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LARGE SUBLATTICES OF A LATTICE

THOMAS P. WHALEY

In this paper we consider a special case of the following two closely related problems of B. Jónsson:

I. For which infinite cardinals m is there an algebra of power m which has finitely many operations and satisfies the descending chain condition for subalgebras.

II. For which infinite cardinals m is there an algebra of power m which has finitely many operations and has no proper subalgebra of power m.

Of course a positive answer to the first problem for a given cardinal always indicates a positive answer to the second for the same cardinal.

The special case we are concerned with is obtained by further restricting the algebras to be lattices. With this restriction we obtain a negative answer to the second problem for any regular cardinal. It follows that the answer to the first question is negative for the class of lattices and for any infinite cardinal m. Actually we obtain a stronger result which shows that in a lattice of power m where m is infinite and regular, there are at most two elements which do not lie in the complement of a sublattice of power m. We give an example to show that regularity is needed here. In a distributive lattice of regular cardinality m every element lies in the complement of some sublattice of cardinality m.

We adopt the conventions of identifying an ordinal with the set of smaller ordinals and of identifying a cardinal m with the smallest ordinal of cardinality m.

The bibliography includes most of the results related to problems I and II.

2. Lattices of regular cardinality. Throughout this paper $\langle L; +, \cdot, \leqq\rangle$ will denote a lattice of power $m \geqq \omega$ in which $x + y$ is the least upper bound of $\{x, y\}$ and $x \cdot y$ is the greatest lower bound of $\{x, y\}$ for any $x, y \in L$. We usually identify such a lattice with the underlying set L. We use the notation $K \subseteqq_s L$ to indicate that K is a sublattice of L. For $x, y \in L$ we let

$$S(x, y) = \{z \in L \mid z + x = y\}$$

and

$$T(x, y) = \{z \in L \mid z \cdot x = y\} .$$

In particular $S(x, x)$ is the principal ideal generated by x and $T(x, x)$ is the principal dual ideal generated by x. We write $S(x)$ for $S(x, x)$ and $T(x)$ for $T(x, x)$. Thus for any $x \in L$, $S(x) \subseteqq_s L$ and $T(x) \subseteqq_s L$. It is easy to check that $S(x, y)$ and $T(x, y)$ are sublattices of L for any $x, y \in L$ if L is distributive. Since for any $x, y \in L$ we have $x + y \in T(x)$ and $y \in S(x, x + y)$, we see that

(1) $L = \bigcup \{S(x, z) \mid z \in T(x)\}$

disjointly and dually

(2) $L = \bigcup \{T(x, z) \mid z \in S(x)\}$

disjointly.

If k is a cardinal, we let M_k be the two-dimensional lattice having k atoms. We indicate this lattice in our diagrams by the figure

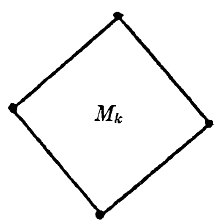

FIGURE 1.

If L has a largest element, we denote this element by 1, and we let 0 indicate the smallest element of L if such an element exists. If L has a largest and a smallest element and $x \in L$, we let $C(x)$ be the set of all complements of x, i.e., $C(x) = S(x, 1) \cap T(x, 0)$.

The following theorem yields some immediate results concerning the problems under investigation. This theorem is also used in almost every proof of this paper.

THEOREM 2.1. *If L is a lattice of power m where m is infinite and regular, then one of the following conditions must hold*:

(i) *There is an $x \in L$, distinct from 1 if 1 exists, with* $|S(x)| = m$.

(ii) *There is an $x \in L$, distinct from 0 if 0 exists, with* $|T(x)| = m$.

(iii) *L has a sublattice isomorphic to M_m.*

Proof. Suppose (i) and (ii) fail. Let x be any element of L other than a possible largest or smallest element. Since (i) and (ii) fail, we have $|S(x) \cup T(x)| < m$. By (1) we have $L = \bigcup \{S(x, y) \mid y \in T(x)\}$ disjointly. Thus $m = |L| = \sum (|S(x, y)|; y \in T(x))$. Since $|T(x)| < m$ and m is regular, there is a $y_0 \in T(x)$ with $|S(x, y_0)| = m$. It is clear that $S(x, y_0) \subseteqq S(y_0)$. Hence $|S(y_0)| = m$ implying that $y_0 = 1$. If $y \in T(x) - \{1\}$, then $|S(y)| < m$ so $|S(x, y)| < m$. Thus

$$\begin{aligned}|L - S(x, 1)| &= |\bigcup \{S(x, y) \mid y \in T(x) - \{1\}\}| \\ &= \sum (|S(x, y)|; y \in T(x) - \{1\}) \\ &< m .\end{aligned}$$

