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CONVOLUTION TRANSFORMS WHOSE INVERSION FUNCTION HAS COMPLEX ROOTS IN A WIDE ANGLE

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In this paper a class of convolution transforms:

(1.1) $$f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$$

whose kernels $G(t)$ **satisfy**

(1.2) $$G(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} [E(s)]^{-1} \cdot e^{st}ds$$

where

(1.3) $$E(s) = \prod_{k=1}^{\infty}(1 - s/a_k) \quad \text{or} \quad E(s) = \prod_{k=1}^{\infty}(1 - s/a_k)\exp(sRe\, a_k^{-1})$$

will be treated. Investigation of properties will be carried for the subclass defined by the restriction on a_k **as follows:**

(a) **For some** ψ, $0 < \psi < \pi/2$

$$\min_{n=0,1,2} |n\pi - \arg a_k| \leqq \psi \quad \text{where} \quad a_k \neq 0 .$$

(b) **For some** $0 < q < 1$

and integer l $\quad |a_{k+l}| \geqq q^{-1}|a_k| \quad$ for all $\quad k \geqq k_0$.

It should be mentioned that the restriction (a) on the argument of a_k is much weaker than those used in other subclasses of convolution transforms defined by (1.1), (1.2) and (1.3) that were investigated before.

I. I. Hirschman and D. V. Widder [4] treated a class of transforms for which $\arg a_k$ tend to either 0 or π. J. Dauns and D. V. Widder [1] and the author [2] studied the case $E(s) = \prod_{k=1}^{\infty}(1 - s^2/a_k^2)$ for which $|\arg a_k| \leqq \psi < \pi/4$, that is: The sequence of roots contains pairs of a_k and $-a_k$. A milder way of coupling was introduced by the author [3]. The question that arises is: Can we relax the restriction on the argument of the a_k's and still have the transforms and their inversion formulae? Of course it was shown [1, p. 442] that in some simple cases the analogous inversion formula to that of Hirschman and Widder does not hold. Examples can be given to show that in some cases (1.2) does not converge. Here a restriction on the growth of the roots is given (b) which assures us of the convergence of (1.2) and helps us to prove that

(1.4) $$\lim_{m\to\infty} P_m(D)f(x) = \prod_{k=1}^{\infty}(1 - a_k^{-1}D)f(x) = \varphi(x) \quad \text{a.e.}$$

where

$$D \equiv \frac{d}{dx}\,.$$

We shall assume for convenience that $|a_k| \leqq |a_{k+1}|$ and also that $E(s) = \prod_{k=1}^{\infty}(1 - s/a_k)$ since treating $E(s) = \prod_{k=1}^{\infty}(1 - s/a_k)\exp(s \operatorname{Re} a_k^{-1})$ would mean only shifting the argument t of $G(t)$ by the number $\sum \operatorname{Re} a_k^{-1}$.

It should be noted that the harder part of the proof is an estimate of $E(s)$ (§ 2) and an estimate on $|G_m(t)| = |P_m(D)G(t)|$ (in § 4) the results achieved for the later had not been published for the nonvoid intersection of our class and the class of variation diminishing transforms.

2. An estimate for $E(s)$.

THEOREM 2.1. *Suppose that* $E(s) = \prod_{k=1}^{\infty}(1 - s/a_k)$ *and the sequence* $\{a_k\}$ *satisfies conditions* (a) *and* (b) *and let* $0 < \eta < \pi/2 - \psi$, *then there exist* $A(n) > 0$ *and* $B(n) > 0$ *such that*

(2.1) $$|E(re^{i\theta})| \geqq (A(n) + B(n)r^{2n})^{1/2}$$

for any n *and* r *uniformly for* $\psi + \eta \leqq \theta \leqq \pi - \psi - \eta$ *and* $\pi + \psi + \eta \leqq \theta \leqq 2\pi - \psi - \eta$.

