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INVARIANT EXTENSIONS OF LINEAR FUNCTIONALS, WITH APPLICATIONS TO MEASURES AND STOCHASTIC PROCESSES

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A theorem is proved slightly stronger than the following. Let G be a set of order-preserving linear operators on a partially-ordered real linear space X, for which there exist sets $G = G_n \supseteqq G_{n-1} \supseteqq \cdots \supseteqq G_0$ with G_0 commutative and such that for $k = 1, \cdots, n$, x in X, g_1 and g_2 in G_k there exist h_1 and h_2 in G_{k-1} satisfying $h_1 g_1 g_2(x) = h_2 g_2 g_1(x)$. If S is a G-invariant subspace such that for all x in X there is an s in S satisfying $s \geqq x$, and f_0 is a G-invariant positive linear functional on S, then f_0 extends to a G-invariant positive linear functional on X. This is used to construct a generalized form of the Banach limit, an ergodic measure on compact Hausdorff spaces, a stationary extension of a relatively stationary stochastic process $x_t (0 \leqq t \leqq \alpha)$ with values in an arbitrary space, and a generalization to arbitrary linear spaces of Krein's extension theorem for positive-definite complex-valued functions.

This paper consists chiefly of one principal theorem (Theorem 2 in §1) on extending positive linear functionals from a subspace S of a linear space X to all of X so as to preserve invariance under a set G of order-preserving linear transformations, together with several applications of that theorem. The set G is assumed to satisfy a condition which we call left-solvability over X, and which is satisfied by every solvable group G. The importance of an algebraic condition like solvability for problems such as this was apparently first recognized by John von Neumann, in a paper [12] in 1929 in which he studied the existence of finitely additive measures invariant under the action of a group of transformations. Our Theorem 2 can readily be seen to be a generalization of a famous extension theorem of Riesz, to which it reduces when $X_1 = X$ and G consists of the identity alone. It also generalizes a lemma of Parthasarathy and Varadhan [10]. A corollary (in §2) which analogously generalizes the Hahn-Banach theorem contains the principal result in a paper by R. P. Agnew and A. P. Morse [1], some of the results in a paper by V. L. Klee [5] and a lemma by M. M. Day [3].

The extension theorem is used in §5 to construct a type of generalized limit for sequences, with larger domain and stronger invariance properties than the familiar Banach limit. In §6 it is used to construct an invariant ergodic measure on compact spaces.

In § 7 we define a general form of stochastic process, whose random variables take values in an arbitrary set, and prove that if such a process on an interval of reals is relatively stationary, it can be extended to the whole real line so as to be stationary. This generalizes a theorem (for real-valued processes) proved by Parthasarathy and Varadhan [10].

In § 8 we digress to define a covariance function for a class of processes somewhat less general than those of § 7 (with values in a real or complex linear space), and to prove a theorem characterizing the functions that are covariances of some process. In particular, the covariances of relatively stationary processes on an interval $(-A, A)$ coincide with the functions that we call positive definite, by a straightforward extension of the meaning of the phrase for complex-valued functions. We use this to generalize a well-known result of M. G. Krein [6] on extending positive-definite complex-valued functions from $(-A, A)$ to $(-\infty, \infty)$.

1. The extension theorems.

In this section we state our principal extension theorems.

Let X be a set, G and H two sets of transformations of X into itself. We shall write g_1g_2 for the composition $g_1 \circ g_2$, and likewise for other compositions.

DEFINITION 1.1. G acts on a subset X_1 of X commutatively to within left H-factors if to each x in X_1 and each g_1 and g_2 in G there correspond h_1 and h_2 in H such that

$$h_1g_1g_2(x) = h_2g_2g_1(x) .$$

DEFINITION 1.2. Let G be a set of transformations acting on a set X, and let X_1 be a subset of X. G is said to be left-solvable over X_1 if there exist sets of transformations $G = G_n \supseteqq G_{n-1} \supseteqq \cdots \supseteqq G_0$ such that for $k = 0, 1, \cdots, n-1, G_{k+1}$ acts on X_1 commutatively to within left G_k -factors, and G_0 is commutative.

The definitions of commutative action to within right H-factors and of right-solvability over X_1 are obvious analogues of (1.1) and (1.2).

In the above definitions and in all later theorems the adjunction of the identity transformation 1 to all sets G, H, G_k, etc. leaves unaltered the properties in (1.1) and (1.2) together with all invariances. So without loss of generality we may and shall assume that all sets G, H, etc., mentioned contain the identity transformation 1.

If G is a semigroup it acts on itself, setting $g(g') = g \circ g'$. We shall say that the semigroup G is *left-solvable* if G is left-solvable over G itself. If G is a semigroup of transformations acting on a set X, the condition that G be left-solvable (over itself) is stronger than the condition that G be left-solvable over X. For then if g_1 and g_2 are in G_{k+1}, there exist h_1, h_2 in G_k such that $h_1g_1g_2(1) = h_2g_2g_1(1)$. This implies that the equation in (1.1) is satisfied for all x, and with an h_1 and h_2 independent of x. M. M. Day defined a concept of left-solvability for semigroups that is slightly stronger than simultaneous left-solvability and right-solvability. The one-sidedness of our condition is not trivial; one of our examples will involve a left-solvable semigroup over X which is not right-solvable over X.

If in (1.2) we add the requirement that all the G_k be groups, then left solvability of G is equivalent to solvability of G as customarily defined, and so is right solvability.

If $\{g_0 = 1, g_1, \cdots, g_m\}$ is a finite group of linear transformations on a linear space X, and we define $T: X \to X$ by setting

$$T_{(x)} = (m + 1)^{-1}[g_0(x) + \cdots + g_m(x)](x \in X) ,$$

we readily see that $g_iT(x) = Tg_i(x) = T^2(x) = T(x)$ for all x. Hence the set $G = \{g_0, \cdots, g_m, T\}$ is a semigroup of transformations. If we take $G_1 = G$, $G_0 = \{1, T\}$, we see that G is both right-solvable and left-solvable over X, the G_0 -factors always being T.

Any finite group $\{g_0 = 1, \cdots, g_m\}$ is similarly contained in a right- and left-solvable semigroup $G = \{g_0, \cdots, g_m, g_{m+1}\}$, where the composition of g_{m+1} with the elements of G is defined by $g_{m+1}g_i = g_ig_{m+1} = g_{m+1}$ $(i = 0, \cdots, m + 1)$.

If G is any set of transformations of a set X into itself (containing as always the identity) G generates a semigroup G^+ as follows.

(1.3) *G^+ consists of all operators of the form $g_1g_2 \cdots g_k$ for all positive integers k and all k-tuples $(g_1, \cdots, g_k)$ of members of G.* We shall often extend sets G to semigroups G^+ without explicit mention of this definition.

For ease of comparison we state our first two theorems together.

THEOREM 1. *Let X be a partially ordered real linear space and S a subspace such that* (1.4) *for each x in X there exists an s in S satisfying $s \geqq x$. Let G and H be sets of order-preserving linear transformations of X into itself such that $H \subseteqq G$ and G acts on X commutatively to within left H-factors. Let S be invariant under G and let $f_0: S \to R$ be a positive G-invariant functional on S. Assume that either*

(i) *H is the identity alone* (*so that* $g_1g_2(x) = g_2g_1(x)$ *for all* x

in X and all g_1, g_2 in G), or else

(ii) *f_0 can be extended to a positive H-invariant linear functional $f_1: X \to R$.*

Then f_0 can be extended to a positive G-invariant linear functional $f: X \to R$.