Similarly, $|L - T(x, 0)| < m$. Combining these inequalities, we get

$$\begin{aligned}|L - C(x)| &= |L - (S(x, 1) \cap T(x, 0))| \\ &\leqq |L - S(x, 1)| + |L - T(x, 0)| \\ &< m .\end{aligned}$$

This holds for any $x \neq 0, 1$. We use this fact to obtain a sequence $\{x_\xi \mid \xi < m\} \subseteqq L - \{0, 1\}$ so that $x_\xi \in C(x_{\xi'})$ whenever $\xi, \xi' < m$ and $\xi \neq \xi'$. Inductively, suppose we have $\beta < m$ and $\{x_\xi \mid \xi < \beta\}$ with this property. Note that

$$\begin{aligned}|L - \bigcap \{C(x_\xi) \mid \xi < \beta\}| &= |\bigcup \{(L - C(x_\xi)) \mid \xi < \beta\}| \\ &\leqq \sum(|L - C(x_\xi)|; \xi < \beta) \\ &< m\end{aligned}$$

since m is regular, $\beta < m$, and for each $\xi < \beta$, $|L - C(x_\xi)| < m$. Now we take $x_\beta \in \bigcap \{C(x_\xi) \mid \xi < \beta\}$. Clearly $\{x_\xi \mid \xi < m\} \cup \{0, 1\}$ is a sublattice of L isomorphic to M_m.

COROLLARY 2.2. *If L is a lattice of cardinality m where m is infinite and regular, then L has a proper sublattice of power m.*

COROLLARY 2.3. *No infinite lattice satisfies the descending chain condition for subalgebras.*

COROLLARY 2.4. *If L is a lattice of cardinality m where m is infinite and regular, then one of the following must hold:*
(i) L has an infinite chain of elements
(ii) L has a sublattice isomorphic to M_m.

Proof. Suppose (ii) fails. We obtain (i) by repeatedly applying (i) and (ii) of Theorem 2.1.

COROLLARY 2.5. (*Well-known*). *No infinite distributive lattice is finite dimensional.*

EXAMPLE 2.6. Suppose m is not regular, say $m = \sum (m_\alpha \mid \alpha \in k)$ where each $m_\alpha < m$ and $k < m$. Consider the lattice.

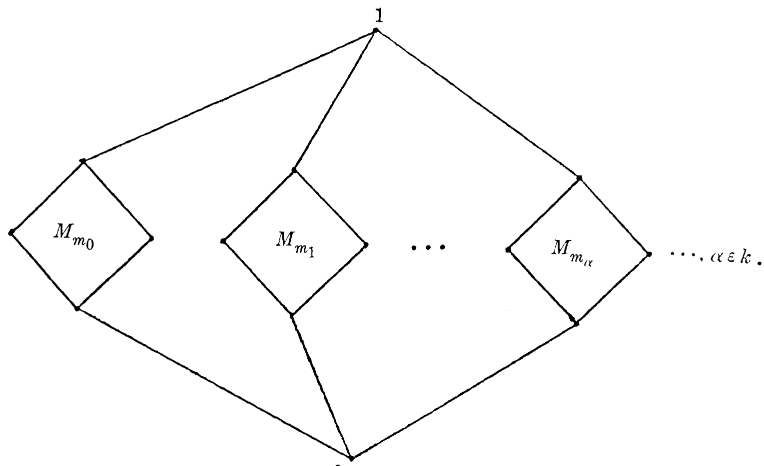

FIGURE 2

It is clear that (i), (ii), and (iii) of Theorem 2.1 fail in this lattice.

3. Separation of elements by large sublattices. Theorem 2.1 leads us to consider how many elements of a lattice of regular cardinality are disjoint from some "large" sublattice. Also, given two elements of such a lattice, is there a "large" sublattice which contains one but not the other? Of course we note that M_k where k is infinite has two elements, 0 and 1, which are in every "large" sublattice and hence may not be so separated. We proceed now to show that this is essentially the only such example.

DEFINITION 3.1. Suppose L is a lattice of power $m \geqq \omega$ and $K \subseteqq_s L$ with $|K| = m$. We say K *separates* x *from* y if $x \in K$ and $y \in L - K$. We say x *can be separated from* y if such a K exists.

LEMMA 3.2. *Suppose L is a lattice of power $m \geqq \omega$ with m regular. If $y \in L$ has the properties*
(i) $|S(y) \cup T(y)| < m$,
(ii) *if $z \in L$ with $|S(z)| = m$, then $y \in S(z)$,*
(iii) *if $z \in L$ with $|T(z)| = m$, then $y \in T(z)$,*
then L has a sublattice isomorphic to M_m having y as an atom.