Proof. Without loss of generality we may assume $|a_k| \leqq |a_{k+1}|$. We define $\varphi_k = \arg a_k$ and have

$$|1 - re^{i\theta}/|a_k|e^{i\varphi_k}|^2 = 1 - 2\frac{r}{|a_k|}\cos(\theta - \varphi_k) + \frac{r^2}{|a_k|^2}$$

$$\geqq 1 - 2\frac{r}{|a_k|}\cos\eta + \frac{r^2}{|a_k|^2} = \left[1 - \left(1 + \frac{1}{2}\tan^2\eta\right)^{-1}\right]$$

(2.2) $$+ \left[\left(1 + \frac{1}{2}\tan^2\eta\right)^{-1} - 2\frac{r}{|a_k|}\cos\eta + \frac{r^2}{|a_k|^2}\cos^2\eta\left(1 + \frac{1}{2}\tan^2\eta\right)\right]$$

$$+ \frac{1}{2}\frac{r^2}{|a_k|^2}\sin^2\eta \geqq \frac{1}{2}\cdot\frac{\tan^2\eta}{1 + \frac{1}{2}\tan^2\eta} + \frac{1}{2}\frac{r^2}{|a_k|^2}\sin^2\eta\,.$$

Therefore

$$\left|\prod_{k=1}^{n}(1 - re^{i\theta}/a_k)\right|^2 \geqq \left(\frac{\frac{1}{2}\tan^2\eta}{1+\frac{1}{2}\tan^2\eta}\right)^n$$

$$+\left(\frac{\sin^2\eta}{2}\right)^n \frac{r^{2n}}{|a_1|^2\cdots|a_n|^2} \equiv A_1(n) + B_1(n)r^{2n} .$$

To complete the proof it is enough to show that there exists a constant $c > 0$ independent of r and θ (in its specified angle) such that:

$$\left|\prod_{n+1}^{\infty}(1 - re^{i\theta}/a_k)\right|^2 \geqq c > 0 .$$

We shall write

$$\prod_{k=n+1}^{\infty}\cdots = \left(\prod_{k=n+1}^{n_1(r)}\cdots\right)\cdot\left(\prod_{k=n_1(r)+1}^{n_2(r)}\cdots\right)\cdot\left(\prod_{k=n_2(r)+1}^{\infty}\cdots\right)$$
$$\equiv I_1(r)\cdot I_2(r)\cdot I_3(r) .$$

Choose $n_1(r)$ as the largest integer satisfying

$$k \leqq n_1(r) , \qquad |a_k| < r/2\cos\eta .$$

If $n_1(r) < n+1$ then $I_1(r) = 1$; otherwise

$$I_1(r) = \left|\prod_{k=n+1}^{n_1(r)}(1 - re^{i\theta}/a_k)\right|^2 \geqq \prod_{k=n+1}^{n_1(r)}\left(1 - \frac{2r}{|a_k|}\cos\eta + \frac{r^2}{|a_k|^2}\right) \geqq 1 .$$

We choose $n_2(r)$ as $n_2(r) = \min\{l : l \geqq n+1,\ k > l$ implise $|a_k| > 4r\cos\eta\}$ and therefore have

$$I_3(r) \geqq \prod_{k=n_2(r)+1}^{\infty}\left(1 - 2\frac{r}{|a_k|}\cos\eta + \frac{r^2}{|a_k|^2}\right) \geqq \prod_{k=n_2(r)+1}^{\infty}\left(1 - 2\frac{r}{|a_k|}\cos\eta\right) .$$

Using condition (b) and the definition of $n_2(r)$ we obtain

$$I_3(r) \geqq \prod_{n=0}^{\infty}\left(1 - \frac{1}{2}q^n\right)^l = c_1(q) > 0 \quad \text{for} \quad 0 < q < 1 .$$

We shall estimate $I_2(r)$ by (2.2) as follows

$$I_2(r) \geqq \prod_{k=n_1(r)+1}^{n_2(r)}\left(\frac{1}{2}\frac{\tan^2\eta}{1+\frac{1}{2}\tan^2\eta} + \frac{r^2}{|a_k|^2}\cdot\frac{\sin^2\eta}{2}\right)$$

$$\geqq \left(\frac{1}{2}\frac{\tan^2\eta}{1+\frac{1}{2}\tan^2\eta}\right)^{n_2(r)-n_1(r)-1}$$

(if $n_1(r) < n+1$ then instead of $n_2(r) - n_1(r) - 1$ we should write $n_2(r) - n - 1$). We can estimate $n_2(r) - n_1(r) - 1$ from above as follows; we shall find the smallest m satisfying $q^{-m} > 4r\cos\eta/r(2\cos\eta)^{-1} = 8\cos^2\eta$ which we call m_0, by (b) $m_0 \cdot l > n_2(r) - n_1(r)$.