THEOREM 2. *Let X be a partially ordered linear space and G a set of order-preserving linear transformations of X into itself. Let $X_n: 0 \leqq n < \bar{n}$ be a set of $\bar{n}$ G-invariant subspaces ($\bar{n}$ may be ∞) such that $X_0 \subseteqq X_1 \subseteqq X_2 \subseteqq \cdots$ and $\cup_n X_n = X$. Assume that for $1 < n < \bar{n}$, either*

(i) *G is left-solvable over X_n, and for each x in X_n there is an s in X_{n-1} such that $s \geqq x$; or else*

(ii) *for each x in X_n there is a g in G such that $g(x) \in X_{n-1}$; and for each x in X_n and g_1, g_2, g_3 in G such that $g_1(x)$ and $g_2g_3(x)$ are in X_{n-1}, there are members h_1, h_2 of G such that*

$$h_1g_1g_2g_3(x) = h_2g_2g_3g_1(x) .$$

Then every G-invariant positive linear functional f_0 on X_0 can be extended to a G-invariant positive linear functional f on X.

2. **Proof of theorem 1.** The left-commutative action of G to within left H-factors enters the proof of Theorem 1 via the following lemma.

LEMMA. *Let X be a linear space. Let G and H ($H \subseteqq G$) be sets of transformations such that G acts on X commutatively to within left H-factors. Let f be an H-invariant linear functional on X. Then for every x in X and every finite sequence $(g_1, \cdots, g_n)$ of elements of G, the value of $f(g_1 \cdots g_n(x))$ is invariant under permutation of the g_i.*

We prove this by induction on n. If $n = 1$, the invariance of $f(g_1 \cdots g_n(x))$ under permutation of the g_i is evident. We assume it true for $n < m$ and show it then holds for $n = m$. It is enough to show invariance under interchange of any two consecutive terms of the sequence of g_i. For all but the last two terms this is an immediate consequence of the induction hypothesis. For the last two, by hypothesis there are members h, h' of H such that $hg_{m-1}g_m(x) = h'g_mg_{m-1}(x)$. Then

$$\begin{aligned} f(g_1 \cdots g_m(x)) &= f(hg_1 \cdots g_{m-2}[g_{m-1}g_m(x)]) \\ &= f(g_1 \cdots g_{m-2}h[g_{m-1}g_m(x)]) \\ &= f(g_m \cdots g_{m-2}h'[g_mg_{m-1}(x)]) \end{aligned}$$

$$= f(h'g_1 \cdots g_{m-2}g_m g_{m-1}(x))$$
$$= f(g_1 \cdots g_{m-2}g_m g_{m-1}(x)) ,$$

which completes the proof.

In proving Theorem 1 we shall use the Hahn-Banach theorem. To construct the appropriate subadditive p we define a subspace.

(2.1) If G is commutative, $N = \{0\}$; if hypothesis (ii) holds, N is the set of all ν in X such that for every finite sequence $(g_1, \cdots g_n)$ of members of G, $f_1(g_1 \cdots g_n(\nu)) = 0$. (Thus, if (ii) holds, N is the largest G-invariant subspace on which f_1 vanishes.)

In either case the following is evident.

(2.2) N is a G-invariant subspace of X. Also

(2.3) If $s \in S$ and there is a ν in N such that $s \geqq \nu$, then $f_0(s) \geqq 0$.

If $N = \{0\}$ this is trivial. Otherwise,

$$f_0(s) = f_1(s) \geqq f_1(\nu) = 0 .$$

(2.4) If $x \in X$, and $(g_1, \cdots, g_n)$ is any finite sequence of elements of G, and $(1', \cdots, n')$ is any permutation of $(1, \cdots, n)$, then

$$g_1 g_2 \cdots g_n(x) - g_{1'}, g_{2'} \cdots g_{n'}(x) \in N.$$

If G is commutative this is trivial. Otherwise it follows at once from the lemma at the beginning of this section.

Now for each x in X we define a set $S[\succ x]$ as follows.

(2.5i) $S[\succ x]$ is the set of all s in S such that for some positive integer n, some ordered n-tuple $(g_1, \cdots, g_n)$ of members of G^+ and some ν in N it is true that

$$s \geqq n^{-1} \sum_{i=1}^{n} g_i(x) + \nu .$$

Also,

(2.5ii) For each x in X, $p(x)$ is defined to be the infimum of $f_0(s)$ for all s in $S[\succ x]$.

Then

(2.6) If x is in X and s' and s'' in S, and $s' \leqq x \leqq s''$, then $f_0(s') \leqq p(x) \leqq f_0(s'')$.

Let s be in $S[\succ x]$; suppose it satisfies the inequality in (2.5 i). Then since the g_i are order-preserving and S is G-invariant,

$$s - n^{-1} \sum_{i=1}^{n} g_i(s') \geqq \nu ,$$

so by (2.3)

$$f_0(s) \geqq n^{-1} \sum_{i=1}^{n} f_0(g_i(s')) = f_0(s') .$$

By (2.5 ii), $p(x) \geqq f_0(s')$. Since $s'' \in S[\succ x]$, $f_0(s'') \geqq p(x)$.

From this and hypothesis (1.4) (applied to x and to $-x$) we see that $p(x)$ is finite-valued. Moreover,

(2.7) if $s \in S$, $p(s) = f_0(s)$.

We next prove

(2.8) p is positively homogeneous and subadditive on X; that is, if $a \geqq 0$ and x and y are in X,

$$p(ax) = ap(x) \text{ and } p(x + y) \leqq p(x) + p(y) .$$

The first statement is trivial. For the second, let ε be positive, and let s_1, s_2 be members of $S[\succ x]$, $S[\succ y]$ respectively such that

$$f_0(s_1) < p(x) + \varepsilon/2, f_0(s_2) < p(y) + \varepsilon/2 .$$

There exist integers m, n, elements $g_1, \cdots, g_n, g'_1, \cdots, g'_m$ of G^+ and elements ν_1, ν_2 of N such that

$$s_1 \geqq n^{-1} \sum_{i=1}^{n} g_i(x) + \nu_1, s_2 \geqq m^{-1} \sum_{j=1}^{m} g'_j(y) + \nu_2 .$$

This implies

(2.9) $m^{-1} \sum_{j=1}^{m} g'_j(s_1) \geqq (mn)^{-1} \sum_{j=1}^{m} \sum_{i=1}^{n} g'_j g_i(x) + m^{-1} \sum_{j=1}^{m} g'_j(\nu_1)$,

(2.10) $n^{-1} \sum_{i=1}^{n} g_i(s_2) \geqq (mn)^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} g_i g'_j(y) + n^{-1} \sum_{i=1}^{n} g_i(\nu_2)$.

By (2.4), for $i \in \{1, \cdots, n\}$ and $j \in \{1, \cdots, m\}$ there is a ν_{ij} in N such that

(2.11) $0 = g'_j g_i(y) - g_i g'_j(y) - \nu_{ij}$.

We multiply each of equations (2.11) by $(mn)^{-1}$ and add the results and (2.9) and (2.10) member by member. The result is, because of (2.1),

(2.12) $m^{-1} \sum_{j=1}^{m} g'_j(s_1) + n^{-1} \sum_{i=1}^{n} g_i(s_2) \geqq (mn)^{-1} \sum_{j=1}^{m} \sum_{i=1}^{n} g'_j g_i(x + y) + \nu_3$,

where $\nu_3 \in N$. Thus the left member is in $S[\succ(x + y)]$, and so

$$\begin{aligned} p(x + y) &\leqq f_0(m^{-1} \sum_{j=1}^{m} g'_j(s_1) + n^{-1} \sum_{i=1}^{n} g_i(s_2)) \\ &= f_0(s_1) + f_0(s_2) \\ &< p(x) + p(y) + \varepsilon . \end{aligned}$$

Since ε is an arbitrary positive number, (2.8) is established.