Proof. As in the proof of Theorem 2.1 there is a $c \in L$ with $|S(y, c)| = m$. Consider any $z \in S(y, c)$, $z \neq c$. By (ii) and (iii) we have $|S(z) \cup T(z)| < m$. Now

$$S(y, c) = \bigcup \{S(z, x) \cap S(y, c) \mid x \in T(z)\} .$$

Thus there is $c' \in L$ with $|S(z, c') \cap S(y, c)| = m$. Since $|S(c')| = |S(z, c')| = m$, we have $y \in S(c')$ by (ii). Thus for $x \in S(z, c') \cap S(y, c)$

$$c' = c' + y = (x + z) + y = (x + y) + z = c + z = c\ .$$

It follows that $| S(y, c) - S(z, c) | < m$. We can now proceed as in the proof of Theorem 2.1 to obtain our desired copy of M_m.

THEOREM 3.3. *Suppose L is a lattice of power m with m infinite and regular. Assume furthermore that $x, y \in L$ with $x \neq y$, that x cannot be separated from y, and that y cannot be separated from x. Then there is a sublattice of L isomorphic to M_m which has one of x, y as the largest element and the other as the smallest.*

Proof. We consider two cases.

Case 1. x is not related to y: First we note that $| S(x) \cup T(x) | < m$. For otherwise $S(x) \cup T(x)$ separates x from y. Now if $z \in L$ and $| S(z) | = m$ we must have $x \in S(z)$. For if not $S(z) \cup T(y)$ separates y from x. Dually, if $| T(z) | = m$, then $x \in T(z)$. By Lemma 3.2 there is a copy of M_m occuring as a sublattice of L and having x as an atom. If y does not belong to this sublattice then x is separated from y. Otherwise we separate x from y by removing y from this sublattice.

Case 2. x is related to y: Without loss of generality, assume that $x < y$. Furthermore, we assume that the conclusion of the theorem is false. We observe that if $z \in L$ with $| S(z) | = m$, then $x, y \in S(z)$. For if $x \notin S(z)$, then $S(z) \cup T(y)$ separates y from x. If $y \notin S(z)$, then $S(z)$ separates x from y. The dual argument gives $x, y \in T(w)$ whenever $| T(w) | = m$. We also note that $| S(x) \cup T(y) | < m$.

Suppose now that $| S(y) \cap T(x) | = m$. Applying Theorem 2.1 to the lattice $S(y) \cap T(x)$ gives

(i) $z \in L$ with $z < y$ and $| S(z) \cap T(x) | = m$,

(ii) $z \in L$ with $x < z$ and $| T(z) \cap S(y) | = m$,

or (iii) there is a copy of M_m occuring as a sublattice of $S(y) \cap T(x)$. Since none of these can happen, we must have $| S(y) \cap T(x) | < m$.

Assume now that $| T(x) | = m$. We know that

$$| (T(x) \cap S(y)) \cup T(y) | < m\ .$$

Also if $z \in T(x)$ with $| S(z) \cap T(x) | = m$, we have $y \in S(z) \cap T(x)$. If $z \in T(x)$ with $| T(z) | = m$, then $z = x$ and $y \in T(z)$. Hence we can apply Lemma 3.2 to the lattice $T(x)$ and the element y to get a copy of M_m occuring as a sublattice of $T(x)$ having y as an atom. We must have x as the smallest element of this sublattice. Thus if we remove y from this sublattice, we separate x from y. This shows that $| T(x) | < m$.

We now apply Lemma 3.2 to L and the element x to get a copy of M_m having x as an atom. If y is not the largest element of this sublattice, then we have separated x from y. If y is the largest element of this sublattice, we just remove x from this sublattice to separate y from x. This final contradiction comes from the assumption that the conclusion of the theorem is false.

COROLLARY 3.4. *In a lattice of power m where m is infinite and regular, there are at most two elements which do not lie in the complement of some sublattice of power m.*

EXAMPLE 3.5. Let k be any infinite cardinal. Take $L(k)$ to be the set of all finite subsets S of k which satisfy the restrictions

(i) if $|S|$ is odd, then $\frac{|S|-1}{2} \subseteqq S$,

(ii) if $|S|$ is even, then $\frac{|S|}{2} \subseteqq S$.

If S is a finite subset of k, we let $\bar{S} = S \cup n$ where n is the smallest member of ω for which $S \cup n \in L(k)$. It is now fairly routine to check that $L(k)$, ordered by set inclusion, is a lattice in which the least upper bound of two elements S and T is $\overline{S \cup T}$, and the greatest lower bound is $S \cap T$.