Combining these results

$$\left|\prod_{k=n+1}^{\infty}(1 - re^{i\theta}/a_k)\right|^{-2} = I_1(r)\cdot I_2(r)I_3(r)$$
$$\geqq \prod_{n=1}^{\infty}\left(1 - \frac{1}{2}q^n\right)^l \cdot \left(\frac{1}{2}\frac{\tan^2\eta}{1+\frac{1}{2}\tan^2\eta}\right)^{m_0\cdot l} = c > 0\,.$$

COROLLARY 2.1.a. *Under assumptions* (a) *and* (b) *the kernel function $G(t)$ satisfies*

$$E(s)^{-1} = \int_{-\infty}^{\infty} e^{-st}G(t)dt \tag{3.2}$$

$G(t) \in C^\infty(-\infty, \infty)$ and

$$G^{(n)}(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{s^n e^{st}ds}{E(s)}\,. \tag{3.3}$$

REMARK 2.1.b. In Theorem 2.1 it is shown that if (a) and (b) are satisfied then for $0 < \eta < \pi/2 - \psi$

$$|E(re^{i\theta})| \geqq C(q, \eta) \tag{3.4}$$

for

$$\left\{\theta : \left|\theta - \frac{\pi}{2}\right| < \frac{\pi}{2} - \psi - \eta\right\} \cup \left\{\theta : \left|\theta - \frac{3\pi}{2}\right| < \frac{\pi}{2} - \psi - \eta\right\}$$

and $C(q,\eta)$ does not depend on r or the smallest $|a_k|$.

3. Asymptotic estimate of $G(t)$.

Define α_1 and α_2 by:

$$\begin{aligned}\alpha_1 &= \max\{\operatorname{Re} a_k, -\infty \,|\, \operatorname{Re} a_k < 0\} \quad \text{and}\\ \alpha_2 &= \min\{\operatorname{Re} a_k, \infty \,|\, \operatorname{Re} a_k > 0\}\,.\end{aligned} \tag{3.1}$$

THEOREM 3.1. *Suppose $E(s) = \prod_{k=1}^{\infty}(1 - s/a_k)$, the sequence $\{a_k\}$ satisfies conditions* (a) *and* (b) *and let $G(t)$ be defined by* (1.2), *then:*

(i) $\alpha_1 = -\infty$ *implies*

$$G^{(n)}(t) = o(e^{ct}) \qquad t \to \infty$$

for all c, $c < o$.

(ii) $\alpha_1 > -\infty$ *implies*

$$G^{(n)}(t) = \sum_{l=1}^{L} P_l(t) \exp(a_{k(l)}t) + o(e^{ct}) \qquad t \to \infty$$

where $\operatorname{Re} a_{k(l)} = \alpha_1$ $1 \leqq l \leqq L(a_{k(j)} \neq a_{k(i)}$ *for* $i \neq j$ *and if* $a_k \neq a_{k(l)}$ $1 \leqq l \leqq L$ *then* $\operatorname{Re} a_k \neq \alpha_1)$ $P_l(t)$ *are polynomials of degree* μ_l *where* $\mu_l + 1$ *is the multiplicity of* $a_{k(l)}$ *in* $\{a_k\}$ *and* $c < \alpha_1$.

(iii) $\alpha_2 = \infty$ *implies*

$$G^{(n)}(t) = o(e^{ct}) \text{ for all } c, c > 0 \, . \qquad t \to -\infty$$

(iv) $\alpha_1 < \infty$ *implies*

$$G^{(n)}(t) = \sum_{l=L+1}^{L+M} P_l(t) \exp(a_{k(l)}t) + o(e^{ct}) \qquad t \to -\infty$$

where $\operatorname{Re} a_{k(l)} = \alpha_2$ *for* $L + 1 \leqq l \leqq L + M$, $P_l(t)$ *are as in* (b) *and* $c > \alpha_2$.

Proof. The proof follows the well established method of Hirschman and Widder [5, p. 108]. In order to use this method it is enough to show that

$$|E(\sigma + i\tau)|^{-1} = 0(|\tau|^{-n}) \qquad |\tau| \to \infty$$

uniformly for $-A \leqq \sigma < A$ for every finite A. By Theorem 2.1 we have for $|\tau|/|\sigma| > \tan(\psi + \eta)$ and therefore for $|\tau| > A\tan(\psi + \eta)$ (where $\eta > 0$ $\psi + \eta < \pi/2$ and ψ is defined in condition (a))

$$|E(\sigma + i\tau)|^{-1} \leqq (A(n) + B(n)|\sigma + i\tau|^{2n})^{-1/2} \leqq B(n)^{-1/2}|\tau|^{-n} \, .$$