By (2.7) and (2.8) we may apply the Hahn-Banach theorem to obtain a linear functional $f: X \to R$ coinciding with f_0 on S and satisfying

(2.13) $f(x) \leqq p(x) \quad (x \in X)$.

It remains to show that f is the G-invariant positive extension we seek. Clearly f is positive, since if $x \geqq 0$ we have by (2.6) and (2.13)

$$f(-x) \leqq p(-x) \leqq f_0(0) = 0 .$$

To show that f is G-invariant we shall need the following lemma, which establishes a strong form of G-invariance for p.

LEMMA. *For all x in X and $g_1, \cdots, g_n$ in G^+, $p(n^{-1}\sum_{i=1}^{n} g_i(x)) = p(x)$.*

Proof. Write $y = n^{-1}\sum g_i(x)$. If s is in $S[\succ y]$, so that $s \geqq m^{-1}\sum_{j=1}^{m} g'_j(y) + \nu$ with $g'_1, \cdots, g'_m$ in G^+ and ν in N, then

$$s \geqq (mn)^{-1}\sum_{j=1}^{m}\sum_{i=1}^{n} g'_j g_i(x) + \nu ,$$

so by definition of $p, f_0(s) \geqq p(x)$ and $p(y) \geqq p(x)$.

Conversely, let s be in $S[\succ x]$. Then there are elements $g'_1, \cdots, g'_m$ in G^+ and ν_1 in N such that $s \geqq m^{-1}\sum_{j=1}^{m} g'_j(x) + \nu_1$, whence

$$\begin{aligned} n^{-1}\sum_{i=1}^{n} g_i(s) &\geqq (mn)^{-1}\sum_{j=1}^{m}\sum_{i=1}^{n} g_i(g'_j(x) + \nu_1) \\ &= (mn)^{-1}\sum_{j=1}^{m}\sum_{i=1}^{n} \{g'_j g_i(x) + [-g'_j g_i(x) + g_i g'_j(x) + g_i(\nu_1)]\} . \end{aligned}$$

The expressions in square brackets are in N by (2.2) and (2.4), so

$$n^{-1}\sum_{i=1}^{n} g_i(s) \geqq m^{-1}\sum_{j=1}^{m} g'_j[n^{-1}\sum_{i=1}^{n} g_i(x)] + \nu' = m^{-1}\sum_{j=1}^{m} g'_j(y) + \nu',$$

where $\nu' \in N$. Hence $n^{-1}\sum g_i(s)$ is in $S[\succ y]$, and $f_0(s) = f_0(n^{-1}\sum g_i(s)) \geqq p(y)$. This implies $p(x) \geqq p(y)$, proving the lemma.

To prove that f is G-invariant let g be in G and n a positive integer. We apply the lemma to $x - g(x)$, with g_1 the identity and $g_j = g^{j-1} (j = 2, \cdots, n)$; the result is

$$\begin{aligned} p(x - g(x)) &= n^{-1}p(x - g(x) + g(x) - g^2(x) + \cdots + g^{n-1}(x) - g^n(x)) \\ &= n^{-1}p(x + [-g^n(x)]) \\ &\leqq n^{-1}[p(x) + p(g^n(-x))] \\ &= n^{-1}[p(x) + p(-x)] , \end{aligned}$$

where in the last equation we have again applied the lemma. Since n is arbitrary, this implies $p(x - g(x)) \leqq 0$, so $f(x - g(x)) \leqq 0$. Repeating the reasoning with $g(x) - x$ in place of $x - g(x)$ yields $f(g(x) - x) \leqq 0$, so $f(g(x)) = f(x)$, and f is G-invariant. The proof of Theorem 1 is complete.

3. **Proof of Theorem 2.** The proof is by induction. We assume that for some $m\,(1 \leqq m < \bar{n})$ there exists a G-invariant positive linear extension f_{m-1} of f_0 to X_{m-1} (which is surely true for $m = 1$) and

we show that f_{m-1} has a G-invariant positive linear extension f_m to X_m.

Suppose first that hypothesis (i) holds for $n = m$. Then there exist sets $G = G_h \supseteqq G_{h-1} \supseteqq \cdots \supseteqq G_0$ with G_0 commutative and G_k acting on X_m commutatively to within left G_{k-1} factors ($k = 1, 2, \cdots, h$). By Theorem 1, (using its hypothesis (i)) f_{m-1} can be extended to a G_0-invariant positive linear functional on X_m. Again by Theorem 1 (using its hypothesis (ii)) f_{m-1} can be extended to a G_1-invariant positive linear functional on X_m, and thus by successive applications of Theorem 1 we obtain a G-invariant positive linear extension of f_{m-1} to X_m.

Suppose next that hypothesis (ii) holds. In order to extend f_{m-1} to X_m we need the following lemma.

LEMMA. *Let G be a set of operators on a partially ordered linear space X_1. Let X_0 be a G-invariant subspace of X_1, such that for each x in X_1, there is a g in G such that $g(x) \in X_0$. Assume that for each x in X_1, and for each g_1, g_2, g_3 in G such that $g_1(x)$ and $g_2g_3(x)$ are in X_0, there exist h_1 and h_2 in G such that $h_1g_1g_2g_3(x) = h_2g_2g_3g_1(x)$. Then every G-invariant linear functional $f_0: X_0 \to R$ has a unique G-invariant linear extension to X_1.*

Proof. Let x be in X_1 and let g_1, g_2 be members of G such that $g_1(x)$ and $g_2(x)$ are both in X_0. By hypothesis there are members h_1, h_2 of G such that $h_1g_1g_2(x) = h_2g_2g_1(x)$. Then

$$f_0(g_1(x)) = f_0(h_2g_2g_1(x)) = f_0(h_1g_1g_2(x)) = f_0(g_2(x)) .$$

Thus $f_0(g(x))$ has a common value for all g in G such that $g(x) \in X_0$. We denote this common value by $f_1(x)$. It evidently coincides with $f_0(x)$ if $x \in X_0$.

If $x \in X_1$ and $g \in G$, there exist members g_1, g_2 of G such that $g_1(x)$ and $g_2g(x)$ are in X_0, and there exist h_1, h_2 in G such that $h_1g_1g_2g(x) = h_2g_2gg_1(x)$. Then

$$\begin{aligned} f_1(g(x)) &= f_0(h_1g_1g_2g(x)) \\ &= f_0(h_2g_2gg_1(x)) \\ &= f_1(x) , \end{aligned}$$

so f_1 is G-invariant.

Let x_1 and x_2 be in X_1, and let g_1, g_2 be members of G such that g_1x_1 and g_2x_2 are in X_0. There are members h_1, h_2 of G such that $h_1g_1g_2x_1 = h_2g_2g_1x_1$. Then, by the G-invariance of X_0 and f_0,

$$\begin{aligned} f_1(x_1) + f_1(x_2) &= f_0(g_1(x_1)) + f_0(g_2(x_2)) \\ &= f_0(h_2g_2g_1(x_1)) + f_0(h_1g_1g_2(x_2)) \end{aligned}$$

$$= f_0(h_1 g_1 g_2[x_1 + x_2])$$
$$= f_1(x_1 + x_2) .$$

Since obviously $f_1(ax) = af_1(x)$ for all real a, f is linear. If f_1' and f_1'' are two G-invariant extensions of f_0 and $x \in X_1$, then with $g \in G$ such that $g(x) \in X_0$ we have

$$f_1'(x) = f_1'(g(x)) = f_0(g(x)) = f_1''(g(x)) = f_1''(x) ,$$

so the extension is unique and the proof of the lemma is complete.

