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ON A PAPER OF RAO

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In this paper, we give an internal proof of Rao's theorem on meromorphic functions of bounded characteristic, i.e., a proof not using uniformization.

In addition, we discuss the classification theory of Riemann surfaces as it pertains to the class O_L of hyperbolic Riemann surfaces which admit no nonconstant Lindelöfian meromorphic functions. In particular, we show that $U_{HB} \subset O_L$ where U_{HB} denotes the class of hyperbolic Riemann surfaces on which there exist at least one bounded MHB minimal function.

We also show that there is no inclusion relation between O_L and O^n_{HD}, n a natural number, where O^n_{HD} denotes the class of hyperbolic Riemann surfaces for which the dimension of the vector lattice HD is at most n.

Finally, we generalize the F. and M Riesz theorem for H_1 of the unit disc to arbitrary open hyperbolic Riemann surfaces.

Let R be hyperbolic and R' be an arbitrary Riemann surface. The mapping $\varphi: R \to R'$ is called a Lindelöfian mapping, if for each $a' \in R'$,

$$G(z, a', \varphi) = \sum_{\varphi(a)=a'} n(a) g_R(z, a)$$

is convergent for $\varphi(z) \neq a'$ where $n(a)$ denotes the multiplicity of φ at a and $g_R(z, a)$ is the Green's function of R with pole at a. If R' is the Riemann sphere, then φ is called a Lindelöfian meromorphic function. Sario and Noshiro [6] have generalized the Nevanlinna theory to the class of meromorphic functions on an arbitrary Riemann surface and have shown that for hyperbolic surfaces, the meromorphic functions of bounded characteristic are precisely the Lindelöfian meromorphic functions. Furthermore, they have shown that a meromorphic function $\varphi(z)$ on a hyperbolic Riemann surface R has bounded characteristic if and only if for each complex number a'

$$\log|\varphi(z) - a'| = G(z, \infty, \varphi) - G(z, a', \varphi) + \varphi_{a'}(z) - \varphi'_{a'}(z)$$

where $\varphi_{a'}(z)$ and $\varphi'_{a'}(z)$ are positive harmonic functions on R.

Let us now turn to the theorem of Rao.

THEOREM 1. *If a nonconstant meromorphic function φ on a hyperbolic Riemann surface R has bounded characteristic, then there exists at most one complex number a' such that the difference between* *the quasibounded components of* $\varphi_{a'}$ *and* $\varphi'_{a'}$ *is constant.*

As stated in the introduction, we now give an internal proof of Rao's theorem. For the proof, we need the following lemmas.

Let Δ_1 denote the set of points in the Martin boundary Δ of R where the Martin function k_b cannot be represented as the sum of two nonproportional potentials. In addition, let χ denote the canonical measure of 1 and f a continuous mapping of R into a compact space. Denote by I_b the class of open sets $G \subset R$ for which $(k_b)_{R-G'}$ the infimum of the class of positive superharmonic functions on R which are quasi everywhere on $R - G$ no smaller than k_b, is a potential. Let $f^{\wedge}(b) = \bigcap_{G \in Ib} \overline{f(G)}$. If $f^{\wedge}(b)$ consists of a single point, we denote the point by $\hat{f}(b)$. We shall now establish the following result.

LEMMA 1. *If s is a singular positive harmonic function on R, then s is defined χ* a.e. *on Δ_1 and is* 0 χ a.e.

Proof. Since s is singular, its quasibounded component is 0. It follows from a result in [2] that s is defined χ a.e. on Δ_1 and that

$$0 = \int_{\Delta 1} \hat{s}(b) k_b d\chi(b) .$$

Since $\hat{s}$ and k_b are positive, it follows that $\hat{s} = 0$ χ a.e.

A different proof of this result can be found in [3]. In addition to Lemma 1, we will use the following, which are proved in [3].

LEMMA 2. *$G(z, a', \varphi)$ has the fine limit* 0 χ a.e. *on* Δ_1

LEMMA 3. *If φ is a nonconstant Lindelöfian meromorphic function on a hyperbolic Riemann surface R, then $\hat{\varphi}$ is defined χ* a.e. *on Δ_1 and the set of points $\hat{\varphi}(b)$ is a set of positive capacity.*

We are now able to prove Theorem 1.