4. **$G_m(t)$ and properties.** Define $G_m(t)$ by

$$E_m(s)^{-1} = \int_{-\infty}^{\infty} e^{-st} G_m(t)dt \tag{4.1}$$

where

$$E_m(s) = \prod_{k=m+1}^{\infty} (1 - s/a_k) \, . \tag{4.2}$$

By Theorem 2.1 and Corollary 2.1.a we have $G_m(t) \in C^\infty(-\infty, -\infty)$ and for $m = 0, 1, 2, \cdots$

$$G_m^{(n)}(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{s^n e^{st}}{E_m(s)} ds \, . \tag{4.3}$$

The asymptotic estimates of Theorem 3.1 for $G_m(t)$ that satisfies condition (a) and (b) will be useful; however the following new estimate will be essential for the proof of the inversion formula.

THEOREM 4.1. *Let conditions* (a) *and* (b) *hold and* $|a_k| \leqq |a_{k+1}|$, *then*

$$(4.4)\qquad \begin{aligned} |G_m(t)| \leqq M_1 |a_{m+1}| \exp\left(-\frac{1}{2}\cos\psi\, |a_{m+1}t|\right) \\ + M_2 |a_{m+2}| \exp\left(-\frac{1}{2}\cos\psi\, |a_{m+2}t|\right). \end{aligned}$$

Proof. We shall divide the proof of (4.4) into two cases

(I) $|a_{m+2}| \leqq 4|a_{m+1}|/\cos\psi$

(II) $|a_{m+2}| \geqq 4|a_{m+1}|/\cos\psi$.

To show (4.4) in Case I we write $d_m = 1/2\cos\psi\, |a_{m+1}|$ and since $d_m < |a_{m+1}|\cos\psi$ we have

$$(4.5)\qquad G_m(t) = \frac{1}{2\pi i}\int_{d_m - i\infty}^{d_m + i\infty} \frac{e^{st}}{E_m(s)}\,ds = \frac{1}{2\pi i}\int_{-d_m - i\infty}^{-d_m + i\infty} \frac{e^{st}}{E_m(s)}\,ds .$$

Using the first integral we have

$$(4.6)\qquad |G_m(t)| \leqq \frac{1}{2\pi} e^{d_m t} \int_{-\infty}^{\infty} |E_m(d_m + i\tau)|^{-1} d\tau .$$

To estimate $G_m(t)$ we have to estimate $E_m(d_m + i\tau)$

$$\begin{aligned} |E_m(d_m + i\tau)| &= \prod_{i=1}^{2} \frac{1}{|a_{m+i}|} |a_{m+i} - d_m - i\tau| \cdot \prod_{k=m+3}^{\infty} \left|1 - \frac{d_m + i\tau}{a_k}\right| \\ &\equiv I_1(\tau)\cdot I_2(\tau) \end{aligned}$$

$$\begin{aligned} &|a_{m+i}|^{-2}|a_{m+i} - d_m - i\tau|^2 \\ &\quad = (\operatorname{Re} a_{m+i} - d_m)^2 |a_{m+i}|^{-2} + (\tau - \operatorname{Im} a_{m+i})^2 |a_{m+i}|^{-2} \\ (4.7)\quad &\quad \geqq \frac{1}{4}\cos^2\psi + (\tau^2 - 2\tau \operatorname{Im} a_{m+i} + (\operatorname{Im} a_{m+i})^2)|a_{m+i}|^{-2} \\ &\quad \geqq \frac{1}{8}\cos^2\psi + \frac{1}{8}\frac{\cot^2\psi}{1 + \frac{1}{8}\cot^2\psi}\tau^2 |a_{m+i}|^{-2} \\ &\quad + \left(\left[1 + \frac{1}{8}\cot^2\psi\right]^{-1}\tau^2 - 2\tau \operatorname{Im} a_{m+i} \right. \\ &\quad \left. + \left(1 + \frac{1}{8}\cot^2\psi\right)(\operatorname{Im} a_{m+i})^2\right)|a_{m+1}|^{-2} \geqq A(\psi) + B(\psi)\tau^2 |a_{m+i}|^{-2} \end{aligned}$$

where $A(\psi) = 1/8\cos^2\psi$ and

$$B(\psi) = \frac{1}{8}\frac{\cot^2\psi}{1 + \frac{1}{8}\cot^2\psi} .$$

Using (4.7) and since $|a_{m+2}| \geqq |a_{m+1}|$

$$\begin{aligned} I_1(\tau) &\geqq \{A(\psi)^2 + 2A(\psi)B(\psi)|a_{m+2}|^{-2}\tau^2 + B(\psi)^2|a_{m+2}|^{-4}\tau^4\}^{1/2} \\ &\geqq k_1(1 + \tau^2|a_{m+1}|^{-2}) . \end{aligned}$$