By this lemma, $f_{m-1}: X_{m-1} \to R$ has a unique G-invariant linear extension f_m to X_m. If $x \in X_m$ and $x \geqq 0$, and $g \in G$ is such that $g(x) \in X_{m-1}$, then $g(x) \geqq 0$, so $f_m(x) = f_{m-1}(g(x)) \geqq 0$.

By use of these two processes we obtain successively G-invariant positive linear functionals $f_0, f_1, f_2, \cdots, f_n$ being defined on X_n and coinciding with f_{n-1} on X_{n-1} for all of the sets $X_n (n < \bar{n})$. We define $f(x)$ to be the common value of $f_n(x)$ for all n such that $x \in X_n$. This clearly is the extension sought.

REMARK. In hypothesis (i) of Theorem 2 the assumption that G is left-solvable over X_n can be replaced by the weaker assumption:

(3.1) There is an infinite ascending sequence of sets $G_0 \subseteqq G_1 \subseteqq \cdots \subseteqq G_n \subseteqq \cdots$ such that G_0 is commutative, each G_k $(k = 1, 2, 3, \cdots)$ acts on X_n commutatively to within left G_{k-1}-factors, and $\cup_i G_i = G$.

We can prove the extension theorem for such a G as follows. Let X^* be the space of all linear functionals $X \to R$ equipped with the topology of pointwise convergence (the weakest topology in which the functionals induced on X^* by X are all continuous). One can easily show that the functionals in X^* which are positive and which extend f_0 on S form a compact (convex) subset, F, of X^*. For each i, the set of functionals in F invariant under the action of G_i is a closed subset F_i, and by Theorem 1 the sets F_i are all nonempty. Hence $F_\infty = \bigcap_{i=1}^{\infty} F_i$ is nonempty, and any functional $f \in F_\infty$ is a positive extension of f_0 on S which is actually G-invariant. The fact that we had a countable sequence of subsets is irrelevant to this argument—any well-ordered ascending family would do.

4. Bounded invariant functionals.

In the literature there are theorems generalizing the Hahn-Banach theorem so as to obtain invariant extension, as Theorem 2 generalized the Riesz theorem. As a first consequence of Theorem 2 we give such a result.

In the following theorem we assume that X is a real linear space and G a semigroup of linear transformations of X into itself, containing the identity. We also assume

(4.1) (i) p is a positively homogeneous subadditive functional on X (i.e., if $x, y \in X$ and $a \geqq 0$ then $p(ax) = ap(x)$ and $p(x+y) \leqq p(x) + p(y)$);

(ii) There is a real number b such that to each x in X and each $\varepsilon > 0$ there corresponds an element $g_{\varepsilon,x}$ of G such that for all g in G,

$$p(gg_{\varepsilon,x}(x)) \leqq bp(x) + \varepsilon .$$

This condition is clearly satisfied if

(4.2) $p(g(x)) \leqq bp(x)$

for all g in G and all x in X.

THEOREM 3. *Let X be a real linear space and G a semigroup of transformations acting on X commutatively to within left G-factors. Let X_1 and $S \subseteqq X_1$ be G-invariant subspaces such that for each x in X there is a g in G for which $g(x) \in X_1$, and such that G is left-solvable over X_1. Let $f_0 \colon S \to R$ be a G-invariant linear functional satisfying $f_0(s) \leqq p(s)$ for all s in S, where p satisfies* (4.1). *Then f_0 has a G-invariant linear extension $f \colon X \to R$ such that*

$$f(x) \leqq bp(x) \text{ for all } x \text{ in } X .$$

Proof. Corresponding to each g in G we define a transformation $g' \colon X \times R \to X \times R$ by setting

$$g'(x, y) = (g(x), y)\ (x \in X, y \in R) .$$

The set G' of all such transformations acts on $X \times R$ commutatively to within left G'-factors, and is left-solvable over $X_1 \times R$.

We partially order $X \times R$ by defining $(x, y) \geqq (0, 0)$ to mean that there exists a γ in G such that $p(g\gamma(x)) \leqq y$ for all g in G. If $(x, y) \geqq (0, 0)$ and $(\bar{x}, \bar{y}) \geqq (0, 0)$, there exist γ and $\bar{\gamma}$ in G such that $p(g\gamma(x)) \leqq y$ and $p(g\bar{\gamma}(\bar{x})) \leqq \bar{y}$ for g in G. There exist h, k in G such that $h\gamma\bar{\gamma}(x) = k\bar{\gamma}\gamma(x)$, whence for all g in G we find

$$p(gh\gamma\bar{\gamma}[x + \bar{x}]) = p(gk\bar{\gamma}\gamma(x)) + p(gh\gamma\bar{\gamma}(\bar{x})) \leqq y + \bar{y} ,$$

and so $(x + \bar{x}, y + \bar{y}) \geqq (0, 0)$. Likewise $(ax, ay) \geqq (0, 0)$ if $a \geqq 0$ and $(x, y) \geqq (0, 0)$, so the elements satisfying $(x, y) \geqq (0, 0)$ satisfy the standard requirements for a positive cone in $X \times R$.

If $(x, y) \geqq (0, 0)$ and $g_1 \in G$, by hypothesis there is a γ in G such that $p(g\gamma(x)) \leqq y$ for all g in G. There are elements h, k of G such that $h\gamma g_1(x) = kg_1\gamma(x)$. Then for all g in G we have

$$p(g[h\gamma][g_1(x)]) = p(gkg_1\gamma(x)) \leqq y ,$$

so $g_1'(x, y) = (g_1(x), y) \geqq (0, 0)$, and g_1' is order preserving on $X \times R$.

Let $S_1 = S \times R$, and on S_1 define f_1 by setting $f_1(s, y) = y - f_0(s)$ $(s \in S, y \in R)$. The set S_1 is obviously G'-invariant, and f_1 is linear and G'-invariant on S_1. Also, if $(s, y) \in S_1$ and $(s, y) \geqq (0, 0)$, then for some γ in G we have $p(g\gamma(s)) \leqq y$ for all g in G, in particular when g is the identity. Then

$$f_0(s) = f_0(\gamma(s)) \leqq p(\gamma(s)) \leqq y ,$$

so $f_1(s, y) \geqq 0$, and f_1 is a positive linear functional on S.

Finally, if $(x, y) \in X \times R$ there is an s_1 in S_1 such that $s_1 \geqq (x, y)$. For s_1 we choose $(0, 1 + y + bp(-x))$. This is in S_1, and if we take γ to be the element $g_{1,-x}$ of (4.1ii) we see that

$$(-x + 0, -y + [1 + y + bp(-x)]) \geqq (0, 0).$$

Now, by Theorem 2, f_1 has a G'-invariant positive linear extension f' to $X \times R$. Since f' is linear on $X \times R$ it can be represented in the form $f'(x, y) = ay - f(x)$, where $a \in R$ and f is a linear functional on X. Since $(0, 1) \in S_1$,

$$\begin{aligned} a = a1 - f(0) &= f'(0, 1) = f_1(0, 1) \\ &= 1 - f_0(0) = 1 . \end{aligned}$$

If $x \in X$ and $g \in G$, then since f' is G'-invariant

$$\begin{aligned} f(g(x)) &= -f'(g(x), 0) \\ &= -f'(g'(x, 0)) \\ &= -f'(x, 0) \\ &= -f(x) , \end{aligned}$$

and f is G-invariant.

