PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 3, 1968

AN EXTENDED FORM OF THE MEAN-ERGODIC THEOREM

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Suppose X is a reflexive Banach space and V is a continuous linear operator in X such that $\| V^n \| \leqq M$ for some $M (n=0, 1, 2, \ldots)$. If N is the null space of $I - V$ and R is the closure of the range of $I - V$, then the mean-ergodic theorem states that

$$\lim_{n\to\infty}\frac{(I+V+\cdots+V^{n-1})x}{n}=Px\ ,$$

where P is the projection associated with N and R; the convergence is in the norm of X. This is pointwise C_1-summability of the sequence $\{V^k\}_{k=0}^{\infty}$ to P, and it suggests a similar theorem for more general Hausdorff summability methods. The purpose of this note is to demonstrate a wide class of operator-valued Hausdorff summability methods which contain the sequence $\{V^k\}_{k=0}^{\infty}$ in their wirkfelder and sum it to certain transforms of the projection operator P. This result shows much more clearly the sense in which convergence actually has meaning for such a sequence $\{V^k\}_{k=0}^{\infty}$.

Denote by $C(X)$ the space of X-valued continuous functions on $[0, 1]$ and by T_1 the bounded linear transformation from $C(X)$ into X given by $T_1(f) = \int_0^1 f(t)dt$. The mean-ergodic theorem states that

$$T_1\left(\sum_{k=0}^{n}\binom{n}{k}(t^k(1-t)^{n-k}V^k\cdot x\right)\xrightarrow[n\to\infty]{} T_1(P\cdot x)\ .$$

In this setting, the main theorem of this paper states a much stronger type of convergence; namely, that for any bounded linear operator T from $C(X)$ into a Banach space Y such that the generating function for T is continuous at 0 and 1, it is true that

$$T\left(\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}V^k\cdot x\right)\xrightarrow[n\to\infty]{} T(P\cdot x)\ .$$

In general one cannot expect much in the way of further relaxations on the operators T, i.e., on the functions which generate such operators. For example if the condition of continuity at 1 is removed, then this allows a generating function $K(t) = 0$ for $t < 1$, $K(1) = 1$ and this generates the Hausdorff method corresponding to ordinary convergence. In general the sequence $\{V^k \cdot x\}$ does not converge.

A nice presentation of the mean ergodic theorem as stated above is to be found in Lorch [2, pp. 54–56]. Suppose Y is a Banach space and $\mu = \{\mu_k\}_{k=0}^{\infty}$ is a sequence of elements of $B[X, Y]$ such that the Hausdorff method $H = \rho\mu\rho$ generated by μ is regular relative to some $L \in B[X, Y]$. (See [1] for notation and terminology. Reference 8 in [1] is reference [3] of this paper.) It follows from [1] that there exists a function K on $[0, 1]$ with values in $B^+[X, Y]$ such that K satisfies the Gowurin ω-property,

$$K(0) = 0, K(1) = L \quad \text{and} \quad \mu_n = \int_0^1 dK(t)\cdot t^n \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

THEOREM. *If K is continuous at $t = 0$ and $t = 1$, then $\{V^k\}_{k=0}^{\infty}$ is pointwise H-summable to LP,* i.e., *$H_n\{V^k\}\cdot x$ converges in the norm of Y to LPx for each $x \in X$.*

The essential ingredient of the proof of the theorem is the following lemma.

LEMMA. *If $\{s_k\}_{k=0}^{\infty}$ is a bounded sequence of elements of a linear normed space S and $0 < a \leqq t \leqq b < 1$, then*

$$\left|\left|\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}(s_k - s_{k+1})\right|\right|_S$$

converges uniformly to zero for $t \in [a, b]$.

Proof of the lemma.[1] Suppose $||s_k|| \leqq N'$ for $k = 0, 1, 2, \cdots$, then set

$$\begin{aligned}
A_n(t) &= \sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}(s_k - s_{k+1}) \\
&= \sum_{k=1}^{n}\left[\binom{n}{k}t^k(1-t)^{n-k} - \binom{n}{k-1}t^{k-1}(1-t)^{n-k+1}\right]s_k \\
&\quad + (1-t)^n s_0 - t^n s_{n+1} \\
&= \sum_{k=1}^{n}\binom{n}{k}t^k(1-t)^{n-k}\left[1 - \frac{k}{n-k+1}\cdot\frac{1-t}{t}\right]s_k \\
&\quad + (1-t)^n s_0 - t^n s_{n+1} \\
&= \frac{1}{t}\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}\left[\frac{t - \frac{k}{n}\cdot\frac{n}{n+1}}{1 - \frac{k}{n}\cdot\frac{n}{n+1}}\right]s_k - t^n s_{n+1} .
\end{aligned}$$

