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# ON CROISOT'S THEORY OF DECOMPOSITIONS

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**Croisot gave a definition of $(m, n)$-regularity which he then showed defined four logically distinct classes of semi-groups. However, semigroups with nilpotent elements did not fall within his classification. Our generalization of $(m, n)^0$-regularity remedies this exclusion; countably many distinct classes of semi-groups are defined.**

**In particular we investigate the structure of semigroups which are $(2, 2)^0$-regular. We show that a semigroup $S$ is in this class precisely when for each $x \in S$ either $x^2 = 0$ or $x^2 \in H_x$. Further, each regular $\mathscr{D}$-class together with 0 of such a semigroup is itself a completely 0-simple semigroup. The $(2, 2)^0$-regularity condition is specialized to that of absorbency: for each $a, b \in S$ either $ab = 0$ or $ab \in (R_a \cap L_b)$. We show that a regular absorbent semigroup is just a mutually annihilating collection of completely 0-simple semigroups.**

A schematic summary of the two classifications is found in Fig. 1 and Fig. 2. We remark now that our classification provides a countable number of open problems; e.g., determining the structure of $(n, n)^0$-regular semigroups for $n > 2$. Moreover, it also remains to treat such $(m, n)^0$-regular semigroups with reciprocity, antireciprocity, and uniqueness conditions (cf. [1], p. 124 or [2], p. 373 ff.).

We finally show that for an absorbent semigroup without 0 Green's relations $\mathscr{L}$ and $\mathscr{R}$ are congruences. It is then shown that a regular simple semigroup $S$ is completely simple if and only if $\mathscr{L}$ and $\mathscr{R}$ are congruence relations on $S$.

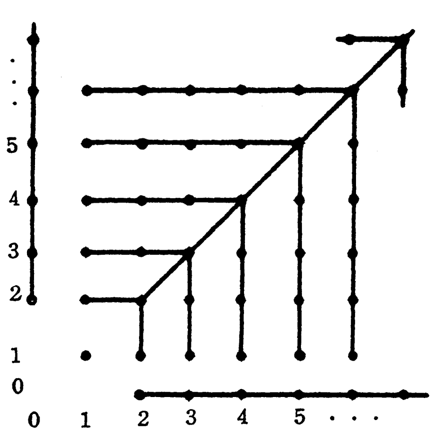


FIG. 1—$(m, n)$-regular classes

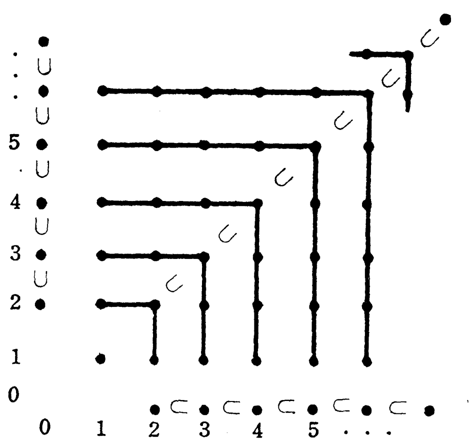


FIG. 2—$(m, n)^0$-regular classes

**1. Preliminaries.** We make use of the notation and terminology of Clifford and Preston [1]. Thus, $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$, and $\mathscr{J}$ will denote Green's relations and $L_a, R_b$, etc., the respective equivalence classes of $a, b \in S$, a given semigroup.

DEFINITION 1.1. Let $m$ and $n$ be nonnegative integers with $m + n > 1$. A semigroup $S$ will be in the class of $(m, n)$-semigroups, written $S \in (m, n)$ if and only if for each $x$ in $S$ there is a $u$ in $S$ such that $x = x^m u x^n$ (where, if necessary $x^0$ is suppressed in the equation); $S$ is then said to be *$(m, n)$-regular*.

