}(x, y)$$

is a homogeneous form of degree k. If not all I_k are identically 0, then there is a largest index $r \geqq 0$ such that I_r is not identically 0. Then

$$0 = I_r(x, y) + I_{r-1}(x, y) + \cdots + I_0(x, y)$$

with $I_r(a, b) \neq 0$ for some a, b. Hence, for all real t,

$$\begin{aligned} 0 &= I_r(ta, tb) + I_{r-1}(ta, tb) + \cdots + I_0(ta, tb) \\ &= t^r I_r(a,b) + t^{r-1} I_{r-1}(a, b) + \cdots + I_0(a, b) . \end{aligned}$$

Since $I_r(a, b) \neq 0$, this equation can hold for at most $r \leqq p-2$ values of t and we have a contradiction. Consequently, each I_k is identically 0 so that each J_j is harmonic. Since J_p is harmonic, we can apply Mann's result [2] (with $A = \theta$) to deduce that $\varphi_p(h, k; J_p)$ is either an indefinite form in h, k or is a constant (actually 0). By Lemma 1 above, $\varphi_p(h, k; J_p)$ is just $J_p(h, k)$ which has exact degree $p \geqq 2$; consequently, it is not a constant and hence is indefinite. Therefore, for suitable h_0, k_0 and h_1, k_1 we have $J_p(h_0, k_0) < 0 < J_p(h_1, k_1)$. Hence

$$P(th_0, tk_0) = t^p J_p(h_0, k_0) + t^{p-1} J_{p-1}(h_0, k_0) + \cdots + J_0(h_0, k_0) < 0$$

if t is positive and large enough. Likewise, $P(th_1, tk_1) > 0$ if t is positive and large enough. Thus P is indefinite.

3. The main results.

We begin with the following result which extends Theorem 2 of [3].

THEOREM 1. *If f is a Lebesgue integrable doubly-harmonic function on a domain $\Omega \subset E_{2m}$, then f is analytic on Ω. And if $A \in \Omega$, then the forms $\varphi_p(H; f)$ are doubly-harmonic functions on E_{2m} such that for each $p \geqq 1$ either the form is indefinite or is identically zero.*

Proof. As remarked earlier, f satisfies the Laplace equation (2). Since it is integrable, the expression $\int_\Omega f(X)\psi(X)dX$, for a test function ψ, defines a distribution as we remarked in the proof of Theorem 1 of [3]. This distribution also satisfies the elliptic equation (2). By Corollary 4.4.1 on p. 114 of Hörmander [1], it follows that the function f is analytic on Ω.

Consequently, the forms $\varphi_p(H; f)$ are defined; and by the corollary of the lemma in [3], these forms are doubly-harmonic on E_{2m}. Suppose that for some $p \geqq 1, \varphi_p(H; f)$ is not indefinite; then we may assume it is always nonnegative. If we fix $h_3, \cdots, h_{2m}$ then $\varphi_p(H;f)$ becomes a harmonic polynomial in h_1, h_2. By Lemma 2, it is either indefinite or is independent of h_1, h_2; since it is nonnegative, it cannot be indefinite and hence must be independent of h_1, h_2. That is, $\varphi_p(H;f)$ depends only $(h_3, \cdots, h_{2m})$. Similarly, it is independent of h_3, h_4 so that it depends only on $(h_5, \cdots, h_{2m})$. Continuing in this way, we see that $\varphi_p(H;f)$ depends only on (h_{2m-1}, h_{2m}); finally, it is independent of (h_{2m-1}, h_{2m}) as well so that $\varphi_p(H;f)$ is actually a constant. This constant is $\varphi_p(\theta;f) = 0$ since $p \geqq 1$. This completes the proof.

THEOREM 2. *If f is a nonconstant Lebesgue integrable doubly-harmonic function on a domain $\Omega \subset E_{2m}$, then it does not assume a maximum at a point in Ω.*

Proof. Suppose, on the contrary, that f assumes a maximum at a point $A \in \Omega$. The preceding theorem shows that f is analytic in Ω; it therefore has a Taylor expansion about A given by

$$f(X) = \sum_{r=0}^{\infty} \varphi_r(X - A; f) .$$

Since f is not a constant, there is some integer $p \geqq 1$ such that $\varphi_r(H;f)$ is identically 0 for each $r = 1, 2, \cdots, p-1$, but $\varphi_p(H;f)$ is not identically 0. Inasmuch as the previous theorem shows that $\varphi_p(H;f)$ is indefinite, there is some $C = (c_1, \cdots, c_{2m})$ such that $\varphi_p(C;f) > 0$. Then $C \neq \theta$ so that $c \equiv \|C\| > 0$. Applying Taylor's theorem with remainder, we have for all B in some neighborhood of A

$$f(B) - f(A) = \sum_{r=1}^{p-1} \varphi_r(B - A; f) + \varphi_p^*(B - A; f) = \varphi_p^*(B - A; f)$$

where, on putting $n = 2m$,

$$\varphi_p^*(H;f) = \sum_{|R|=q} \frac{1}{R!} h_1^{r_1} \cdots h_n^{r_n} (D_X^{|R|} f)_{X=D}$$

and $D = A + \vartheta(B - A)$, $0 < \vartheta < 1$, is a suitable point on the line segment joining A and B. If we define

$$\varepsilon = \frac{1}{2} \cdot \frac{p!}{(nc)^p} \varphi_p(C;f) ,$$

then there is a $\delta > 0$ such that, for all R with $|R| = p$,

$$|(D_X^{|R|} f)_{X=Y} - (D_X^{|R|} f)_{X=A}| < \varepsilon$$

whenever $\|Y - A\| < \delta$. Consequently, on taking $B = A + \lambda C$ and $0 < \lambda < \delta/c$, we find

$$\begin{aligned}|\varphi_p^*(\lambda C; f) - \varphi_p(\lambda C; f)| &\leqq \varepsilon \sum_{|R|=p} \frac{1}{R!}(\lambda\,|\,c_1\,|)^{r_1} \cdots (\lambda\,|\,c_n\,|)^{r_n} \\ &\leqq \frac{\varepsilon}{p!}(\lambda\,|\,c_1\,| + \cdots + \lambda\,|\,c_n\,|)^p \leqq \frac{\varepsilon}{p!}(\lambda n c)^p = \frac{1}{2}\lambda^p\varphi_p(C; f)\,.\end{aligned}$$

Since $f(A)$ is maximal,

$$\begin{aligned}0 \geqq f(A + \lambda C) - f(A) = \varphi_p^*(\lambda C; f) &\geqq \varphi_p(\lambda C; f) - \frac{1}{2}\lambda^p\varphi_p(C; f) \\ &= \lambda^p\varphi_p(C; f) - \frac{1}{2}\lambda^p\varphi_p(C; f) \\ &= \frac{1}{2}\lambda^p\varphi_p(C; f) > 0\end{aligned}$$

for all sufficiently small $\lambda > 0$. This contradiction proves the theorem.

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