$$||A_n(t)||_S \leqq \frac{1}{t}\left\|\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k}\left[\frac{t - \frac{k}{n}\cdot\frac{n}{n+1}}{1 - \frac{k}{n}\cdot\frac{n}{n+1}}\right]s_k\right\|_S + t^n ||s_{n+1}||$$

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[1] The proof presented here is incorrect. See part 2 for a corrected proof.

where $0 < a \leqq t \leqq b < 1$, and hence

$$\| A_n(t) \|_S \leqq \frac{N'}{t} \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n} \cdot \frac{n}{n+1} \right|}{1 - \frac{k}{n} \cdot \frac{n}{n+1}} + t^n \cdot N' .$$

Let $f_n(x, t)$ be given by

$$f_n(x, t) = \left| x - t \cdot \frac{n}{n+1} \right| \Big/ \left(1 - t \cdot \frac{n}{n+1} \right)$$

and $C_n(t)$ by

$$C_n(t) = \frac{1}{t} B_n[f_n(x, t)] \big|_{x=t}$$

where B_n denotes the n-th Bernstein polynomial. The above inequality may now be written

$$\| A_n(t) \|_S \leqq N' \, | C_n(t) | + t^n \cdot N'$$

and the second term converges uniformly to zero for $t \in [a, b]$.

The first term is treated as follows. By a direct calculation it can be shown that for each $x \in [0, b]$, the collection $\{f_n(x, t)\}$ is equi-uniformly continuous in t for $t \in [0, b]$, that is to say, if $\varepsilon > 0$, then there exists $\delta > 0$ such that $| f_n(x, s) - f_n(x, t) | < \varepsilon/2$ for all $s, t \in [0, b]$ such that $| s - t | < \delta$ and for all n.

Consider a fixed $t \in [0, b]$ and set $A = \{k : |k/n - t| < \delta\}$ and $B = \{0, 1, \cdots, n\} - A$. Then

$$\begin{aligned} &|B_n[f_n(x, t)] - f_n(x, t)| \\ &\leqq \left(\sum_A + \sum_B\right) \left| \binom{n}{k} t^k (1-t)^{n-k} \left\{ \frac{\left(x - \frac{k}{n} \cdot \frac{n}{n+1} \right)}{\left(1 - \frac{k}{n} \cdot \frac{n}{n+1} \right)} - f_n(x, t) \right\} \right| \end{aligned}$$

$$\sum_A \leqq \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \cdot \frac{\varepsilon}{2} = \frac{\varepsilon}{2} .$$

Set $Q = \max_{0 \leqq t, x \leqq b} f_n(x, t)$ for $n = 0, 1, 2, \cdots$ and the second term can be treated as follows:

$$\sum_B \leqq 2Q \sum_B \binom{n}{k} t^k (1-t)^{n-k} \frac{(k - nt)^2}{n^2 \delta^2} \leqq \frac{2Q}{n^2 \delta^2} \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} (k - nt)^2$$

which, as is well known, converges uniformly to zero for $t \in [0, 1]$. Hence, there exists an integer N_0 such that $\sum_B < \varepsilon/2$ for $n > N_0$ and further such that $| B_n[f_n(x, t)] - f_n(x, t) | < \varepsilon$ for $n > N_0$, both inequalities holding uniformly for $0 \leqq t \leqq b$. Collecting all these items together yields

$$\lim_{n \to \infty} C_u(t) = \frac{1}{t} \frac{|t - t|}{1 - t} = 0$$

uniformly for $t \in [a, b]$, and hence $\| A_n(t) \|_S \to 0$ uniformly on $[a, b]$.

Proof of the theorem. Let

$$T_n = H_n\{V^k\}_{k=0}^{\infty} = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} \mu_k V^k = \int_0^1 dK(t) \cdot \left[\sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} V^k \right].$$

Since N, R is a complementary pair in X, it is sufficient to investigate the behavior of T_n on each of these sets.