These conditions were shown to fall into four logically distinct classes ([2], p. 370) which are summarized in Fig. 1; the equivalent classes, represented by lattice points, are connected. One readily sees that a semigroup $S \in (m, n)$ can contain no nilpotent elements (other than zero) whenever either $m$ or $n \geqq 2$. The following definition, it will be seen, permits the investigation of semigroups with nilpotent elements which were not previously considered.

Since we can always adjoin a zero to a semigroup, $S$, (cf. [1], p. 4) we will consider $S$ to have a zero, 0, in what follows.

DEFINITION 1.2. Let $m$ and $n$ be nonnegative integers with $m + n > 1$. A semigroup $S$ will be in the class of $(m, n)^0$-semigroups, written $S \in (m, n)^0$, if and only if for each $x \in S$ one of the following holds

(1) $m > 0$ and $x^m = 0$,

(2) $n > 0$ and $x^n = 0$,

(3) $x = x^m u x^n$ for some $u \in S$ where $x^0$ is suppressed in the equation when necessary.

We will say that $S$ is *$(m, n)^0$-regular* whenever $S \in (m, n)^0$ and that $S$ is *$n^0$-regular* when $S \in (n, n)^0$.

REMARK 1.3. We readily conclude from (1.2.3) that $(m, n) \subseteqq (m, n)^0$. Indeed, if $S$ is a semigroup with no nilpotent elements (other than perhaps 0) we have $S \in (m, n)$ if and only if $S \in (m, n)^0$.

PROPOSITION 1.4. (1) $(0, q)^0 \subset (0, n)^0$ for $n > q \geqq 2$.

(2) $(1, q)^0 \subset (1, n)^0$ for $n > q \geqq 2$.

(3) $(p, q)^0 \subseteqq (m, n)^0$ for $m \geqq p \geqq 2, n \geqq q \geqq 2$; the containment is proper if the inequality is strict for the larger of $m$ and $n$.

*Proof.* The cyclic semigroup $S = \{0, a, a^2, \cdots, a^{n-1}\}$ with $a^n = 0$ suffices to demonstrate the proper containment for all three statements since $S$ is in the larger class but not the smaller.

We now prove (1), the proof of (2) and (3) is similar. Let $S \in (0, q)^0$ and let $x \in S$. If $x^q = 0$ then $x^n = 0 (n > q)$ and we are done. If $x^q \neq 0$ then there is a $u \in S$ such that $x = ux^q$. Now suppose $n = m(q-1) + p$ where $0 \leqq p < q - 1$. Then $x = (ux)x^{q-1} = u(ux^q)x^{q-1} = (u^2xx^{q-1})x^{q-1} = u^2xx^{2(q-1)} = \cdots = u^{m+1}xx^{(m+1)(q-1)} = vx^n$ upon reassociating after $m$ substitutions. Thus for each $x \in S$ either $x^n = 0$ or $x = vx^n$ for some $v \in S$. Whence $S \in (0, n)^0$ and the proof is complete.

The latter part of the above proof is essentially that of [2] Propriété 3. The technique illustrated in the above proof, treating the three conditions of (1.2) by cases and carefully checking nilpotency, is used in proving many of the following theorems. Whenever this is the case we will omit the proof and refer the reader to the appropriate reference in Croisot [2].

THEOREM 1.5. $(1, n)^0 = (n, 1)^0$ for $n \geqq 2$.

*Proof.* Cf. [2], Theorem 2.

COROLLARY 1.6. $(n, n)^0 = (1, n)^0 = (n, 1)^0$ for $n \geqq 2$.

PROPOSITION 1.7.

$$(m, n)^0 = (p, n)^0 \quad \text{and} \quad (n, p)^0 = (n, m)^0 \qquad \text{for } 2 \leqq p \leqq m \leqq n\,.$$

*Proof.* By (1.4.3) we already have $(p, n)^0 \subseteqq (m, n)^0$. Conversely, suppose we have $S \in (m, n)^0$ and let $x \in S$. If either $x^p = 0$ (then $x^m = 0$) or $x^n = 0$ we are done; if not then $x^n \neq 0$ implies $x^m \neq 0$ ($m \leqq n$) and since $S \in (m, n)^0$ we can find a $u$ in $S$ such that $x = x^m u x^n = x^p(x^{m-p}u)x^n$ whence we can conclude that $S \in (p, n)^0$. It now follows that $(p, n)^0 = (m, n)^0$. The other equality is a direct dualization.