To estimate $I_2(\tau)$ we recall from Remark 2.1.b that when $d_m + i\tau = re^{i\theta}$ and $\psi + \eta \leqq \theta \leqq 2/\pi$ or $3\pi/2 \leqq \theta \leqq 2\pi - \psi - \eta$, that is when $|\tau| \geqq d_m \tan(\psi + \eta)$, $I_2(\tau) \geqq c > 0$.

When $|\tau| \leqq d_m \tan(\psi + \eta)$ we choose $n_1 \geqq m + 3$ such that $|a_{n_1}| \geqq 2d_m(1 + \tan(\psi + \eta))$ and therefore

$$\begin{aligned} I_2(\tau) &\geqq \left\{\prod_{k=m+3}^{n_1} A(\psi)\right\}^{\frac{1}{2}} \cdot \prod_{k=n_1+1}^{\infty}\left(1 - \frac{d_m(1 + \tan(\psi + \eta))}{|a_k|}\right) \\ &\geqq A(\psi)^{(n_1-m-3)/2} \cdot \prod_{n=0}^{\infty}\left(1 - \frac{1}{2}q^n\right)^l \end{aligned}$$

and since by condition (b) $n_1 - m - 3$ is bounded regardless of m $I_2(\tau) \geqq c_1$.

Therefore (4.6) and (4.7) yield

$$\begin{aligned} |G_m(t)| &\leqq C_2 e^{d_m t}\int_{-\infty}^{\infty}[1 + |a_{m+1}|^{-2}\tau^2]^{-1}d\tau \\ &\leqq M_1|a_{m+1}|\exp\left(\frac{1}{2}\cos\psi\,|a_{m+1}|\,t\right) . \end{aligned}$$

This estimation though correct for all t is valuable only for $t \leqq 0$; for $t \geqq 0$ we obtain the result taking the second integral of (4.5) into consideration.

In the Case II, $|a_{m+2}| > 4|a_{m+1}|/\cos\psi$, therefore

$$|\operatorname{Re} a_{m+1}| \leqq |a_{m+1}| < \frac{1}{4}|\operatorname{Re} a_{m+2}| .$$

To prove our result for $t \leqq 0$ we use the method of Theorem 3.1 and obtain

$$G_m(t) = a_{m+1}\frac{e^{a_{m+1}t}}{E_{m+1}(a_{m+1})} + \int_{k_m-i\infty}^{k_m+i\infty}\frac{e^{st}ds}{E_m(s)} \tag{4.8}$$

when $\operatorname{Re} a_{m+1} < 0$ and

$$G_m(t) = \int_{k_m-i\infty}^{k_m+i\infty}\frac{e^{st}ds}{E_m(s)} \tag{4.9}$$

when $\operatorname{Re} a_{m+1} > 0$; where $k_m = 1/2|a_{m+2}|\cos\psi$.

Using (4.8) and (4.9) we obtain

$$(4.10)\qquad |G_m(t) \leqq |a_{m+1}| \cdot \frac{e^{|\mathrm{Re}\, a_{m+1}|t}}{E_{m+1}(a_{m+1})} + e^{k_m t}\int_{-\infty}^{\infty}[E_m(k_m + i\tau)]^{-1}d\tau$$

$$|E_{m+1}(a_{m+1})| \geqq \prod_{k=2}^{\infty}\left(1 - \left|\frac{a_{m+1}}{a_{m+k}}\right|\right) \geqq \prod_{n=0}^{\infty}\left(1 - \frac{1}{2}q^n\right)^l .$$

Considerations already used in this theorem show that

$$|E_{m+2}(k_m + i\tau)| \geqq C > 0 .$$

Since $|a_{m+1}| \leqq |a_{m+2}|$ and $|\mathrm{Re}\, a_{m+1} - k_m| \geqq 1/2k_m > |a_{m+1}|$ we have

$$\left|1 - \frac{k_m + i\tau}{a_{m+1}}\right| \cdot \left|1 - \frac{k_m + i\tau}{a_{m+2}}\right| \geqq (A(\psi) + B(\psi)\tau^2|a_{m+2}|^{-2}) .$$

Recalling that

$$E_m(k_m + i\tau) = E_{m+2}(k_m + i\tau) \cdot \left(1 - \frac{k_m + i\tau}{a_{m+1}}\right)\left(1 - \frac{k_m + i\tau}{a_{m+2}}\right)$$

the proof of Case II for $t \leqq 0$ follows immediately. The proof when $t \geqq 0$ is similar taking $-k_m$ instead of k_m.

