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OSCILLATION CRITERIA FOR ELLIPTIC EQUATIONS

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Conditions on the coefficients of a linear elliptic partial differential equation will be obtained which are sufficient for the equation to be oscillatory in certain unbounded domains. The criteria obtained in the first three theorems involve integrals of suitable majorants of the coefficients while the criterion in Theorem 4 involves limits of these majorants at infinity. We also obtain a nonoscillation criterion involving similar limits.

Oscillation criteria of both limit type and integral type will be obtained for the linear elliptic partial differential equation

$$(1) \qquad Lu \equiv \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + bu = 0$$

in unbounded domains R in n-dimensional Euclidean space E^n. Our theorems constitute extensions of several well-known one-dimensional oscillation theorems of Kneser-Hille [6] (limit type), Leighton [8], Moore [10], and Wintner [13] (integral type). A special case of Theorem 4 below was obtained by Glazman [4, 5] when L is the Schrödinger operator and R coincides with E^n. Analogues of Theorem 1 were obtained by Kreith [7] and Swanson [12] in the case that one variable is separable and R is limit cylindrical, i.e., contains an infinitely long cylinder.

Points in E^n are denoted by $x = (x^1, x^2, \cdots, x^n)$ and differentiation with respect to x^i is denoted by D_i, $i = 1, 2, \cdots, n$. The functions a_{ij} and b involved in (1) are assumed to be real-valued and continuous on $R \cup \partial R$, and the matrix (a_{ij}) is supposed to be symmetric and positive definite in R (ellipticity condition). A "solution" of (1) is defined in the usual way [1, 12].

We assume that R contains the origin and that R is large enough at ∞ in the x^n direction to contain the cone $C_\alpha = \{x \in E^n : x^n \geqq |x| \cos \alpha\}$ for some α, $0 < \alpha \leqq \pi$. The boundary ∂R of R is supposed to have a piecewise continuous unit normal vector at each point. The following notations will be used:

$$R_r = R \cap \{x \in E^n : |x| > r\}\,; \qquad S_r = \{x \in R \cup \partial R : |x| = r\}\,.$$

A bounded domain $N \subset R$ is said to be a *nodal domain* of a nontrivial solution u of (1) if and only if $u = 0$ on ∂N. The differential equation (1) is said to be *oscillatory* in R if and only if there exists a nontrivial solution u_r of (1) with a nodal domain in R_r for all $r > 0$. It follows from the n-dimensional analogue of Sturm's separation theorem [1] that *every* solution of an oscillatory differential equation vanishes at some point in R_r for all $r > 0$.

Let $\wedge(x)$ denote the largest eigenvalue of the matrix $(a_{ij}(x))$, $x \in R$. A *majorant* of (a_{ij}) is a positive-valued function $f \in C^1(0, \infty)$ such that

$$f(r) \geqq \max_{x \in S_r} \wedge(x) \qquad (0 < r < \infty) .$$

The function g defined by

$$g(r) = \min_{x \in S_r} b(x) \qquad (0 < r < \infty) \tag{2}$$

is called a majorant of $b(x)$.

Let A, B be the functions in R defined by the equations $A(x) = f(|x|)$, $B(x) = g(|x|)$, respectively. We shall obtain oscillation theorems for equation (1) by comparing (1) with the separable equation

$$\sum_{i=1}^{n} D_i(AD_iv) + Bv = 0 . \tag{3}$$

Let $r, \theta_1, \theta_2, \cdots, \theta_{n-1}$ denote hyperspherical polar coordinates [9, p. 58], defined as follows:

$$\begin{cases} x_1 = r \prod_{i=1}^{n-1} \sin\theta_i , \qquad x_n = r\cos\theta_1 , \\ x_i = r\cos\theta_{n-i+1} \prod_{j=1}^{n-i} \sin\theta_j , \qquad i = 2, 3, \cdots, n-1 . \end{cases}$$

By writing (3) in terms of these coordinates, we find that (3) has solutions (in particular) of the form

$$v(x) = \rho(r)\varphi(\theta_1) , \qquad 0 \leqq r < \infty , \qquad 0 \leqq \theta_1 \leqq \alpha , \tag{4}$$

where ρ and φ satisfy the ordinary differential equations

$$\frac{d}{dr}\left[r^{n-1}f(r)\frac{d\rho}{dr}\right] + r^{n-1}[g(r) - \lambda_\alpha r^{-2}f(r)]\rho = 0 , \tag{5}$$

$$\frac{d}{d\theta_1}\left[\sin^{n-2}\theta_1 \frac{d\varphi}{d\theta_1}\right] + \lambda_\alpha \varphi \sin^{n-2}\theta_1 = 0 , \tag{6}$$

respectively. For $0 < \alpha < \pi$, we choose λ_α to be the smallest number for which (6) has a nontrivial solution φ on $0 \leqq \theta_1 \leqq \alpha$ satisfying $\varphi(\alpha) = 0$. It is well-known [2] that λ_α exists as the smallest eigenvalue of a singular Sturm-Liouville problem. To be specific, we shall suppose that the corresponding eigenfunction has been normalized by the condition $\varphi(0) = 1$. For $\alpha = \pi$, we choose $\lambda_\alpha = 0$ and $\varphi(\theta_1) \equiv 1$.