COROLLARY 1.8.

$$(1, n)^0 = (m, n)^0 = (n, m)^0 = (n, 1)^0 \qquad \text{for } n \geqq m \geqq 2\,.$$

PROPOSITION 1.9. $(n, n)^0 = (0, n)^0 \cap (n, 0)^0$ for $n \geqq 2$.

*Proof.* Cf. [2], Theorem 1.

PROPOSITION 1.10. If $S$ is $n^0$-regular ($n \geqq 2$) then for each $x$ in $S$ with $x^n \neq 0$ there is an idempotent $e$ of the form $e = x^{n-1}v = ux^{n-1}$ ($u$ and $v$ in $S$) which is $\mathscr{H}$-equivalent to $x$. Moreover, $ux^{2n-3}v$ is then the group inverse of $x$ with respect to $e$.

*Proof.* If $x^n \neq 0$ then since $S \in (0, n)^0 \cap (n, 0)^0$ by (1.9) we can find a $u$ and $v$ in $S$ such that $x = ux^n = x^n v$. Then $ux^{n-1} = (ux^{n-2})x = (ux^{n-2})x^n v = (ux^n)(x^{n-2})v = x(x^{n-2}v) = x^{n-1}v$ and one can verify directly that $e = ux^{n-1} = x^{n-1}v$ is an idempotent. Clearly $ex = x = xe$ and it follows that $e\mathscr{H}x$. Again, direct calculation shows that $x(ux^{2n-3}v) = (ux^{2n-3}v)x = e$. Since $ux^{2n-3}v = ux^{n-1}x^{n-2}v = x^{n-1}vx^{n-2}v$ and $ux^{2n-3}v = ux^{n-2}x^{n-1}v = ux^{n-2}ux^{n-1}$ we can conclude $ux^{2n-3}v\mathscr{H}e\mathscr{H}x$ and the result follows.

We conclude this section with several propositions which are readily demonstrated.

PROPOSITION 1.11. $(n, n)^0 = (1, n)^0 \cap (n, 0)^0$ for $n \geqq 2$.

PROPOSITION 1.12. If $S$ is regular then $S \in (1, n)^0$ if and only if $S \in (0, n)^0$.

PROPOSITION 1.13. If $S \in (0, n)^0$ and $x^n \neq 0$ then $x$ is not nilpotent.

*Proof.* Briefly, if $S \in (0, n)^0$ and $x \in S$ with $x^n \neq 0$ there is a $u \in S$ such that $x = ux^n$. Then $x = uxx^{n-1} = u(ux^n)x^{n-1} = u^2xx^{2n-2} = \cdots = u^k xx^{kn-k}$. Since $kn - k = k(n-1) \geqq k(n \geqq 2)$ and $x \neq 0$ it is now clear that $x$ can not be nilpotent.

PROPOSITION 1.14. $(m, 0)^0 \cap (0, n)^0 \subseteqq (m, n)^0$ for $m, n \geqq 2$.

*Proof.* Let $S \in (m, 0)^0 \cap (0, n)^0$ and let $x \in S$. Suppose $n \geqq m$. If $x^m \neq 0$ then $x$ is not nilpotent (1.13). Thus there are $u, v \in S$ such that $x = x^m v = ux^n$. Then $ux^{n-1} = ux^{n-2}x^m v = ux^n x^{m-2}v = x^{m-1}v$. Hence $x = x^m v = xx^{m-1}v = xux^{n-1} = (x^m v)ux^{n-2}ux^n$. Now if $x^m = 0$ there is nothing else to show. Whence in either case $S \in (m, n)^0$.

