PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 3, 1968

PICK'S CONDITIONS AND ANALYTICITY

A. C. HINDMARSH

Let $w(z)$ be a function in the open upper half plane (UHP) with values in UHP, and let $P_n = (d_{ij})$ be the $n \times n$ matrix of difference quotients

$$d_{ij} = \frac{w(z_i) - \overline{w(z_j)}}{z_i - \bar{z}_j}$$

formed from any n points $z_1, z_2, \cdots, z_n \in$ UHP. It was shown by G. Pick that if $w(z)$ is also analytic in UHP, then the P_n are all nonnegative definite Hermitian matrices (denoted $P_n \geqq 0$). In what follows, two converse results are derived.

(1) If D is a domain in UHP, $w(z)$ is continuous in D and has values in UHP, and $P_3 \geqq 0$ for all choices of the $z_1, z_2, z_3 \in D$, then $w(z)$ is analytic in D. It is well known that the condition $P_2 \geqq 0$ does not imply anything of this sort, but corresponds only to a distance-shrinking property of $w(z)$ in the noneuclidean geometry of UHP.

(2) If w is as before, but $P_n \geqq 0$ for all n and all $z_1, \cdots, z_n \in D$, i.e., $\{w(z) - \overline{w(\zeta)}\}/(z - \bar{\zeta})$ is a nonnegative definite kernel in D, then $w(z)$ is analytic in D and has an analytic extension to UHP whose values are in UHP.

The central idea of result (1) is to consider the kernel $K(z, \zeta) = \{w(z) - \overline{w(\zeta)}\}/(z - \bar{\zeta})$ for z, ζ in a neighborhood of a point $z_0 \in D$ and to interpret the 3rd Pick condition $P_3 \geqq 0$ locally at z_0, thereby deriving coefficient inequalities for K at (z_0, z_0). This idea is made explicit in the following lemma on general kernels:

LEMMA. *Let D be an open set in $\boldsymbol{R}^n$, and let*

$$K(u, v) = K(u_1, \cdots, u_n; v_1, \cdots, v_n)$$

be a C^2 kernel defined for $u, v \in D$, with $K(u, v) = \overline{K(v, u)}$. If $K \geqq 0$ of order $n + 1$ in D, i.e., $(k_{ij}) \geqq 0$ for the $(n + 1) \times (n + 1)$ matrix with elements $k_{ij} = K(u^i, u^j)$ formed from any $n + 1$ points $u^0, u^1, \cdots, u^n \in D$, then for each $u \in D$ we have

$$M(u) = \begin{pmatrix} K & K_{v_j} \\ K_{u_i} & K_{u_i v_j} \end{pmatrix}\Bigg|_{(u,u)} \geqq 0\ .$$

Here K_{v_j} refers to the row vector $(K_{v_1} K_{v_2} \cdots K_{v_n})$, K_{u_i} to a similar column vector, and $K_{u_i v_j}$ to an $n \times n$ matrix. Subscripts on K denote partial differentiation.

Proof. Fix $u \in D$. For small positive h, let $u^i = (u_1^i, \cdots, u_n^i)$, where $u_k^i = \begin{cases} h & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$. Then let $K(h)$ be the $(n+1) \times (n+1)$ matrix (k_{ij}), $0 \leqq i, j \leqq n$, $k_{ij} = K(u + u^i, u + u^j)$. For all small h, $K(h) \geqq 0$. Now form $\tilde{K}(h) = (\tilde{k}_{ij})$ where

$$\tilde{k}_{00} = k_{00},\ \tilde{k}_{0j} = \frac{k_{0j} - k_{00}}{h},\ \tilde{k}_{i0} = \frac{k_{i0} - k_{00}}{h},\ \tilde{k}_{ij} = \frac{k_{ij} + k_{00} - k_{0j} - k_{i0}}{h^2} \qquad (i, j \geqq 1).$$

If K, K_{u_i}, etc., denote the value and various derivatives of K at (u, u), then we have

$$k_{00} = K,\ k_{0j} = K + hK_{v_j} + \frac{h^2}{2}K_{v_jv_j} + o(h^2)\ ,$$

$$k_{i0} = K + hK_{u_i} + \frac{h^2}{2}K_{u_iu_i} + o(h^2)\ ,$$

$$k_{ij} = K + h(K_{u_i} + K_{v_j}) + \frac{h^2}{2}(K_{u_iu_i} + 2K_{u_iv_j} + K_{v_jv_j}) + o(h^2)\ ,$$

and so, as $h \to 0$,

$$\tilde{k}_{00} = K,\ \tilde{k}_{0j} = K_{v_j} + o(1),\ \tilde{k}_{i0} = K_{u_i} + o(1),\ \tilde{k}_{ij} = K_{u_iv_j} + o(1) \qquad (i, j, \geqq 1)\ .$$