If $x \in X$ and $\varepsilon > 0$, by (4.1) we have for all g in G

$$bp(x) + \varepsilon \geqq p(gg_{\varepsilon,x}(x)) ,$$

so $(x, bp(x) + \varepsilon) \geqq 0$. Since f' is positive,

$$0 \leqq f'(x, bp(x) + \varepsilon) = bp(x) + \varepsilon - f(x) .$$

Since ε is arbitrary, $f(x) \leqq bp(x)$.

REMARK 1. In the most important case, in which $p(x) + p(-x) > 0$ for some x, the freedom of b to be any real number is rather illusory. For with the f of the conclusion, we have $0 = f(x) + f(-x) \leqq b[p(x) + p(-x)]$, so $b \geqq 0$. Furthermore, if $b < 1$ the only f satisfying the conditions of the conclusion is $f = 0$. For suppose $0 \leqq b < 1$; let x be in X and $\varepsilon > 0$. By repeated use of (4.1) we can find a sequence $g_1, g_2, g_3, \cdots$ of members of G such that

$$p(g_1(x)) \leqq bp(x) + \varepsilon/4 ,$$

$p(g_n g_{n-1} \cdots g_1(x)) \leqq bp(g_{n-1}g_{n-2} \cdots g_1(x)) + \varepsilon/2^{n+1}(n = 2, 3, \cdots)$. Then

$$\begin{aligned} f(x) &= f(g_n g_{n-1} \cdots g_1(x)) \\ &\leqq bp(g_n g_{n-1} \cdots g_1(x)) \\ &\leqq b^{n+1}p(x) + \varepsilon(2^{-2} + 2^{-3} + \cdots + 2^{-n-1}) . \end{aligned}$$

By choosing a large n we find $f(x) \leqq \varepsilon$, whence $f(x) \leqq 0$. Likewise $f(-x) \leqq 0$, so $f(x) = 0$ for all x.

REMARK 2. This corollary generalizes the principal result in a paper by Agnew and Morse [1], and also generalizes and sharpens two of the corollaries in a paper by Klee [5].

5. An extension of the Banach limit. We now use Theorem 2 to show that an extension of the classical Banach limit can be defined for a large class of sequences of numbers, including all those that are bounded and many that are not. Let X^* be the space of all sequences of real numbers. On X^* we define linear transformations T, H and I_r $(r = 1, 2, 3, \cdots)$ as follows. T is the translation operation defined for $x = (x_1, x_2, \cdots)$ by

(5.1) $T(x) = (x_2, x_3, \cdots, x_{n+1}, \cdots)$.

H is the Hölder-mean operator,

(5.2) $H(x) = (x_1, [x_1 + x_2]/2, \cdots, [x_1 + \cdots + x_n]/n, \cdots)$.

I_r is the r-fold iteration operation, each member of the sequence being repeated r times; thus

(5.3) $I_2(x) = (x_1, x_1, x_2, x_2, x_3, x_3, \cdots)$.

It is clearly hopeless to try to define an extension of the ordinary limit to all of X^*. We shall consider several subspaces:

X_0 is the space of all convergent sequences with limit 0,

X_c is the space of all convergent sequences,

X_b is the space of all bounded sequences.

For each positive integer k, X_k is the space of all x in X^* such that $H^k x \in X_b$.

$X_{o(n)}$ is the set of all $x = (x_1, x_2, \cdots)$ in X^* such that $|x_n| = o(n)$.

$X = X_{o(n)} \cap [\bigcup_k^\infty X_k]$.

Clearly $X_0 \subset X_c \subset X_b \subset X \subset X^*$. Also, T, H and the I_r all map X_0 into itself, so if we define two sequences to be equivalent if their difference is in X_0, the operations T, H and I_r extend uniquely to $Y^* = X^*/X_0$. We also define $Y_b = X_b/X_0$, etc. Clearly Y_c and Y_b are invariant under T, H and the I_r, and Y_k is invariant under H.

We define $y \geqq 0$ for $y \in Y^*$ to mean that there is a sequence $x = (x_1, x_2, \cdots)$ in the class y such that $x_i \geqq 0$ for all i.

Let now x be such that $|x_n| = o(n)$. Given any $\varepsilon > 0$, there is an n' such that $|x_n| < \varepsilon n$ if $n > n'$, so

(5.4) $|(x_1 + \cdots + x_n)/n| < (x_1 + \cdots + x_{n'})/n + \varepsilon(n+1)/2 < \varepsilon n$
provided that n is large enough. Hence $X_{o(n)}$ is invariant under H. It is clearly invariant under T.

With the same notation we readily compute that for $r = 1, 2, 3, \cdots$

(5.5) $1 \cdot I_r T(x) = T^{r-1} T I_r(x)$.

Let us write

$$z' = TH(x) - HT(x) ,$$
$$z'' = I_r H(x) - HI_r(x) .$$

If for each positive integer n we define h, k by $n = hr - k$ $(0 \leqq k < r)$, the n-th terms of z' and z'' are respectively

$$z'_n = x_1/(n+1) - (x_1 + \cdots + x_{n+1})/n(n+1) ,$$
$$z''_n = (k/n)[x_h - (x_1 + \cdots + x_n)/h] ,$$

and these tend to 0 if $|x_n| = o(n)$. By repeated application of this result, we see that for every k the sequences $TH^k(x) - H^kT(x)$ and $I_rH^k(x) - H^kI_r(x)$ have limits 0. In particular, if $H^k(x)$ belongs to X_b, so do $H^kT(x)$ and $H^kI_r(x)$; so H, T and the I_r all map $X_k \cap X_{o(n)}$ into itself. Also, if we denote by G the semigroup generated by H, T and the I_r and define G_0 to be $\{1, T, T^2, \cdots\}$, we see that G acts on $Y_{o(n)}$ commutatively to within left G_0-factors, and $Y_{o(n)}$, Y, Y_b and Y_c are all G-invariant.

For each s in Y_c let $f_0(s)$ be the common value of $\lim_n x_n$ for all sequences x representing s. We apply Theorem 2 with X, X_0, X_1, X_2 replaced by Y, Y_c, Y_b, Y respectively, and obtain a positive linear functional $f_1\colon Y \to R$ that is invariant under T, H and all the I_r. This defines a functional on X, which we also call f_1, by setting $f_1(x)$ equal to $f_1(y)$ where y is the member of Y that contains x.

It is possible to extend this still further. It can be shown that each X_k is invariant under T as well as under H, and that if $x \in X_k$ and $N \geqq k$ then

$$H^NTx - TH^Nx \in X_0 .$$

Hence for each k the semigroup generated by T and H is left solvable over Y_k, since if A and B are in the semigroup and $x \in X_k$,

$$H^kABx - H^kBAx \in X_0 .$$

By Theorem 2, the functional lim can be extended from Y_c to be linear, positive and T- and H- invariant on $\cup_k Y_k$, hence (as above) on $\cup_k X_k$. The details of the proof are too lengthy to justify publication in this Journal, but the authors undertake to furnish a duplicated copy of the proof in full detail to any one who requests one within a reasonable number of years.

We have thus attained the following theorem.

