PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 2, 1969

THE δ^2-PROCESS AND RELATED TOPICS II

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This paper considers three transforms of a complex series Σa_n: namely, (1) Aitken's δ^2-transform Σb_n, (2) Lubkin's W-transform Σc_n, and (3) a closely related transform Σd_n which the author calls the $W1$-transform and for which $\sum_0^n d_k = \sum_0^{n+1} c_k$. If $a_{n-1} \neq 0$, set $r_n = a_n/a_{n-1}$. If, moreover, Σa_n converges, define $T_n = (a_n + a_{n+1} + \cdots)/a_{n-1}$ and let $MR(\Sigma a_n)$ be the class of all series converging more rapidly to the sum $S = \Sigma a_n$ than Σa_n. Some of the results proven in this paper are as follows:

(1) If $b_n/a_n \to 0$, then the three conditions (i) $\Sigma b_n \in MR(\Sigma a_n)$, (ii) $\Sigma c_n \in MR(\Sigma a_n)$, and (iii) $\Sigma d_n \in MR(\Sigma a_n)$ are equivalent.

(2) $\Sigma b_n \in MR(\Sigma a_n)$ if and only if $\Delta T_n \to 0$.

(3) If $|r_n| \leqq \rho < 1$ for all sufficiently large n, then the three conditions (i) $\Sigma b_n \in MR(\Sigma a_n)$, (ii) $\Delta r_n \to 0$, and (iii) $b_n/a_n \to 0$ are equivalent.

Samuel Lubkin has given several sufficient conditions for $\Sigma b_n \in MR(\Sigma a_n)$ in case Σa_n is a real series. The third result above contains a generalization of one of his results to the complex plane while relaxing some of his hypothesis.

The following results on complex products are also proven:

(4) If the sequence $\{1/a_n - 1/a_{n-1}\}$ is bounded, then the product $\Pi_0^\infty (1 + a_n)$ diverges.

(5) Suppose that $|r_n| \leqq \rho < 1$ for all sufficiently large n and $a_n \neq -1$ for all n. Then a necessary and sufficient condition for the δ^2-transform to accelerate the convergence of the infinite product $\Pi_0^\infty (1 + a_n)$ is that $\Delta r_n \to 0$.

The notations and definitions set forth in Tucker [2] will be used in this paper. In particular, $S_n = a_0 + a_1 + \cdots + a_n$, $\Sigma a_n = \sum_0^\infty a_n$, and $S = \Sigma a_n$ if Σa_n is convergent. Given a second series $\Sigma a'_n$ we use the notation $S'_n = a'_0 + \cdots + a'_n$, $r'_n = a'_n/a'_{n-1}$ for $a'_{n-1} \neq 0$, $S' = \Sigma a'_n$ and $T'_n = (S' - S'_{n-1})/a'_{n-1}$ for $a'_{n-1} \neq 0$. Likewise, given a "transform sequence" $\{\alpha_n\}$, α_n complex, we set $S_{\alpha n} = S_n + a_{n+1}\alpha_{n+1}$ for $n \geqq 0$, $a_{\alpha 0} = S_{\alpha 0} = a_0 + a_1\alpha_1$, and $a_{\alpha n} = S_{\alpha n} - S_{\alpha(n-1)}$ for $n \geqq 1$.

The transform sequences associated with the δ^2, W, and $W1$ transforms are defined respectively as follows:

(i) $\alpha_n = 1/(1 - r_n)$, $n \geqq 1$,

(ii) $\alpha_1 = -a_0/a_1$; $\alpha_n = (1 - r_{n-1})/(1 - 2r_n + r_{n-1}r_n)$, $n \geqq 2$,

(iii) $\alpha_n = (1 - r_{n+1})/(1 - 2r_{n+1} + r_n r_{n+1})$, $n \geqq 1$.

Whenever division by zero occurs in (i), we set $\alpha_n = 0$. We do likewise for (ii) and (iii). As in Tucker [2], we retain the notation

$\{\delta_n\}$ for the δ^2-transform sequence, and if "$*$" denotes any relation, the notation "$*.$" means that $*$ holds for all sufficiently large n and "$*:$" means that $*$ holds for infinitely many positive integers n.

In what follows, the author is generally interested in the interrelationships between the conditions (1) $\Sigma b_n \in MR(\Sigma a_n)$, (2) $\Sigma c_n \in MR(\Sigma a_n)$, (3) $\Sigma d_n \in MR(\Sigma a_n)$, (4) $b_n/a_n \to 0$, (5) $\Delta T_n \to 0$, (6) $\Delta r_n \to 0$, (7) $|r_n| \leqq. B$ for some B, and (8) $0 < B \leqq. |1 - r_n|$ for some B. Also, the notation Σb_n, Σc_n and Σd_n specified in the first paragraph for the respective δ^2, W and $W1$ transforms will not be used in what follows. Instead, the appropriate $\Sigma a_{\delta n}$ or $\Sigma a_{\alpha n}$ notation will be employed.

The following two theorems, the second in particular, are helpful when investigating acceleration.