THEOREM 4.2. *If conditions* (a) *and* (b) *are satisfied and* $|a_k| \leqq |a_{k+1}|$, *then*

$$(4.4)\qquad |G'_m(t)| \leqq \sum_{i=1}^{3} N_i |a_{m+i}|^2 \exp\left(-\frac{1}{2}\cos\psi\, |a_{m+i}t|\right) .$$

Proof. Since $(1 - a_k^{-1}D)G_m(t) = G_{m+1}(t)$ we have

$$G'_m(t) = -a_{m+1}G_{m+1}(t) + |a_{m+1}||G_m(t)| .$$

Using Theorem 4.1 for both m and $m + 1$ we obtain

$$\begin{aligned}|G'_m(t)| &\leqq |a_{m+1}||G_{m+1}(t)| + |a_{m+1}||G_m(t)| \\ &\leqq M_1|a_{m+1}|^2 \exp\left(-\frac{1}{2}\cos\psi\,|a_{m+1}t|\right) \\ &\quad + M_2|a_{m+1}||a_{m+2}|\exp\left(-\frac{1}{2}\cos\psi\,|a_{m+2}t|\right) \\ &\quad + M_1|a_{m+1}||a_{m+2}|\exp\left(-\frac{1}{2}\cos\psi\,|a_{m+2}t|\right) \\ &\quad + M_2|a_{m+1}||a_{m+3}|\exp\left(-\frac{1}{2}\cos\psi\,|a_{m+3}t|\right)\end{aligned}$$

which yields (4.4) easily.

REMARK 4.1.a. If in Theorems 4.1 and 4.2 the restriction, $|a_{m+1}| < L|a_m|$ for some $L > 1$, is added the proofs become obviously shorter and involve only the first term (in each theorem).

REMARK 4.1.b. It can be proved that if conditions (a) and (b) are satisfied and the multiplicity of a_{m+1} and a_{m+2} in $\{a_k\}$ is one then:

(a) $$|G_m(t)| \leqq M|a_{m+1}| \exp(-|\mathrm{Re}\, a_{m+1}||t|)$$

and

(b) $$|G'_m(t)| \leqq K|a_{m+1}|^2 \exp(-|\mathrm{Re}\, a_{m+1}||t|) + N|a_{m+1}||a_{m+2}| \exp(-|\mathrm{Re}\, a_{m+2}||t|) .$$

These results are better than those of Theorem 4.1 and 4.2 but the proof I have uses those theorems. Since Theorems 4.1 and 4.2 are sufficient for the inversion result, I will not prove here these generalizations.

5. Inversion theorems. The results we shall obtain will correspond to the following two different situations: (1) Both α_1 and α_2 are finite. (2) Either α_1 or α_2 is non finite. (α_1 and α_2 were defined in § 3).

THEOREM 5.1. *Suppose*:
(1) *Conditions* (a) *and* (b) *are satisfied.*
(2) *The constants* α_1 *and* α_2 *are finite,* $\left|\int_0^t \varphi(v)dv\right| \leqq Ke^{(\alpha_2-\varepsilon)t}$ *for* $t \geqq 0$ *and* $\left|\int_t^0 \varphi(v)dv\right| \leqq Ke^{(\alpha_1+\varepsilon)t}$ *for* $t \leqq 0$ *for some* $\varepsilon > 0$, *and* $\varphi(t) \in L_1$ (A, B) *for all* A, B *satisfying* $-\infty < A < B < \infty$.
(3) *At a point* x $\int_0^h [\varphi(x+y) - \varphi(x)]dy = o(h)$ $h \to 0$. *Then*

(5.1) $$\lim_{m\to\infty} P_m(D)f(x) = \varphi(x) .$$

Proof. By Theorem 3.1 and assumption 2 we derive the uniform convergence in an interval $a \leqq x \leqq b$ of $\int_{-\infty}^{\infty} G^{(m)}(x-t)\varphi(t)dt$ and therefore