THEOREM 4. *On the space* $\bigcup_{k=0}^{\infty} X_k$ *of all sequences* $x = (x_1, x_2, \cdots)$ *of real numbers such that for some nonnegative integer k the sequence* $H^k x$ *of k-fold iterated Hölder means is bounded, there exists a positive linear functional* f_1 *such that* $f_1(x)$ *is the limit of the sequence* $H^k x$ *whenever the latter exists. It also has the invariance properties*

(5.6) *for all* x *in* $\cup_k X_k$, $f_1(hx) = f_1(x)$ *for all h in the semigroup generated by H and T;*

(5.7) *for all* $x = (x_1, x_2, \cdots)$ *in* $\cup_k X_k$ *such that* $|x_n| = o(n)$, $f_1(gx) = f_1(x)$ *for all g in the semigroup generated by H, T and the* $I_r (r = 1, 2, 3, \cdots)$.

6. **Invariant measures.** In topological dynamics the existence of an invariant measure or mean is often an important condition. (Cf., for example, Chapter VI of the book of Nemytskii and Stepanov [9]). Suppose that X is a set, G a semigroup of mappings of X into itself, and μ a measure on a family $\mathscr{M}$ of subsets of X (called measurable sets) such that if $A \in \mathscr{M}$ and $g \in G$ then $g^{-1}(A) \in \mathscr{M}$. A measurable set A is *invariant* if $\mu(A \cup g^{-1}(A) - A \cap g^{-1}(A)) = 0$ for all g in G; and the measure μ is *ergodic* if for every invariant measurable subset A of X, either $\mu(X) = 0$ or $\mu(X - A) = 0$. In 1937 Kryloff and Bogoliuboff [7] proved that if X is a compact metric space and G a one-parameter group of homeomorphisms of X, there is a Baire measure μ on X invariant under the action of G and ergodic (cf. [7], or [9], pp. 486–519). The same result is known when G is the semigroup (cyclic and commutative) generated by a single continuous map (not necessarily a homeomorphism) of X into itself. A recent paper of Schwartz [11] proves the corresponding result for the case in which G is a topological group and either G or X is connected. We prove below a theorem containing all of these. First we prove a theorem on the existence of invariant means in a more general context. A *mean* on the space $B(X)$ of bounded real-valued functions on X is a positive linear functional M on $B(X)$ such that $M(1) = 1$. A mean M is invariant if $M(f \circ g) = M(f)$ for all f in $B(X)$ and g in G.

THEOREM 5. *Let X be a set; let G be a semigroup of transformations on X containing the identity and right-solvable over G itself.*

Then there exists an invariant mean on the space $B(X)$ of bounded real-valued functions on X.

For each g in G we define a transformation of $B(X)$ into itself (which we also call g) as follows: $g(f)$ is the function such that $gf(x) = f(g(x))$ $(x \in X)$. Then $(g_1g_2)(f) = g_2(g_1(f))$, since $[g_2(g_1(f))](x) = (g_1(f))(g_2(x)) = f(g_1[g_2(x)])$. Since G acts right-solvably on itself there is a sequence $G = G_n \supseteqq G_{n-1} \supseteqq \cdots \supseteqq G_0$ with G_0 commutative and such that if k is one of the numbers $1, \cdots, n$ and g_1 and g_2 are in G_k, there are members h_1, h_2 of G_{k-1} such that $g_1g_2h_2\gamma = g_2g_1h_1\gamma$ for all γ in G, in particular for γ the identity. Then $[g_1g_2h_2](x) = [g_2g_1h_1](x)$ for all x in X, whence $h_2g_2g_1(f) = h_1g_1g_2(f)$ for all f in $B(X)$. This implies that G is left-solvable over $B(X)$. Now for X_0 we choose the set of constant functions, and for $s = c$ we define $M(s) = c$. By Theorem 2, M extends to a G-invariant positive linear functional over $B(X)$, proving the theorem.

Let us now specialize this by requiring X to be a compact Hausdorff space and G to be a semigroup of continuous transformations. With the assumptions of Theorem 5 there is a G-invariant mean M on $B(X)$. We restrict M to the space $C(X)$ of functions continuous on X. By the Riesz representation theorem there is a Baire measure μ on X such that for all f in $C(X)$ we have

$$M(f) = \int_X f(x)\mu(dx) .$$

Thus we have proved part of the following theorem.

THEOREM 6. *Let X be a compact Hausdorff space. Let G be a semi-group of continuous transformations on X containing the identity, and such that G is right-solvable over G itself. Then there is a Baire measure μ on X which is G-invariant and ergodic.*

The invariance of μ is a rather immediate consequence of the invariance of M. To show that μ can be chosen to be ergodic, in the dual space of $C(X)$ with the weak* topology we consider the set I of invariant means. This is convex, and in the weak* topology it is compact, since it is a closed subset of the unit ball. By the Krein-Milman theorem, I has at least one extreme point. Let M be such an extreme point, with corresponding measure μ. If there is an invariant set A with

$$\lambda_1 = \mu(A) > 0 \quad \text{and} \quad \lambda_2 = \mu(X - A) > 0 ,$$

then for each Baire set B we define

$$\mu_1(B) = \mu(B \cap A)/\lambda_1, \mu_2(B) = \mu(B \cap [X - A]/\lambda_2 .$$

These are invariant measures with $\mu_1(X) = \mu_2(X) = 1$, and $\mu = \lambda_1\mu_1 + \lambda_2\mu_2$. This is impossible since μ is an extreme point of I, so μ is ergodic.

7. **Extension of stochastic processes.** In this section we shall give a nontraditional meaning to the expression "stochastic process", by permitting finitely-additive set-functions to be used as probability measures. Let Y and T be nonempty sets, and let Σ be an algebra of subsets of Y. In the space $X = Y^T$ of functions from T to Y we define $\mathscr{A} = \mathscr{A}(T, \Sigma)$ to be the algebra of subsets of X consisting of all finite unions of finite intersections of sets of the form $\{x \in X: x(t_1) \in A_1\}$ with $t_1 \in T$ and $A_1 \in \Sigma$. The sets belonging to $\mathscr{A}$ will be called *figures*. Let P be a nonnegative additive set-function on $\mathscr{A}$ such that $P(X) = 1$. Then the triple $(X, \mathscr{A}, P)$ will be called a *weak stochastic process*.

A function f on X is *based on* a subset T_0 of T if for every x and x' in X such that $x(t) = x'(t)$ for all $t \in T_0$ it is also true that $f(x) = f(x')$.

Now suppose that T is an interval $[a, b]$ on the real line. Then we say that the weak stochastic process $(X, \mathscr{A}(T, \Sigma), P)$ is *relatively stationary* if the following condition holds: whenever $t_1, \cdots, t_k \in T$ and τ is a real number such that $t_1 - \tau, \cdots, t_k - \tau$ are all in T, and $A_1, \cdots, A_k$ are in Σ, we have

$$P\{x \in X: x(t_1) \in A_1, \cdots, x(t_k) \in A_k\}$$
$$= P\{x \in X: x(t_1 - \tau)(\in A_1, \cdots, x(t_k - \tau)) \in A_k\} .$$

Our principal theorem on extension of stochastic processes is the following:

THEOREM 7. *Let T_0 be an interval and let $(Y^{T_0}, \mathscr{A}(T_0, \Sigma), P_0)$ be a relatively stationary weak stochastic process. Then there is a stationary weak process $(Y^T, \mathscr{A}(T, \Sigma), P)$, where T is the whole real line, which extends $(Y^{T_0}, \mathscr{A}(T_0, \Sigma), P_0)$; that is, for all figures A based on T_0, $P_0(A) = P(A)$.*

Proof. We shall discuss the case in which T_0 is a closed interval $[a, b]$; open or half-open intervals call for only trivial changes.