OBSERVATION 1.15. The reverse inclusion of (1.14) is false. For if $n > m$ then $S = \{0, a, \cdots, a^{n-1}\}$ with $a^n = 0$ is $(m, n)^0$-regular but $S \notin (m, 0)^0$.

REMARK 1.16. As Croisot has pointed out the Baer-Levi semigroup which consists of all one-to-one mappings of a denumerable set into itself which "miss" a denumerable subset and the usual composition ($\beta\alpha$: first $\alpha$ then $\beta$) furnishes an example of a semigroup which is $(0, n)^0$-regular but neither $(n,0)^0$-regular nor $(1, 0)^0$-regular. An anti-isomorphic semigroup shows that $(n, 0)^0 \nsubseteq (0, n)^0$.

Fig. 2 schematically connects equivalent classes of $(m, n)^0$-regular semigroups which are represented (as in Fig. 1) by lattice points. Inclusions are as shown.

OPEN QUESTION 1.17. We raise the following problem for consideration: If $S$ is $n^0$-regular ($n > 2$) and $x \in S$ with $x^n \neq 0$ then $x\mathscr{H}e$ for some $e^2 = e \in S$ (1.10). Then $ex = xe = x$. However, for arbitrary $y \in S$ we may have $g^2 = g$ and $gy = yg = y$ and also $f^2 = f$ and $fy = yf = y$. Find necessary and sufficient conditions to imply $f = g$.

From the above equations when $y = x$ and (1.10) we can deduce $fe = ef = e$ so that "each nonzero idempotent of $S$ is primitive" and "$S$ has no nilpotent elements" are sufficient conditions for the uniqueness of a left-right idempotent identity for each $x \in S \in (n, n)^0$.

2. **0-semiprime conditions and $2^0$-regularity.** In [2] Croisot was able to show [Theorem 1] that a semigroup $S$ is the union of groups if and only if $S \in (0, 2) \cap (2, 0)$ and that $(2, 2) = (0, 2) \cap (2, 0)$. This section will be devoted to developing results analogous to these and those of § 4.1 and § 4.2 in [1].

Since in the sequel we will be especially concerned with three types of regularity we single them out here

DEFINITION 2.1. (1) A semigroup $S$ is said to be *left* 0-*regular* if $S \in (0, 2)^0$. Dually

(2) A semigroup $S$ is said to be *right* 0-*regular* if $S \in (2, 0)^0$.

(3) A semigroup $S$ is said to be *intra*-0-*regular* if for each $x$ in $S$ either $x^2 = 0$ or $x = ux^2v$ for some $u$ and $v$ in $S$.

We now give a generalized definition for semiprimeness:

DEFINITION 2.2. (1) An ideal $I$ (any type) of a semigroup (with 0) is said to be 0-*semiprime* if $x \in I$ whenever $x^2 \in I \backslash \{0\}$.

(2) A semigroup $S$ is said to be [*left, right*] 0-*semiprime* whenever every [left, right] two-sided ideal of $S$ is 0-semiprime.

PROPOSITION 2.3. Let $S$ be a semigroup with 0. Then the following are equivalent.

(1) $S$ is [left, right] intra-0-regular.

(2) $S$ is [left, right] 0-semiprime.

(3) If $x \in S$ and $x^2 \neq 0$ then $[x\mathscr{L}x^2, x\mathscr{R}x^2]x\mathscr{J}x^2$.

(4) If $x \in S$ and $x^2 \neq 0$ then $[x \in Sx^2, x \in x^2S]x \in Sx^2S$.

*Proof.* The equivalences follow immediately from their respective definitions in much the same fashion as their restrictive counterparts (e.g., left regular) of [1], § 4.1 (or cf. [2], Propriétés 5, 6 p. 365).

PROPOSITION 2.4. All left, right and two-sided ideals of a semigroup $S$ are 0-semiprime if and only if $S$ is $2^0$-regular.

*Proof.* This follows directly from (1.9) with $n = 2$ and (2.3).