Proof. Suppose the conclusion of the theorem is false. Then for some function φ of bounded characteristic on R and two complex numbers $a'_1 \neq a'_2$,

$$\log |\varphi(z) - a'_i| = G(z, \infty, \varphi) - G(z, a_i, \varphi) + k_i + s_i(z) - s'_i(z)$$

for $i = 1$ and 2. s_i and s'_i are singular positive harmonic functions on R and k_i is a constant.

Since φ has bounded characteristic, it is Lindelöfian, and we deduce from Lemmas 1, 2, and 3 that

$$\log |\hat{\varphi}(b) - a_i'| = k_i , \qquad i = 1, 2,$$

for almost all $b \in \Delta_1$.

Hence the points $\hat{\varphi}(b)$ lie on both circles

$$|\zeta - a_i'| = e^{k_i}, \qquad i = 1, 2, a_1' \neq a_2'.$$

In view of the second part of Lemma 3, this is a contradiction and the theorem is proved.

2. We now turn to the class O_L. Rao [5] has shown that $O_{HB} \subset O_L$. We shall prove the much stronger result that $U_{HB} \subset O_L$. In addition, we shall prove a stronger form of Rao's Corollary 2.

Let φ be an analytic mapping from a Riemann surface R into a Riemann surface R'. φ is called a Fatou mapping if φ has a continuous extension to a mapping from the Wiener compactification of R into the Wiener compactification of R'.

It is shown in [2] that every Lindelöfian mapping is a Fatou mapping. It is also shown in [2] that if $R \in U_{HB}$, there exists no nonconstant Fatou mapping of R into a parabolic surface. As immediate consequences of these results, we obtain the following:

THEOREM 2. $U_{HB} \subset O_L$.

THEOREM 3. *If there exists a nonconstant Lindelöfian map φ from a hyperbolic Riemann surface R into a parabolic Riemann surface R', then $R \notin U_{HB}$.*

3. Constantinescu and Cornea [2] have shown that if R is of class U_{HB}, then $R \in U_{HD}$ where U_{HD} denotes the class of Riemann surfaces on which there exist at least one bounded MHD minimal function. Punch out n pairwise disjoint discs $F_1, \cdots, F_n$ from R and reflect $R - \bar{F}_i$ about ∂F_i for $i = 1, \cdots, n$. Weld the reflections to $R - \bigcup_{i=1}^n F_i$ and denote the resulting surface by W. If $R \in U_{HB}$, then $W \in U_{HB}$. Since R is also of class U_{HD}, it follows that the Royden harmonic boundary of W possesses at least $n+1$ points of positive harmonic measure and hence that $W \notin O_{HD}^n$. But by Theorem 2, $W \in O_L$. Hence $O_L \not\subset O_{HD}^n$. Since $O_{HD}^n \not\subset O_L$, we have established the following result.

THEOREM 4. *There is no inclusion relation between O_{HD}^n and O_L.*

We remark that Theorem 4 contains the result of Rao that there is no inclusion relation between O_{HD} and O_L.

It would be interesting to know if Theorem 4 can be strengthened to read that there is no inclusion relation between U_{HD} and O_L. The author has investigated this question but has been unable to settle it.

4. Let us now turn to the class H_1 of analytic functions on an open hyperbolic Riemann surface R whose moduli possess a harmonic majorant. We shall prove the following result.

THEOREM 5. *If $f \in H_1$, then f is defined χ a.e. on Δ_1 and*

$$f = \int_{\Delta 1} f(b) k_b d\chi_b \ .$$

Proof. Since $f \in H_1$, it follows that the Ref and the Imf can be written as the difference of quasibounded harmonic functions. Hence

$$\mathrm{Ref} = \int_{\Delta 1} \widehat{\mathrm{Ref}}\,(b) k_b d\chi(b)$$

and

$$\mathrm{Imf} = \int_{\Delta 1} \widehat{\mathrm{Imf}}\,(b) k_b d\chi(b) \ .$$

Thus

$$f = \int_{\Delta 1} \hat{f}(b) k_b d\chi(b).$$

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