We first define simple functions: Map Y^T into Y^{T_0} by the restriction map π, namely if $x: T \to Y$, $\pi(x): T_0 \to Y$ is defined by $\pi(x) = x \mid T_0$. The set of inverse images under π of sets in $\mathscr{A}(T_0, \Sigma)$ will be called $\mathscr{A}_0(T, \Sigma)$. For $A \in \mathscr{A}_0(T, \Sigma)$ with image $A_0 \in \mathscr{A}(T_0, \Sigma)$ we define $P(A) = P_0(A_0)$. We wish to extend this P to all figures of X. The members of $\mathscr{A}_0(T, \Sigma)$ form a proper subclass of the class of all figures $\mathscr{A}(T, \Sigma)$; they are the figures based on subsets of T_0. A *simple function* is a function on Y^T having finitely many values, assumed on disjoint sets belonging to $\mathscr{A}(T, \Sigma)$. Observe that a simple function is based on T_0 if and only if each of its sets of constancy belongs to $\mathscr{A}_0(T, \Sigma)$. Observe also that there is a one-to-one correspondence between finitely additive measures P on X and positive linear functionals on the space of simple functions where if P is a measure we denote the corresponding functional by $\int f dP$.

Let us define S_0 to be the class of simple functions based on T. Then we define S_1 to be the class of simple functions that can be represented as a finite sum $f_1 + \cdots + f_k$ in which each f_j is a simple function based on a translate of T_0. This class S_1 is clearly invariant under translations; that is, if we define $U_\tau(f)$ by

$$[U_\tau(f)](x) = f(\Theta_\tau(x)) \quad \text{where} \quad [\Theta_\tau(x)](t) = x(t - \tau) \ ,$$

then for all functions f in S_1 and all real τ, $U_\tau(f) \in S_1$.

Let us define S_0 to be the (linear, but not translation-invariant) space of simple functions based on T_0, and S the space of all simple functions. We have defined a linear functional $\int f dP$ on S_0; we wish to find a translation-invariant extension to S. We first define an extension to S_1, and for this we need a lemma.

LEMMA. *If $f_1, \cdots, f_n$ are in S_0, and there exist real numbers $\tau_1, \cdots, \tau_n$ such that $\sum_{i=1}^{n} U_{\tau_i} f_i \geqq 0$, then*

$$\int (f_1 + \cdots + f_n) dP \geqq 0 \ .$$

The proof comes fairly directly out of Parthasarathy and Varadahan [10]; we include it for completeness. We proceed by induction; the case $n = 1$ is clear. Suppose that the assertion of the lemma holds for an integer n; we shall prove that it holds for $n + 1$. Assume then that $f_1, \cdots, f_{n+1}$ are in S_0, $\tau_1 < \tau_2 < \cdots < \tau_{n+1}$ are reals and $\sum_{j=1}^{n+1} U_{\tau_j} f_j \geqq 0$. Since f_{n+1} is based on a subset $t'_1, \cdots, t'_k$ of $[a, b]$, $U_{\tau_{n+1}} f_{n+1}$ is based on $t'_1 + \tau_{n+1}, \cdots, t'_k + \tau_{n+1}$. We denote by $t'_{m+1} + \tau_{n+1}, \cdots, t'_k + \tau_{n+1}$ the members of this set (if any) which exceed $b + \tau_n$. The sum $U_{\tau_1} f_1 + \cdots + U_{\tau_n} f_n$ is based on a subset T^* of $[a + \tau_1, \cdots, b + \tau_n]$, hence is independent of the values $x(t'_{m+1} + \tau_{n+1}), \cdots, x(t'_k + \tau_{n+1})$. Given any $x \in X$ and any set of points $y_{m+1}, \cdots, y_k$ in Y there is an $\tilde{x} \in X$ such that $\tilde{x}(t) = x(t)$ on T^* and $\tilde{x}(t'_j + \tau_{n+1}) = y_j$ for $j = m + 1, \cdots, k$. From the first of these equations we see that

$U_{\tau_j} f_j(\tilde{x}) = U_{\tau_j} f_j(x), j = 1, \cdots, n$. So by the hypothesis of the lemma

$$U_{\tau_1} f_1(x) + \cdots + U_{\tau_n} f_n(x) + U_{\tau_{n+1}} f_{n+1}(\tilde{x}) \geqq 0 .$$

We define another functional g in X as follows: For each $x \in X$ let $g(x)$ be the least of the (finitely many) values of $U_{\tau_{n+1}} f_{n+1}(z)$, where z is any member of X such that $z(t'_j + \tau_{n+1}) = x(t'_j + \tau_{n+1})$ for $j = 1, \cdots, m$. Then g is based on $t'_1 + \tau_{n+1}, \cdots, t'_m + \tau_{n+1}$, and is easily seen to be a simple function.

By the previous inequality,

$$\sum_{j=1}^{n} U_{\tau_j} f_j(x) + g(x) \geqq 0 \text{ for all } x \text{ in } X .$$

If we write this as

$$\sum_{j=1}^{n-1} U_{\tau_j} f_j(x) + [U_{\tau_n} f_n(x) + g(x)] \geqq 0 ,$$

we notice that the expression in brackets defines a simple function based on $[a + \tau_n, b + \tau_n]$, so $f_n + U_{-\tau_n} g$ is in S_0. Hence by the induction hypothesis

$$\sum_{j=1}^{n-1} \int f_j dP + \int [f_n + U_{-\tau_n} g] dP \geqq 0 .$$

But the difference $U_{\tau_{n+1}} f_{n+1}(x) - g(x)$ is based on $[a + \tau_{n+1}, b + \tau_{n+1}]$ and is simple, and by definition of g it is nonnegative. Hence $f_{n+1} - U_{-\tau_{n+1}} g$ is in S_0 and is $\geqq 0$, so

$$\int f_{n+1} dP - \int U_{-\tau_{n+1}} g dP \geqq 0 .$$

Finally, both $U_{-\tau_n} g$ and $U_{-\tau_{n+1}} g$ are based on $[a, b]$ and are translates of each other, and the process is relatively stationary on $[a, b]$, so

$$\int U_{-\tau_{n+1}} g dP - \int U_{-\tau_n} g dP = 0 .$$

Adding this equality and the previous two inequalities, we obtain the lemma.

By changing sign we can prove that the lemma holds with $\leqq 0$ in place of $\geqq 0$, hence it holds with $= 0$ in place of $\geqq 0$. This implies that if $f \in S_1$ and $f_1, \cdots, f_n$ are members of S_0 and $\tau_1, \cdots, \tau_n$ are real numbers such that $f = U_{\tau_1} f_1 + \cdots + U_{\tau_n} f_n$, the sum $\sum_{j=1}^{n} \int f_j dP$ is uniquely determined by f and is independent of representation.

We can therefore define a functional L_1 on S_1 by the rule

$$L_1\left(\sum_{i=1}^{n} U_{\tau_i} f_i\right) = \sum_{i=1}^{n} \int f_i dP$$

where the f_i are in S_0. This functional is clearly linear and nonnegative on S_1, and is invariant under U_τ for all real τ. By Theorem 1, L_1 has a nonnegative linear extension L to the space S of all simple functions, and L is invariant under all U_τ. Hence L defines a weak stochastic process on Y^T which is stationary and is an extension of $(Y^{T_0}, \mathscr{A}(T_0, \Sigma), P_0)$.