Suppose $f \in N$, i.e., $Vf = f$, then

$$\begin{aligned} T_n f &= \int_0^1 dK(t) \cdot \left[\sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} V^k f \right] = \int_0^1 dK(t) \cdot \left[\sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f \right] \\ &= \int_0^1 dK(t) \cdot f = [K(1) - K(0)] f = Lf = LPf . \end{aligned}$$

Now suppose $f \in R$ and $\varepsilon > 0$, then there exists g and h such that $f = g - Vg + h$ where $\| h \| < \varepsilon / 4[1 + W_0^1 K] M$. For this f,

$$\begin{aligned} T_n f = &\int_0^1 dK(t) \cdot \sum_{k=0}^{n} \binom{n}{k} t^k (1 - t)^{n-k} [V^k g - V^{k+1} g] \\ &+ \int_0^1 dK(t) \cdot \sum_{k=0}^{n} \binom{n}{k} t^k (1 - t)^{n-k} V^k h = I + II . \end{aligned}$$

$$\begin{aligned} \| II \|_Y &\leqq W_0^1 K \cdot \max_{0 \leqq t \leqq 1} \left| \sum_{k=0}^{n} \binom{n}{k} t^k (1 - t)^{n-k} \right| \max_{0 \leqq k \leqq n} \| V^k h \|_X \\ &\leqq W_0^1 K \cdot M \cdot \varepsilon / 4[1 + W_0^1 K] M < \frac{\varepsilon}{4} \qquad \text{for all } n . \end{aligned}$$

$$\begin{aligned} \| I \|_Y = \| I \|_{Y^{**}} &= \left\| \int_0^a + \int_a^b + \int_b^1 \right\|_{Y^{**}} \\ &\leqq \left\| \int_0^a \right\|_{Y^{**}} + \left\| \int_a^b \right\|_{Y^{**}} + \left\| \int_b^1 \right\|_{Y^{**}} . \end{aligned}$$

It is necessary to regard the norms on the right as Y^{**} norms because these integrals may exist only as elements in Y^{**} and not as elements in Y (see the remarks following Theorem 1 [3, p. 950].)

$$\left\| \int_0^a dK(t) \cdot \left[\sum_{k=0}^{n} \binom{n}{k} t^k (1 - t)^{n-k} [V^k g - V^{k+1} g] \right\|_{Y^{**}} \leqq W_0^a K \cdot 2M \cdot \| g \|_X$$

and

$$\left\| \int_b^1 dK(t) \cdot \left[\sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} [V^k g - V^{k+1} g] \right\|_{Y^{**}} \leqq W_b^1 K \cdot 2M \cdot \| g \|_X .$$

Since K is assumed continuous at $t = 0$ and $t = 1$, there are values for a and b sufficiently near, but distinct from 0 and 1 respectively, such that each of $W_0^a K$ and $W_b^1 K$ less than $\varepsilon/8M[1 + ||g||]$. With these values of a and b, there is n sufficiently large, by the above lemma, that

$$\max_{a \leqq t \leqq b} \left|\left| \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} [V^k g - V^{k+1} g] \right|\right|_X \leqq \varepsilon/2[1 + W_a^b K] .$$

Collecting all this together yields

$$|| T_n f ||_Y \leqq \varepsilon$$

for all n sufficiently large. Thus

$$\lim_{n \to \infty} || T_n f ||_Y = \lim_{n \to \infty} || T_n f - LPf || = 0$$

since

$$LPf = \theta_Y$$

and this completes the proof.

In case that $Y \equiv X$ and H is regular relative to I, then H sums $\{V^k\}_{k=0}^{\infty}$ to P. In particular, any regular scalar-valued Hausdorff method whose generating function K is continuous at $t = 0$ and $t = 1$ will sum $\{V^k\}_{k=0}^{\infty}$ to P. The case treated in [2], corresponds to the case here where $K(t) = tI$, i.e., the C_1 method. The following example illustrates the theorem for a nonscalar-valued Hausdorff method.