THEOREM 1. *Equation* (1) *is oscillatory in* R *if* R *contains a cone* C_α $(\alpha > 0)$, *and* $(a_{ij}), b$ *have majorants* f, g, *respectively, such that*

$$(7) \qquad \int_1^\infty \frac{dr}{r^{n-1}f(r)} = +\infty \quad \textit{and} \quad \int_1^\infty r^{n-1}[g(r) - \lambda_\alpha r^{-2} f(r)]dr = +\infty\ .$$

THEOREM 2. *Equation* (1) *is oscillatory in* R *if* R *contains a cone* C_α $(\alpha > 0)$, *and* $(a_{ij}), b$ *have majorants* f, g, *respectively, such that*

$$(8) \qquad \int_1^\infty \frac{dr}{r^{n-1}f(r)} < \infty \quad \textit{and} \quad \int_1^\infty r^{n-1}h_n^m(r)[g(r) - \lambda_\alpha r^{-2} f(r)]dr = +\infty\ ,$$

for some number $m > 1$, *where* $h_n(r) = \int_r^\infty dt/t^{n-1}f(t)$.

THEOREM 3. *Suppose that* R *contains the cone* C_α *for some* $\alpha > 0$, *and that* $\wedge(x)$ *is bounded in* R. *Then equation* (1) *is oscillatory in* R *for* $n = 2$ *if*

$$(9) \qquad \int_1^\infty r[g(r) - \lambda_\alpha r^{-2} f(r)]dr = +\infty\ ,$$

and for $n \geqq 3$ *if there exists a number* $\delta > 0$ *such that*

$$(10) \qquad \int_1^\infty r^{1-\delta}[g(r) - \lambda_\alpha r^{-2} f(r)]dr = +\infty\ ,$$

where $g(r)$ *is given by* (2).

In the case $n = 1$, (1) is oscillatory if (10) holds with $\delta = 1$ (Leighton-Wintner theorem).

THEOREM 4. *Suppose that* R *contains the cone* C_α *for some* $\alpha > 0$, *and that* $\wedge(x)$ *is bounded in* R, *say* $\wedge(x) \leqq \wedge_1, x \in R$. *Then equation* (1) *is oscillatory in* R *if*

$$(11) \qquad \liminf_{r\to\infty} r^2 g(r) > \wedge_1[\lambda_\alpha + (n-2)^2/4]\ .$$

In particular, (11) reduces to Glazman's criterion [5]

$$\liminf_{r\to\infty} r^2 g(r) > (n-2)^2/4$$

if $-L$ is the Schrödinger operator $-\nabla^2 - b(x), x \in E^n$. For $n = 1$ and $a_{11}(x) = 1$, Theorem 4 reduces to the classical Kneser-Hille theorem [6].

THEOREM 5. *Suppose that L is uniformly elliptic in R_s for some $s > 0$, i.e., there exists a number $\wedge_0 > 0$ such that $\sum_{i,j} a_{ij}(x)z^i z^j \geqq \wedge_0 |z|^2$ for all $x \in R_s$, $z \in E^n$. Let $g_0(r)$ denote the maximum of $b(x)$ for $x \in S_r$, $0 < r < \infty$. Then equation (1) is nonoscillatory in R if*

$$\limsup_{r\to\infty} r^2 g_0(r) < (n-2)^2 \wedge_0/4 . \tag{12}$$

Proofs. The hypotheses (7) imply that the ordinary differential equation (5) is oscillatory in $0 < r < \infty$ by the Leighton-Wintner oscillation theorem [8, 13]. Let $\rho(r)$ be a nontrivial solution of (5) with zeros at $r = \delta_1, \delta_2, \cdots$, where $\delta_k \uparrow \infty$. If φ is an eigenfunction of (6) with boundary condition $\varphi(\alpha) = 0$ corresponding to the eigenvalue λ_α, the function v defined by (4) is a solution of the comparison equation (3) with nodal domains in the form of "truncated cones"

$$C_{\alpha k} = \{x \in E^n : x^n > |x| \cos \alpha,\ \delta_k < |x| < \delta_{k+1}\} ,$$
$$0 < \alpha < \pi, \qquad k = 1, 2, \cdots ,$$

with piecewise smooth boundaries.