REMARK. For any given property of stochastic processes, one can ask whether the extended process guaranteed by Theorem 7 has the property (or can be required to have the property) if the original one does. If the space Y of values is a metric space, the notion of stochastic continuity in measure is very easily adapted to weak stochastic processes and it does extend in the way discussed. As usual, given any set E in X, we define $P^*(E)$ to be the infimum of $P(A)$ for all sets A in $\mathscr{A}$ that contain E. Then the process $(X, \mathscr{A}, P)$ is continuous in measure, or stochastically continuous, at a point t_0 if

$$\lim_{t \to t_0} P^*\{x \in X: d(x(t), x(t_0)) > \varepsilon\} = 0$$

for each positive ε. Clearly if this property holds for $(Y^{T_0}, \mathscr{A}(T_0, \Sigma), P_0)$ in Theorem 7 it also holds for $(Y^T, \mathscr{A}(T, \Sigma), P)$.

A more difficult question is that of countable additivity. It is not clear whether if the original process, viewed as a measure, is countably additive we can conclude that the extended one is. We content ourselves with proving this in a special case.

COROLLARY 5. *If in Theorem 7 we require that Y be a locally compact separable metric space and Σ be the σ -algebra generated by compact subsets of Y, and if the original measure P_0 was countably additive, then so is the extended P.*

Proof. To prove the measure countably additive it suffices to show that if $B_1, B_2, \cdots$ are measurable and $B_1 \supseteqq B_2 \supseteqq \cdots$ and $\bigcap_{i=1}^{\infty} B_i = \phi$ then $PB_i \to 0$ (Cf. Loève [8, p. 89]). (In this case the probability measure on $\mathscr{A}(T, \Sigma)$ extends to a countably additive probability measure on the σ -algebra in Y^T generated by $\mathscr{A}(T, \Sigma)$.). For every finite subset T_* of T there is a natural restriction map $Y^T \to Y^{T_*}$ and we can think of Y^{T_*} as a finite product of copies of Y, one for each point in T_*. There is a standard theorem due to Kolmogorov (cf. Loève [8, p. 93]) which states that if Y is a interval on the real line and if the measures on the spaces Y^{T*} satisfy certain consistency conditions (trivially satisfied here) then the extended measure P is countably additive. The crux of the proof is that in any of the spaces Y^{T_*}, the measure of a set can be approximated arbitrarily closely by the measure of a compact set contained in it. Since this is guaranteed in this case (the spaces Y^{T_*} are locally compact metric spaces) the usual proof is valid. This completes the proof of the corollary.

8. Covariances and a theorem of M. Krein. From Theorem 7 we can deduce a generalization of a well-known theorem of M. Krein on the extension of positive definite functions. We first need to generalize a known characterization of covariances, given for real or complex processes in (Doob [4], p. 72). The generalization requires a somewhat lengthy proof; we here present an abbreviated version, and undertake to furnish the proof in full detail to any one who requests it in a reasonable number of years.

We use the notation of § 7, and add the following hypotheses. K is the real field or the complex field, Y is a linear space over K, Λ is a linear aggregate of linear functionals $\lambda\colon Y \to K$, and Λ^* is the set of all linear functionals $f\colon \Lambda \to K$. Σ is the algebra of subsets of Y generated by the half-spaces $\{y \in Y\colon R\lambda(y) \geqq c\}$ with $\lambda \in \Lambda$ and c real.

For fixed λ in Λ and t in T, $(\lambda(x(t))\colon x \in X)$ is a function from X to K, and its integral and that of $|\lambda(x(t))|^2$ can be defined by an obvious limit process. We assume that the latter integral is finite; the former integral then exists, and is denoted by $M(t, \lambda)$. By standard arguments the covariance R, whose value at each (t_1, t_2) in $T \times T$ is the sesquilinear function

$$R(t_1, t_2\colon \lambda_1, \lambda_2) = \int_X [\lambda_1(x(t_1)) - M(t_1, \lambda_1)]\overline{[\lambda_2(x(t_2)) - M(t_2, \lambda_2)]}dP ,$$

exists, and if $(t_1, \lambda_1), \cdots, (t_n, \lambda_n)$ are in $T \times \Lambda$ the matrix with the elements

$$r_{ij} = R(t_i, t_j; \lambda_i, \lambda_j) \qquad (i, j = 1, \cdots, n) \tag{8.1}$$

is nonnegative definite Hermitian. We now state the converse.

THEOREM 8. *With the preceding notation, let* $R\colon T \times T \times \Lambda \times \Lambda \to K$ *be sesquilinear on* $\Lambda \times \Lambda$ *for each* (t_1, t_2) *in* $T \times T$; *let the matrix with elements* (8.1) *be nonnegative definite Hermitian; and let* $M\colon T \to \Lambda^*$ *be a function such that for each* t *in* T *there is an* $x(t)$ *in* $X(= Y^T)$ *such that* $M(t, \lambda) = \lambda(x(t))$ $(\lambda \in \Lambda)$. *Then there is a weak stochastic process* $(X, \mathscr{A}(T, \Sigma), P)$ *with mean value* M *and covariance* R. *Also, if* K *is the complex field, we can choose* P *so that*

$$\int_X \lambda_1(x(t_1))\lambda_2(x(t_2))dP = M(t_1, \lambda_1)M(t_2, \lambda_2)$$

for all t_1, t_2 *in* T *and* λ_1, λ_2 *in* Λ.

The general case is easily deduced from the case $M = 0$. We choose a Hamel base H for Λ. Then R defines a function on $(T \times H) \times (T \times H)$ that satisfies the hypotheses of the theorem (Doob [4], p. 72), so there is a Gaussian process on $T \times H$ with that function as covariance. There remains the verification of numerous details to show that the probability measure on $K^{T \times H}$ corresponding to that Gaussian process can be used to define a weak stochastic process (i.e., a finitely additive measure on Λ^T) with R as covariance.

Note that in the above situation, if T is the real line and if the process is stationary, then the covariance $R(t_1, t_2)$ depends only on the difference $t_1 - t_2$, and can be thought of as a function of one real variable (whose value at each point is a sesquilinear functional). We now apply Theorem 7 to prove an extension theorem for such functions.

DEFINITION. Let Λ be a linear space over K and $R^*(t)$ a function assigning to each real number in the interval $(-A, A)$ (where $0 < A \leqq \infty$) a sesquilinear functional $\Lambda \times \Lambda \rightarrow K$. Then R^* is *positive definite* if for each finite set $(t_1, \lambda_1) \cdots (t_n, \lambda_n)$ of elements of $[0, A) \times \Lambda$ the matrix with coefficients

$$R^*(t_i - t_j; \lambda_i, \lambda_j)$$

is nonnegative definite Hermitian.

THEOREM 9. *If R^* is a positive definite function (in the sense of the above definition) on the interval $(-A, A)$, then R^* extends to a positive definite function on the real line.*

This is an immediate application of Theorems 7 and 8 and the above remark. If Λ is one dimensional, then "positive definite" in our sense agrees with the classical definition, so this theorem generalizes the theorem of Krein on extensions of complex-valued positive definite functions. We can also show that if $R^*(t, \lambda_1, \lambda_2)$ is continuous at $t = 0$ for all fixed λ_1, λ_2, then the extension is continuous on $(-\infty, \infty)$ for all λ_1 and λ_2 in Λ.

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