THEOREM 2.5. *If $S$ is a semigroup with 0 then $S$ is $2^0$-regular if and only if $x^2 = 0$ or $x^2 \in H_x$ for each $x \in S$.*

*Proof.* Assume $S$ is $2^0$-regular. Let $x \in S$. If $x^2 \neq 0$ then there is a $u$ in $S$ such that $x = x^2ux^2$. It follows that $x \mathscr{H} x^2$ and $x^2 \in H_x$.

Conversely, suppose $x^2 = 0$ or $x^2 \in H_x$. In the former case (1.2.1) it satisfied; in the latter case $H_x$ is a group ([1], Theorem 2.16) and the equation $x = x^2ux^2$ is the solvable for $u$ in $H_x$. Thus in either case $S$ is $2^0$-regular.

COROLLARY 2.6. *If $S$ is $2^0$-regular then all the irregular elements of $S$ lie in $\mathscr{D}$-classes, $D$, which square to zero, i.e., $D^2 = \{0\}$.*

*Proof.* Let $D$ be an irregular $\mathscr{D}$-class ([1], § 2.3) of $S$. Let $a$ and $b$ be elements of $D$ and let $x \in R_b \cap L_a$. By (2.5) since $S$ is $2^0$-regular, $x^2 = 0$ or else $H_x$ would be a group ([1], Theorem 2.16) and $D$ then would not be irregular. Now by [1], Theorem 2.4, $L_aR_b \subseteqq D'$ a $\mathscr{D}$-class. Since $x^2 = 0 \in L_aR_b$, $D'$ must be the zero $\mathscr{D}$-class $\{0\}$ and $ab = 0$. It thus follows that $D^2 = \{0\}$.

THEOREM 2.7. *Let $S$ be a $2^0$-regular semigroup and suppose $D$ is a nonzero regular $\mathscr{D}$-class union $\{0\}$. Then $D$ is itself a completely 0-simple semigroup.*

*Proof.* We first show that $D$ is a semigroup. Indeed, proceeding as in (2.6) we will show that if $a, b \in D\backslash\{0\}$ either $ab = 0$ or $ab \in R_a \cap L_b$. Thus let $a, b \in D\backslash\{0\}$. Then $L_aR_b \subseteqq D'$, where $D'$ is a $\mathscr{D}$-class ([1], Theorem 2.4). Let $c \in R_b \cap L_a$ which is nonempty (cf. [1], p. 48) so that we have $c^2 \in L_aR_b \subseteqq D'$. If $c^2 = 0$ then $D' = \{0\}$ since $D'$ is a $\mathscr{D}$-class and thus $ab = 0 \in D$. But if $c^2 \neq 0$ then $c^2 \in H_c$ by (2.5) since $S$ is $2^0$-regular. By [1], Theorem 2.16 $H_c = R_b \cap L_a$ is a group. Hence by [1], Theorem 2.17 we have $ab \in R_a \cap L_b \subseteqq D$. If either $a$ or $b$ is 0 then surely $ab \in D$. In any case $ab \in (R_a \cap L_b) \subseteqq D$ or $ab = 0 \in D$.

Since $D\backslash\{0\}$ is regular it contains nonzero idempotents by [1], Lemma 1.13 and hence $D^2 \neq 0$. We will now show that each nonzero idempotent in $D$ is primitive (cf. [1], p. 76). We must show that if $0 \neq e \leqq f$, then $e = f$. In fact we prove more. We shall show for any two idempotents $e, f$ in $D$, even $ef = fe \neq 0$ implies $e = f$. As above, if $ef \neq 0$ then we must have $ef \in R_e \cap L_f$. Likewise $fe \neq 0$ implies $fe \in R_f \cap L_e$. Hence $ef = fe \in R_e \cap L_e \cap R_f \cap L_f = H_e \cap H_f$. Since $ef = fe$ is an idempotent and a group can contain at most one idempotent it follows by [1], Theorem 2.16 that $e = ef = fe = f$.