Thus v has a nodal domain $C_{\alpha k} \subset R_p$ for all $p > 0$; in fact, for arbitrary $p > 0$, choose k large enough so that $\delta_k \geqq p$, and clearly $x \in C_{\alpha k}$ implies that $|x| > \delta_k \geqq p$ and $x \in C_\alpha \subset R$, so that $x \in R_p$. Since

$$\sum_{i,j=1}^{n} a_{ij}(x)z^i z^j \leqq \wedge(x)|z|^2 \leqq f(r)|z|^2 = A(x)|z|^2 , \qquad z \in E^n ,$$

and $b(x) \geqq g(|x|) = B(x)$, equation (1) majorizes equation (3). It then follows from a known comparison theorem [11, p. 514] that the smallest eigenvalue μ of the problem

$$-Lw = \mu w \text{ in } C_{\alpha k} , \qquad w = 0 \text{ on } \partial C_{\alpha k}$$

satisfies $\mu \leqq 0$. Let $M_{\alpha k t} = \{x \in C_{\alpha k} : \delta_k < |x| < t\}$, $\delta_k < t \leqq \delta_{k+1}$, and let $\mu(t)$ denote the smallest eigenvalue of the problem

$$-Lw = \mu(t)w \text{ in } M_{\alpha k t} , \qquad w = 0 \text{ on } \partial M_{\alpha k t} .$$

Since $\mu(t)$ is monotone nonincreasing in $\delta_k < t \leqq \delta_{k+1}$ [3], and since $\mu(\delta_{k+1}) \leqq 0$ and $\lim_{t\to\delta_k+} \mu(t) = +\infty$, there exists a number T in $(\delta_k, \delta_{k+1}]$ such that $\mu(T) = 0$. This means that $M_{\alpha k T}$ is a nodal domain of a nontrivial solution u_k of (1), and since $M_{\alpha k T} \subset C_{\alpha k} \subset R_p$ for arbitrary $p > 0$ provided k is sufficiently large, equation (1) is oscillatory in R. This completes the proof of Theorem 1.

To prove Theorem 2, we use Moore's oscillation theorem [10, p. 127] to deduce that the ordinary differential equation (5) is oscillatory in $0 < r < \infty$ on account of the hypotheses (8). The remainder of the proof follows that of Theorem 1 without change.

If $\wedge(x)$ is bounded in R, say $\wedge(x) \leqq \wedge_1$, $x \in R$, we can choose $f(r) = \wedge_1$, $0 \leqq r < \infty$. Then, for $n = 2$, the first condition (7) is fulfilled and hence the first statement of Theorem 3 follows from Theorem 1. For $n \geqq 3$, the first condition (8) is fulfilled, and $h_n(r) = r^{2-n}/(n-2)\wedge_1$. By hypothesis there exists a number $\delta > 0$ such that (10) holds. Let $m = 1 + \delta/(n-2)$. Then one easily checks that the condition (10) implies the second condition (8), and hence the second statement of Theorem 3 follows from Theorem 2.

The hypothesis (11) of Theorem 4 implies that there exist constants r_0 and γ such that

$$r^2 g(r) > \gamma > \wedge_1[\lambda_\alpha + (n-2)^2/4]$$

provided that $r > r_0$. We then compare (5) with the Euler equation

$$\frac{d}{dr}\left[\wedge_1 r^{n-1}\frac{d\rho}{dr}\right] + (\gamma - \wedge_1\lambda_\alpha)r^{n-3}\rho = 0 \ , \tag{13}$$

with solutions $\rho = r^\beta$, where β satisfies

$$\beta^2 + (n-2)\beta + \gamma/\wedge_1 - \lambda_\alpha = 0 \ .$$

Since $\gamma > \wedge_1[\lambda_\alpha + (n-2)^2/4]$, equation (13) is oscillatory in (r_0, ∞). Then also (5) is oscillatory by Sturm's comparison theorem on account of the hypotheses

$$f(r) = \wedge_1 \ , \qquad r^{n-1}[g(r) - \lambda_\alpha r^{-2} f(r)] > (\gamma - \wedge_1\lambda_\alpha)r^{n-3} \ .$$

The proof of Theorem 4 is now completed in the same way as that of Theorem 1.

To prove Theorem 5, suppose to the contrary that (1) is oscillatory in R. Under the stated hypotheses, it is easily checked that (1) is majorized by the equation

$$\sum_{i=1}^n \wedge_0 D_i^2 v + B_0(x)v = 0 \qquad (B_0(x) = g_0(|x|),\ x \in R) \ , \tag{14}$$

and hence there exists a nodal domain $N_r \subset R_r$ of some nontrivial solution of (14) for all $r > 0$ (by an argument similar to that used in the proof of Theorem 1). Then *every* solution of (14) vanishes at some point of $N_r \cup \partial N_r$ by the n-dimensional analogue of Sturm's separation theorem [1]. However, (14) has radial solutions $v(x) = \rho(r)$ $(r = |x|)$, where ρ satisfies the ordinary differential equation (the analogue of (5))

$$\wedge_0 \frac{d}{dr}\left(r^{n-1}\frac{d\rho}{dr}\right) + r^{n-1}g_0(r)\rho = 0 \,. \tag{15}$$

The hypothesis (12) implies that there exist constants r_0 and γ such that

$$r^2 g_0(r) < \gamma < (n-2)^2 \wedge_0/4$$

for $r > r_0$. Thus the Euler equation

$$\wedge_0 \frac{d}{dr}\left(r^{n-1}\frac{d\rho}{dr}\right) + \gamma r^{n-3}\rho = 0$$

is nonoscillatory, and also (15) is nonoscillatory by Sturm's comparison theorem. This means that there exists a solution $v(x) = \rho(r)$ of (14) and a number r_0 such that $v(x)$ is free of zeros in R_r for all $r > r_0$, and the contradiction establishes Theorem 5.

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