Suppose $X = Y = H$, a Hilbert space. Suppose also that K is a bounded resolution of the identity such that $K(0) = 0$, $K(1) = I$, K is continuous at 0 and 1 in the operator norm, and K satisfies the Gowurin ω-property. The approximating sums for integrals of the form $\int_0^1 t^n dK(t)$ converge to the integral in the operator norm [2], hence they converge in the sense given by Tucker [3]. Consider the moment sequence $\{\mu_n\}_{n=0}^{\infty}$ given by $\mu_n = \int_0^1 t^n dK(t)$. As shown in [2], μ_1 is a self-adjoint operator in H, and if we denote it by A, it follows that $\mu_n = A^n (n = 0, 1, 2, \cdots)$ where $\mu_0 = K(1) = A^0 = I$. If $\{V^n\}_{n=0}^{\infty}$ is a sequence of operators as given in the theorem, and $H(\mu)$ is the Hausdorff summability method generated by $\{\mu_n\} = \{A^n\}$, then

$$\lim_{n \to \infty} \sum_{k=0}^{n} \binom{n}{k} (\Delta^{n-k} A^k) V^k x = Px$$

for all $x \in H$, the limit being taken in the norm of H.

Part 2

It has been pointed out that the proof of the lemma given above is incorrect. It can be corrected in the following manner. As given,

$$\| A_n(t) \|_s \leqq \frac{N'}{t} \sum_{k=0}^{n} \left(\binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} \right) + t^n N' .$$

Proceed as follows. For $0 < a \leqq t \leqq b < 1$

$$\| A_n(t) \|_s \leqq \frac{N'}{a} \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} + b^n N' .$$

Suppose $\varepsilon > 0$ and pick δ such that $0 < \delta < \{(1-b)\varepsilon/2/(1+\varepsilon|2)\}$. For $t \in [a, b]$, set

$$A_t = \left\{ k : \left| t - \frac{k}{n+1} \right| < \delta \right\} \quad \text{and} \quad B_t = \left\{ k : \left| t - \frac{k}{n+1} \right| \geqq \delta \right\} .$$

Then

$$\sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}}$$

$$= \left(\sum_{A_t} + \sum_{B_t} \right) \left\{ \binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} \right\} .$$

Consider the sums separately.

$$\sum_{A_t} \leqq \sum_{A_t} \binom{n}{k} t^k (1-t)^{n-k} \frac{\delta}{1 - b - \delta} \leqq \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \frac{\varepsilon}{2} = \frac{\varepsilon}{2} .$$

$$\sum_{B_t} = \frac{1}{1-t} \sum_{B_t} \binom{n}{k} \frac{n+1}{n+1-k} t^k (1-t)^{n-k+1} \left| t - \frac{k}{n+1} \right| .$$

For $k \in B_t$,

$$\left| \frac{k}{n+1} - t \right| \leqq 1 \leqq \frac{((n+1)t - k)^2}{\delta^2 (n+1)^2} ,$$

so

$$\sum_{B_t} \leqq \frac{1}{(1-t)\delta^2(n+1)^2} \sum_{k=0}^{n} \binom{n}{k} \frac{n+1}{n+1-k} t^k (1-t)^{n-k+1}[(n+1)t-k]^2$$
$$= \frac{1}{(1-t)\delta^2(n+1)^2} \sum_{k=0}^{n+1} \binom{n+1}{k} t^k (1-t)^{n-k+1}[(n+1)t-k]^2$$
$$= \frac{1}{(1-t)\delta^2} \sum_{k=0}^{n+1} \binom{n+1}{k} t^k (1-t)^{n-k+1}\left(t - \frac{k}{n+1}\right)^2$$
$$= \frac{1}{(1-t)\delta^2} \cdot \frac{t(1-t)}{n+1} \leqq \frac{b}{(n+1)\delta^2} .$$

Collecting this together gives

$$\| A_n(t) \|_s \leqq \frac{N'}{a}\left(\frac{\varepsilon}{2} + \frac{b}{(n+1)\delta^2}\right) + b^n N' , \qquad \text{for } 0 < a \leqq t \leqq b < 1 ,$$

which proves the lemma.

BIBLIOGRAPHY

1. L. C. Kurtz and D. H. Tucker, *Vector-valued summability methods on a linear normed space,* Proc. Amer. Math. Soc. **16** (1965), 419-428.
2. E. R. Lorch, *Spectral Theory,* Oxford University Press, New York, 1962.
3. D. H. Tucker, *A representation theorem for a continuous linear transformation on a space of continuous functions,* Proc. Amer. Math. Soc. **16** (1965), 946-953.

Received July 12, 1967. This research was done while the authors were participating in the ARMU Differential Equations Symposium in Boulder, Colorado.

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