It remains to show that $D$, as a semigroup is 0-simple. Indeed we will show that if $S^1a = S^1b$ for $a$ and $b$ in $D\backslash\{0\}$ then $D^1a = D^1b$. Suppose $a\mathscr{L}b$ in $S$. Then either $a = b$ and the result follows immediately or there is an $x$ in $S$ such that $xa = b$. Since $a$ and $b$ are regular we can find, [1], Lemma 1.13, $\mathscr{R}$-equivalent idempotents $e$ and $f$ for $a$ and $b$ respectively. Similarly let $g$ be an $\mathscr{L}$-equivalent idempotent for $a$ and $b$. From $xa = b$ and [1], Lemma 2.14 we deduce that $fxea = b$. By [1], Theorem 2.18 we can find an inverse $a'$ for $a$ in $L_e \cap R_g$. Hence $0 \neq fxe = (fxe)e = (fxea)a' = ba'$ and so $ba' \neq 0$. Thus by the first part of the proof $fxe = ba' \in R_b \cap L_{a'} \subseteqq D\backslash\{0\}$. So the equation $za = b$ is solvable in $D$. Likewise $yb = a$ is also solvable in $D$. This clearly is sufficient to show $D^1a = D^1b$. Dually $cS^1 = dS^1$ implies $cD^1 = dD^1$ for $c$ and $d$ in $D\backslash\{0\}$. It follows that $D$ is 0-bisimple and hence 0-simple. Whence we have shown that $D$ is completely 0-simple, and the proof is complete.

The following is a generalization of Theorem 4.6 in [1]. We note that Theorems 4.2, 4.3 and 4.4 of [1] are also capable of generalization in an obvious manner (cf. (2.3)) by inserting "0" in the appropriate spots and permitting null subsemigroups along with the groups.

THEOREM 2.8. *Let $S$ be a regular semigroup with* 0. *Then the following are equivalent*:

(1) *$S$ is $2^0$-regular.*

(2) *$S$ is the* 0-*disjoint union of subsemigroups which are themselves completely* 0-*simple semigroups.*

(3) *$S$ is left* 0-*regular and right* 0-*regular.*

(4) *All left and right ideals of $S$ are* 0-*semiprime.*

*Proof.* (1) $\Leftrightarrow$ (2). If $S$ is $2^0$-regular then by (2.7) each nonzero regular $\mathscr{D}$-class union 0 is a completely 0-simple semigroup. Since $S$ is regular by our overall assumption and $\mathscr{D}$ is an equivalence relation we have a 0-disjoint union of the $\mathscr{D}$-classes of $S$ of the type desired in (2).

Conversely, if we assume (2) holds then for any $x \in S$, $x$ and $x^2$ both belong to a subsemigroup which is completely 0-simple. In such subsemigroups $x^2 \in H_x \cup \{0\}$ by [1], Theorem 2.52. (Note: $H_x$ here denotes an $\mathscr{H}$-class with respect to Green's relation defined on the subsemigroup.) If $x^2 \neq 0$ then $H_x$ is a group [1], Theorem 2.16 and in this case $H_x$ is easily seen to be contained in an $\mathscr{H}$-class of $S$, hence $x^2 \in H_x^S$ where $H_x^S$ is the $\mathscr{H}$-class of $x$ in $S$. If $x^2 = 0$ then $x^2 \in H_x^S \cup \{0\}$.

In either case we can conclude by (2.5) that $S$ is $2^0$-regular.

(1) $\Leftrightarrow$ (3). This is just (1.9) with $n = 2$.

(1) $\Leftrightarrow$ (4). This is just (2.4).

**3. Absorbent semigroups.** Even though the gross structure of a $2^0$-regular semigroup was determined in (2.9) and (2.7) we are still unable to predict the location of the product of any two elements which do not lie within a single $\mathscr{D}$-class. However [1], Theorem 2.5 2.2 does suggest one fruitful subclass of the $2^0$-regular semigroups for investigation. Formally:

DEFINITION 3.1. (1) A semigroup $S$ with zero, 0, will be called *absorbent* if for any two elements $a$ and $b$ we have either $ab \in (R_a \cap L_b)$ or $ab = 0$.