(5.2) $$\frac{d^n}{dx^n} f(x) = \int_{-\infty}^{\infty} G^{(n)}(x-t)\varphi(t)dt$$

and

$$P_m(D)f(x) = \int_{-\infty}^{\infty} G_m(x-t)\varphi(t)dt .$$

To complete the proof we remember that $\int_{-\infty}^{\infty} G_m(t)dt = 1$ and therefore

$$|P_m(D)f(x) - \varphi(x)| = \left|\int_{-\infty}^{\infty} G_m(x-t)[\varphi(t) - \varphi(x)]dt\right|$$
$$= \left|\left\{\int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\infty}\right\} G'_m(x-t)\alpha(t)dt\right| = I_1 + I_2 + I_3$$

where $\alpha(t)$ is given by $\alpha(t) = \int_x^t [\varphi(v) - \varphi(x)]dv$. Using (3) one can choose δ so that for $(x-t) \leqq \delta$ $|\alpha(t)| \leqq \varepsilon|x-t|$ and therefore

$$|I_2| \leqq \varepsilon \int_{x-\delta}^{x+\delta} |G'_m(x-t)||x-t|\,dt = \varepsilon \int_{-\delta}^{\delta} |G'_m(v)||v|\,dv$$
$$\leqq \varepsilon\, 2 \sum_{i=1}^{3} N_i |a_{m+i}|^2 \cdot \int_0^{\delta} v \exp\left(-\frac{1}{2}\cos\psi\, |a_{m+i}|\, v\right)dv$$
$$\leqq \varepsilon\, 2 \sum_{i=1}^{3} N_i \left\{\delta\, |a_{m+i}| \exp\left(-\frac{1}{2}\cos\psi\, |a_{m+i}|\, \delta\right) \cdot 2 \cdot (\cos\psi)^{-1}\right.$$
$$\left. + 4(\cos\psi)^{-2}\right\}.$$

For any fixed δ $|a_{m+i}| \exp(-1/2 \cos\psi\, |a_{m+i}|\, \delta) = o(1)$ $m \to \infty$. Using (2) $|\alpha(t)| \leqq K e^{(\alpha_2 - \varepsilon)t}$ for $t \geqq 0$ and $|\alpha(t)| \leqq K e^{(\alpha_1 + \varepsilon)t}$ for $t \leqq 0$ and therefore

$$|I_3| \leqq K e^{(\alpha_1+\varepsilon)(x+\delta)} \int_{x+\delta}^{\max(x+\delta,0)} \sum_{i=1}^{3} N_i |a_{m+i}|^2 \exp\left(\frac{1}{2}\cos\psi\, |a_{m+i}|\, t\right)$$
$$+ \sum_{i=1}^{3} K N_i |a_{m+i}|^2 \int_{\max(x+\delta,0)}^{\infty} \exp\left(-\frac{1}{2}\cos\psi\, |a_{m+1}|\, t\right) e^{(\alpha_2-\varepsilon)t} dt .$$

Since $\lim_{x\to\infty} xe^{-ax} = 0$ for $a > o$ we obtain $I_3 = o(1)m \to \infty$ and similarly $I_2 = o(1)m \to \infty$.

REMARK 5.1.a. Condition (2) can be replaced by a milder condition (2*) when there are only simple roots on $\operatorname{Re} z = \alpha_1$ and $\operatorname{Re} z = \alpha_2$. (2*) $|\alpha(t)| \leqq K\chi(t)e^{\alpha_2 t}$ for $t \leqq 0$ and $|\alpha(t)| \leqq K\chi(t)e^{\alpha_1 t}$ for $t \geqq 0$ where $\chi(t) > 0$ and $\int_{-\infty}^{\infty} \chi(t)dt < \infty$. In the proof the only change is in showing the uniform convergence (on a finite interval) of (5.2).

If for some $G(t)$ $\alpha_1 = -\infty$ then for $G^*(t), G^*(t) \equiv G(-t)$ $\alpha_2 = \infty$ and vice versa. We shall treat therefore such kernels for which $\alpha_1 = -\infty$. For the inversion result we shall need the following lemma.

LEMMA. 5.2. *If conditions* (a) *and* (b) *are satisfied and* $\alpha_1 = -\infty$, *then* $G(t) = 0$ *for* $t \geqq 0$.