But $K(h) \geqq 0 \Leftrightarrow \tilde{K}(h) \geqq 0$, because the change $K \to \tilde{K}$ in the associated quadratic form corresponds to the invertible linear change of coordinates in C^{n+1} given by $X_0 = \tilde{X}_0 - (\sum_1^n \tilde{X}_i)/h$, $X_i = \tilde{X}_i/h$ $(i \geqq 1)$. Hence we conclude that $\lim_{h\to 0} \tilde{K}(h) = M(u) \geqq 0$.

We wish to apply the lemma to the case of a kernel $K(z, \zeta)$ defined for $z, \zeta \in D$, D being an open set in the plane, with $K \in C^2$, and $K(z, \zeta) = \overline{K(\zeta, z)}$. If we have $K \geqq 0$ of order 3 in D, i.e., $(K(z_i, z_j)) \geqq 0$ for the 3×3 matrix formed from $z_1, z_2, z_3 \in D$, we deduce that

$$N(z) = \begin{pmatrix} K & K_\xi & K_\eta \\ K_x & K_{x\xi} & K_{x\eta} \\ K_y & K_{y\xi} & K_{y\eta} \end{pmatrix}\Bigg|_{(z,z)} \geqq 0 \qquad (z = x + iy, \zeta = \xi + i\eta)$$

for $z \in D$, by applying the lemma to $J(u, \mathrm{v}) = K(u_1 + iu_2, v_1 + iv_2)$ with $n = 2$. Further, by a change of coordinates given by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -i/2 \\ 0 & 1/2 & i/2 \end{pmatrix},$$

we obtain

$$AN(z)A^* = M(z) = \begin{pmatrix} K & K_{\bar\zeta} & K_\zeta \\ K_z & K_{z\bar\zeta} & K_{z\zeta} \\ K_{\bar z} & K_{\bar z\bar\zeta} & K_{\bar z\zeta} \end{pmatrix}\Bigg|_{(z,z)} \geqq 0 .$$

To apply this last result to the present problem, let D be an open set in UHP, let $w(z)$ be given in D with values in UHP and with $P_3 \geqq 0$ in D, and suppose first that $w \in C^2$. Then $K(z, \zeta) = \{w(z) - \overline{w(\zeta)}\}/(z - \bar\zeta)$ is an admissible kernel, and we are led to the 3×3 coefficient matrix $M(z) = (m_{ij}) \geqq 0$. Putting $A = z - \bar\zeta$, $B = w(z) - \overline{w(\zeta)}$, the required derivatives of $K = B/A$ at (z, ζ) are

$$K_\zeta = \frac{AB_\zeta - A_\zeta B}{A^2} = -\frac{\overline{w_{\bar z}(\zeta)}}{A}, \quad K_{\bar z} = \frac{w_{\bar z}(z)}{A}, \quad K_{z\zeta} = \frac{\overline{w_{\bar z}(\zeta)}}{A^2},$$

$$K_{\bar z\zeta} = 0 , \qquad \text{etc.}$$

But $M(z) \geqq 0$ implies in particular that

$$0 \leqq m_{22}m_{33} - |m_{23}|^2 = K_{z\bar\zeta}K_{\bar z\zeta} - |K_{z\zeta}|^2|_{(z,z)} = -|K_{z\zeta}(z, z)|^2 .$$

Hence $K_{z\zeta}(z, z) = 0$, and so $w_{\bar z}(z) = 0$. I.e., the Cauchy-Riemann Equations hold in D, and $w(z)$ is analytic in D.

In order to remove the assumption $w \in C^2$, we use a standard mollification argument. In a neighborhood of $z_0 \in D$, we approximate the continuous function $w(z)$ by mollified functions $w_\delta(z)$, such that $w_\delta \in C^2$ and $w_\delta \to w$ uniformly in a neighborhood of z_0. Since the property $P_3 \geqq 0$ is additive and positive-homogeneous in w, we see also that $P_3 \geqq 0$ for each w_δ as well as for w. We therefore know that w_δ is analytic in a neighborhood of z_0. By uniform convergence, so is w. Since z_0 was arbitrary, $w(z)$ is analytic throughout D.