(2) A collection of subsemigroups $\{S_\alpha\}_{\alpha \in \mathscr{A}}$ of a semigroup $S$ with 0 will be called *mutually annihilating* if $0 \in S_\alpha$ for each $\alpha \in \mathscr{A}$ and if $S_\alpha S_\beta = \{0\} = S_\beta S_\alpha$ for $\alpha \neq \beta$.

OBSERVATION 3.2. If we let $a = b$ in (3.1.1) we readily see (using (2.5)) that an absorbent semigroup is $2^0$-regular since $R_a \cap L_a = H_a$.

PROPOSITION 3.3. A regular nonzero $\mathscr{D}$-class union $\{0\}$ of an absorbent semigroup is itself a completely 0-simple semigroup.

*Proof.* This follows immediately from (2.7) by (3.2).

LEMMA 3.4. *Let $S$ be an absorbent semigroup. Then the collection of $\mathscr{D}$-classes union $\{0\}$ of $S$ is mutually annihilating.*

*Proof.* It readily follows from (3.2), (2.6) and (2.7) that each $\mathscr{D}$-class union $\{0\}$ is a semigroup. If $a \not\mathscr{D} b$ then $R_a \cap L_b = \varnothing$ by a remark on [1], p. 48. It then follows from the definition of absorbency that $ab = 0$ and thus $(D_a \cup \{0\})(D_b \cup \{0\}) = \{0\}$.

THEOREM 3.5. *A semigroup $S$ with 0 is absorbent if and only if it is$2$ $^0$-regular, and the collection of its $\mathscr{D}$-classes union $\{0\}$ is mutually annihilating.*

*Proof.* One implication follows directly from (3.2) and (3.4).

Conversely, suppose $S$ is $2^0$-regular and that the product of any two distinct, $\mathscr{D}$-classes is $\{0\}$. Let $a$ and $b$ be given. If $a \not\mathscr{D} b$ then $ab = 0$. On the other hand, if $a \mathscr{D} b$ and $D_a$ is irregular then we have $ab = 0$ by (2.6); but if $D_a$ is regular then $D_a \cup \{0\}$ is completely 0-simple (2.7) and we have $ab \in (R_a \cap L_b)$ or $ab = 0$ by [1], Theorem 2.5 2.2

and $S$ is absorbent. This completes the proof.

COROLLARY 3.6. *A regular semigroup with* 0 *is absorbent if and only if it is mutually annihilating collection of completely* 0-*simple semigroups with a common zero.*

COROLLARY 3.7. *If $S$ is an absorbent semigroup then $\mathscr{J} = \mathscr{D}$ on $S$. Indeed each $\mathscr{D}$-class union $\{0\}$ is an ideal.*

*Proof.* Since $S$ is absorbent the last statement is immediate since the $\mathscr{D}$-classes are mutually annihilating by (3.4). Suppose now that $b \neq 0$ and $a \mathscr{J} b$. Then we can find $u, v \in S^1$ such that $a = ubv$. Now we have seen above that $D_b \cup \{0\}$ is an ideal and from $a \mathscr{J} b, b \neq 0$ it follows that $a \neq 0$. Thus from $a = ubv$ we can conclude $a \in D_b$. Whence $a \mathscr{D} b$ and the proof is complete since $J_0 = D_0 = \{0\}$.

In order to investigate the irregular $\mathscr{D}$-classes of an absorbent semigroup we need the following definition:

DEFINITION 3.8. An ideal $I$ of a semigroup $S$ is called $[0-]$ *prime* if and only if $ab \in I[ab \in I \backslash \{0\}]$ implies $a$ or $b$ belongs to $I$.