It is clear that the atoms of this lattice are the sets of the form $\{\xi\}$ where $\xi \in k$. Also, we can check that any sublattice which contains infinitely many atoms contains each $n \in \omega$.

Suppose now that m is an infinite cardinal with $cf(m) = k < m$. Then there are cardinals $m_\alpha < m$ for $\alpha \in k$ so that $\sum (m_\alpha \mid \alpha \in k) = m$. We obtain a lattice L from the following diagram by letting the elements $\{\xi\}$ for $\xi \in k$ generate $L(k)$ as a sublattice:

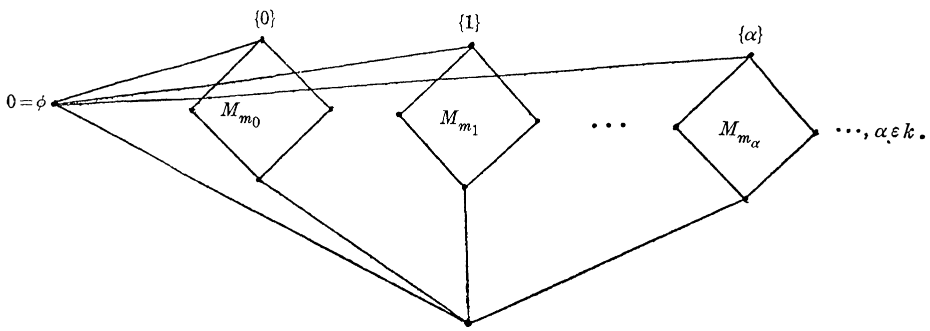

FIGURE 3

Then any sublattice of L of power m must contain infinitely many of the elements $\{\alpha\}$ with $\alpha \in k$. This shows that each $n \in \omega$ is in such a sublattice. Thus there are infinitely many elements of L which lie in each sublattice of power m.

LEMMA 3.6. *If L is a distributive lattice of power $m \geqq \omega$, then for any $y \in L$ we have $|S(y) \cup T(y)| = m$.*

Proof. Define a map ϕ of L into $S(y)xT(y)$ by $\phi(x) = (x \cdot y, x + y)$. Since L is distributive, ϕ is one-to-one (cf [1]). Hence $|S(y)| \cdot |T(y)| = m$; so $|S(y)| = m$ or $|T(y)| = m$.

THEOREM 3.7. *If L is a distributive lattice of regular power $m \geqq w$, then each element of L lies in the complement of some sublattice of power m.*

Proof. Suppose x_0 is a member of each sublattice of power m. By Lemma 3.6 $|S(x_0) \cup T(x_0)| = m$. We assume that $|S(x_0)| = m$. Take any $x_1 \in S(x_0)$ with $x_1 \neq x_0$. Applying Lemma 3.6 to the lattice $S(x_0)$ and the element x_1 gives

$$|S(x_1) \cup (T(x_1) \cap S(x_0))| = m .$$

Clearly we must have $|T(x_1) \cap S(x_0)| = m$. Let Y be the set of all elements of $S(x_0)$ which are not related to x_1. Suppose $|Y| = m$. Then since

$$Y = \bigcup \{T(x_1, z) \cap Y \mid z \in S(x_1) - \{x_1\}\} ,$$

we must have $|T(x_1, z_0) \cap Y| = m$ for some $z_0 < x_1$. Thus $|T(x_1, z_0) \cap S(x_0)| = m$. However, since L is distributive, $T(x_1, z_0) \cap S(x_0)$ is a sublattice of L not containing x_0. Hence we must have $|Y| < m$ or $|S(x_0) - T(x_1)| < m$. This holds for any $x_1 < x_0$.

Suppose $\xi < m$ and we have $\{x_\beta \mid \beta < \xi\}$ having the property that $1 \leqq \beta < \beta'$ implies that $x_\beta < x_{\beta'} < x_0$. Now

$$\begin{aligned} |S(x_0) - \cap \{T(x_\beta) \mid \beta < \xi\}| &= |\bigcup \{S(x_0) - T(x_\beta) \mid \beta < \xi\}| \\ &\leqq \sum (|S(x_0) - T(x_\beta)|; \beta < \xi) \\ &< m \end{aligned}$$

since m is regular. We take $x_\xi \in \cap \{S(x_0) \cap T(x_\beta) \mid \beta < \xi\}$, $x_\xi < x_0$. In this way we get $\{x_\xi \mid \xi < m\}$ a chain of elements of L. But then $\{x_\xi \mid 1 \leqq \xi < m\}$ is a sublattice of L not containing x_0. This contradiction completes the proof.

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VANDERBILT UNIVERSITY AND
FISK UNIVERSITY