Proof. Let $[1 - s/a_i]^{-1} = \int_{-\infty}^{\infty} e^{-st} g_i(t)dt$, then since $\operatorname{Re} a_i > 0$

$$g_i(t) = \begin{cases} a_i e^{a_i t} & t < 0 \\ 0 & t > 0 \end{cases}$$

Define $G_m^*(t) = g_1 * g_2 * \cdots * g_m(t)$, it is clear that $G_m^*(t) = 0$ for $t > 0$ (by induction) and that

$$G_m^*(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} \Big[\prod_{k=1}^{m} \Big(1 - \frac{s}{a_k}\Big)\Big]^{-1} ds .$$

We have, for all m

$$G(t) = \int_{-\infty}^{\infty} G_m^*(v) G_m(t - v) dv = \int_{-\infty}^{0} G_m^*(v) G_m(t - v) dv$$

$$\begin{aligned} |G_m^*(t)| &\leqq |g_1(\ )| * |g_2(\ )| * \cdots * |g_m(t)| \\ &= \prod_{i=1}^{m} \frac{|a_i|}{\operatorname{Re} a_i} h_1 * \cdots * h_m(t) \end{aligned}$$

where

$$h_i(t) = \begin{cases} \operatorname{Re} a_i \exp(\operatorname{Re} a_i t) & t < 0 \\ 0 & t > 0 \end{cases}.$$

It is well known that $|h_1 * \cdots * h_m(t)| \leqq \min_{1 < k \leqq m} \operatorname{Re} a_k$ (see [5, p. 138]) and that for $m \geqq m_0$ $\min_{1 < k \leqq m} \operatorname{Re} a_k = \alpha_2$. Therefore for $m \geqq m_0$

$$|G_m^*(t)| \leqq \frac{\alpha_2}{\cos^m \psi} \quad (|a_i| \leqq (\cos \psi)^{-1} |\operatorname{Re} a_i|) .$$

Since we have

$$\begin{aligned} |G(t)| &\leqq \alpha_2 \cdot (\cos \psi)^{-m} \int_{-\infty}^{0} |G_m(t - v)| dv \\ &= \alpha_2 (\cos \psi)^{-m} \cdot \int_{t}^{\infty} |G_m(v)| dv \end{aligned}$$

using Theorem 4.1 we obtain for $t > 0$

$$\begin{aligned} |G(t)| \leqq \alpha_2 (\cos \psi)^{-m} \cdot 2(\cos \psi)^{-1} \Big[& M_1 \exp\Big(-\frac{1}{2} \cos \psi \, |a_{m+1}| |t|\Big) \\ & + M_2 \exp\Big(-\frac{1}{2} \cos \psi \, |a_{m+2}| |t|\Big)\Big] . \end{aligned}$$

Condition (b) implies for every $t \neq 0$

$$(\cos \psi)^{-m} \exp\Big(-\frac{1}{2} \cos \psi \, |a_{m+i}| |t|\Big) = o(1) \qquad m \to \infty$$

for $i = 1, 2, \cdots$. Therefore $G(t) = o(1)$ $m \to \infty$ for $t > 0$ and being independent of m $G(t) = 0$ for $t > 0$. Since $G(t) \in C^\infty(-\infty, \infty)$ $G(t) = 0$ for $t = 0$ also.

THEOREM 5.3. *Suppose*:

(1) *Conditions* (a) *and* (b) *are satisfied.*

(2) $\alpha_1 = -\infty$, *$\varphi(t)$ is defined for $t \geqq M$ and $\varphi(t) \in L_1(M, R)$ for all $R < \infty$ and* $\int_M^t \varphi(t)dt \leqq Ke^{(\alpha_2 - \varepsilon)t}$.

(3) *Conditions* (3) *of* Theorem 5.1 *is satisfied.*

Then for $x > M$

$$\lim_{m \to \infty} P_m(D)f(x) = \varphi(x) . \tag{5.3}$$

Proof. The proof is almost identical to that of Theorem 5.1, but for the convergence of

$$\int_{-\infty}^{\infty} G^{(n)}(x - t)\varphi(t)dt \quad \text{and} \quad \int_{-\infty}^{\infty} G_m(x - t)\varphi(t)dt$$

we have to use also Lemma 5.2 (remembering that $G_m(t)$ satisfies also conditions (a) and (b) and $\alpha_1 = -\infty$ and therefore $G_m(t) = 0$ for $t \geqq 0$).

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