REMARK 3.9. One readily observes that an ideal $I$ of $S$ is $[0-]$ prime if and only if $S \backslash I[(S \backslash I) \cup \{0\}]$ is a subsemigroup.

DEFINITION 3.10. A semigroup $S$ is said to be *ideally irregular* if the subset, $U$ of irregular elements and 0 is an ideal.

PROPOSITION 3.11. An absorbent semigroup $S$ is ideally irregular. Indeed $U$ is a 0-prime ideal.

*Proof.* Clearly $U$ is the union of the irregular $\mathscr{D}$-classes and $\{0\}$. That $U$ is an ideal follows from (3.7). By (3.3) each regular $\mathscr{D}$-class union $\{0\}$ is a semigroup. Since the $\mathscr{D}$-classes of $S$ are mutually annihilating (3.4) $(S \backslash U) \cup \{0\}$ is seen to be a subsemigroup. (3.9) completes the proof. (Indeed, in exactly the same manner we can show that each $\mathscr{D}$-class union $\{0\}$ of an absorbent semigroup is a 0-prime ideal.)

PROPOSITION 3.12. If $S$ is an absorbent semigroup and $i$ an irregular element of $S$ then $D_i = \{i\}$.

*Proof.* Suppose that there is an $i \neq 0$ and $i'$ ($i$ irregular) such that $i' \mathscr{L} i$ and $i' \neq i$. Then there is an $x$ such that $i = xi'$. Either

$x \in D_{i'} = D_i$ and then $xi' = 0$ by (3.5) and (2.6) or $x \notin D'_i$ and $xi' = 0$ by (3.5). In either case $i = 0$, a contradiction. Thus $L_i = \{i\}$. Dually $R_1 = \{i\}$. It follows immediately that $D_i = \{i\}$.

OPEN QUESTION 3.13. If $S$ is a $2^0$-regular semigroup are there necessary and sufficient conditions for $U \subset S$ to be an ideal [prime ideal]?

We conclude the paper with a brief investigation of the relationship between absorbency and those semigroups for which $\mathscr{L}$ and $\mathscr{R}$ are congruence relations.

PROPOSITION 3.14. If $S\backslash\{0\}$ is a subsemigroup of $S$ and if $S$ is an absorbent semigroup then $\mathscr{L}$ and $\mathscr{R}$ are congruences.

*Proof.* The follows directly from the definition of absorbency.

The converse of (3.14) is false. In an infinite cyclic semigroup, which is far from being absorbent, Green's relations are just equality and thus trivially congruences. However, if $S$ is regular we do have the following decomposition:

PROPOSITION 3.15. Let $S$ be a regular semigroup. If $\mathscr{L}$ and $\mathscr{R}$ are congruences on $S$ then $S$ is the union of groups.

*Proof.* We will show that each $\mathscr{H}$-class of $S$ is a group. Let $H_a$ be given. Then since $a$ is regular there are $\mathscr{L}$ and $\mathscr{R}$ equivalent idempotents $e$ and $f$ respectively [1], Lemma 1.13. Using [1], Theorem 2.14, and the hypothesis we have $a = ae\mathscr{L}a^2$ and $a = fa\mathscr{R}a^2$. But this implies $a\mathscr{H}a^2$. Whence by [1], Theorem 2.16, $H_a$ is a group.

The following theorem provides a partial converse of both the above proposition (since completely simple semigroups are unions of groups [1], Theorem 4.6) and Theorem 2.51 of [1].

THEOREM 3.16. *A regular* $[-0]$ *simple semigroup* $S$ *is completely simple* [*with adjoined* 0] *if and only if* $\mathscr{L}$ *and* $\mathscr{R}$ *are congruences on* $S$.

*Proof.* Suppose $\mathscr{L}$ and $\mathscr{R}$ are congruences. Then by (3.15) $S$ is the union of groups. Application of [1], Theorem 4.5, then shows that $S$ is completely simple.

The converse is given by [1] Ex. 9, p. 83.

We conclude by remarking that regular absorbent semigroups will be further investigated in [3].

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