From the above proof, it is clear that the hypotheses in statement (1) are considerably stronger than they need be. First, the fact that only $m_{22}m_{33} - |m_{23}|^2 \geqq 0$ was used means that $P_3 = (k_{ij})$ need only be nonnegative definite on the subspace $L_3 = \{(X_i) \in C^3 : \sum X_i = 0\}$ of complex dimension 2. For, in the notation of the proof of the lemma, the latter condition is equivalent to

$$\begin{pmatrix} \tilde k_{11} & \tilde k_{12} \\ \tilde k_{21} & \tilde k_{22} \end{pmatrix} \geqq 0 .$$

The analogous form of the lemma, in which $(K(u^i, u^j)) \geqq 0$ on L_{n+1} for $u^0, u^1, \cdots, u^n \in D \Rightarrow (K_{u_i\bar v_j}(u, u)) \geqq 0$, is similarly proved. Secondly, there is now no need for the values of $w(z)$ to lie in UHP. These two alterations mean that the analyticity result holds when $w(z)$ is a continuous "infinitesimal transformation" of the class of maps of

D satisfying $P_3 \geqq 0$, i.e., $w(z) = \partial f_t(z)/\partial t|_{t=0}$, where f_t, $0 \leqq t \leqq t_0$, is a family of functions in D satisfying $P_3 \geqq 0$ in D for all t, and $f_0(z) = z$. The class of such $w(z)$ is in fact characterized by the condition $P_3 \geqq 0$ on L_3 (and likewise for general n). The positivity hypothesis could also be weakened from a global condition to a local one, but since D is arbitrary and analyticity is a local property, this would be a trivial alteration. To summarize, we state:

THEOREM 1. *Let $w(z)$ be a continuous function in an open subset D of* UHP. *If, for all $z_1, z_2, z_3 \in D$, the 3×3 matrix of difference quotients $d_{ij} = \{w(z_i) - \overline{w(z_j)}\}/(z_i - \bar{z}_j)$ satisfies $(d_{ij}) \geqq 0$ on the subspace $\{(X_i) \in C^3 : \sum X_i = 0\}$, then $w(z)$ is analytic in D.*

It should be noted here that result (1), in the weaker form, can also be easily proven from Pick's Theorem (below). However, the latter requires a proof that considerably more involved than that given here for Theorem 1.

The statement (2) gives a characterization of the class P of "positive" functions, analytic in UHP with values in UHP. It says that all of Pick's conditions together imply that w is the restriction to D of a P function. The proof depends on the following:

PICK'S THEOREM. *If $z_1, \cdots, z_n, w_1, \cdots, w_n \in$ UHP and $P_n = (d_{ij}) \geqq 0$ for the $n \times n$ matrix of difference quotients $d_{ij} = (w_i - \bar{w}_j)/(z_i - \bar{z}_j)$, then there is a function $f \in P$ for which $f(z_i) = w_i$ for $1 \leqq i \leqq n$.*

Now if $w(z)$ is continuous in D and $K(z, \zeta) = \{w(z) - \overline{w(\zeta)}\}/(z - \bar{\zeta})$ is nonnegative definite (of infinite order) in D, we can choose a dense sequence (z_i) from D and apply Pick's Theorem for each n. Because P is a normal family, the P functions so gotten have a normally convergent subsequence, and the analytic limit agrees with w in D. We thus obtain

THEOREM 2. *Let $w(z)$ be a continuous function in a domain $D \subset$ UHP with values in* UHP. *If $\{w(z) - \overline{w(\zeta)}\}/(z - \bar{\zeta})$ is a nonnegative definite kernel in D, then w is analytic in D and has an analytic extension to* UHP *whose values are in* UHP.

I wish to take this opportunity to express my deep gratitude for Prof. Loewner's guidance and my sorrow at his loss.

I wish to take this opportunity to express my sorrow at the loss of Professor Charles Loewner, who, as my thesis advisor, inspired the work represented in this paper.

Reference

1. G. Pick, *Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden,* Math. Annalen **77** (1915), 7-23.

Received March 6, 1968. This work was supported by a N. S. F. Graduate Fellowship. The results herein are part of a doctoral thesis, for which the research was performed under the late Professor Charles Loewner.

Stanford University
Stanford, California