THEOREM 1. *Suppose that Σa_n is a complex series, $\{b_n\}$ is a complex sequence, and $\Sigma a'_n$ is a series with partial sums $S'_n =. S_n + b_{n+1}$. Then $\Sigma a'_n \in MR(\Sigma a_n)$ if and only if $b_{n+1} \sim S - S_n \to 0$.*

Proof. If either condition holds, then

$$S - S_n =. S - S'_n + b_{n+1} \neq. 0 ,$$

so that $b_{n+1}/(S-S_n) + (S-S'_n)/(S-S_n) =. 1$. Thus $(S-S'_n)/(S-S_n) \to 0$ and $S - S_n \to 0$, if and only if, $b_{n+1}/(S - S_n) \to 1$ and $S - S_n \to 0$; but this is equivalent to $b_{n+1} \sim S - S_n \to 0$.

From Theorem 1, we see that the class of all sequences $\{c_n\}$ such that $\Sigma a'_n \in MR(\Sigma a_n)$, where $S'_n = S_n + c_{n+1}$, is completely determined by one such sequence $\{b_n\}$; the required condition being that $c_n \sim b_n$. Similarly, we now show that if $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, then $\Sigma a_{\beta n} \in MR(\Sigma a_n)$, if and only if $\beta_n \sim \alpha_n$.

THEOREM 2. *Suppose that $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$. Then $\Sigma a_{\beta n} \in MR(\Sigma a_n)$ if and only if $\beta_n \sim \alpha_n$.*

Proof. From Theorem 1, $a_{n+1}\alpha_{n+1} \sim S - S_n \to 0$. Hence, from Theorem 1, $\Sigma a_{\beta n} \in MR(\Sigma a_n)$ if and only if $a_{n+1}\beta_{n+1} \sim S - S_n$, and this is equivalent to $a_{n+1}\beta_{n+1} \sim a_{n+1}\alpha_{n+1}$, that is, $\beta_{n+1} \sim \alpha_{n+1}$.

LEMMA 3. *If $(1 - r_n)(1 - r_{n+1}) \neq 0$, then $a_{\delta n}/a_n = 1/(1 - r_{n+1}) - 1/(1 - r_n) = r_{n+1}/(1 - r_{n+1}) - r_n/(1 - r_n) = (r_{n+1} - r_n)/(1 - r_n)(1 - r_{n+1})$.*

Proof. Since $r_n \neq 1$ and $r_{n+1} \neq 1$, we have $\delta_n = 1/(1 - r_n)$ and $\delta_{n+1} = 1/(1 - r_{n+1})$. Thus, $a_{\delta n}/a_n = (a_n + a_{n+1}\delta_{n+1} - a_n\delta_n)/a_n = 1 + r_{n+1}\delta_{n+1} - \delta_n = r_{n+1}/(1 - r_{n+1}) + 1 - 1/(1 - r_n) = r_{n+1}/(1 - r_{n+1}) - r_n/(1 - r_n) = [r_{n+1}(1 - r_n) -$

$r_n(1-r_{n+1})]/(1-r_n)(1-r_{n+1}) = (r_{n+1}-r_n)/(1-r_n)(1-r_{n+1}) = 1/(1-r_{n+1}) - 1/(1-r_n)$.

We now establish a relationship between the δ^2-transform and the $W1$-transform.

THEOREM 4. *Suppose that* $a_{\delta n}/a_n \to 0$. *Then* $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ *if and only if* $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, *where* $\alpha_n =. (1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1})$.

Proof. Suppose that $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. From Lemma 3,

$$\begin{aligned}1 - 2r_{n+1} + r_n r_{n+1} &=. (1-r_n)(1-r_{n+1}) - (r_{n+1} - r_n) \\ &=. (1-r_n)(1-r_{n+1}) \cdot [1 - (r_{n+1}-r_n)/(1-r_n)(1-r_{n+1})] \\ &=. (1-r_n)(1-r_{n+1})(1 - a_{\delta n}/a_n) \neq. 0 \ .\end{aligned}$$

Hence, $\alpha_n/\delta_n =. (1-r_n)(1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1}) =. 1/(1-a_{\delta n}/a_n) \to 1$. From Theorem 2, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$.

Suppose that $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$. Then $r_n \neq. 1$, so that

$$\alpha_n/\delta_n =. 1/(1-a_{\delta n}/a_n) \to 1$$

and, from Theorem 2, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.

The same type of relationship is now established between the δ^2-transform and the W-transform.

THEOREM 5. *Suppose that* $a_{\delta n}/a_n \to 0$. *Then* $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ *if and only if* $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, *where* $\alpha_n =. (1-r_{n-1})/(1-2r_n+r_{n-1}r_n)$.

Proof. Suppose that $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. As in the proof of Theorem 4,

$$1 - 2r_n + r_{n-1}r_n =. (1-r_{n-1})(1-r_n)[1 - a_{\delta(n-1)}/a_{n-1}] \neq. 0 \ .$$

Hence,

$$\alpha_n/\delta_n =. (1-r_{n-1})(1-r_n)/(1-2r_n+r_{n-1}r_n) =. 1/(1-a_{\delta(n-1)}/a_{n-1}) \to 1 \, .$$

From Theorem 2, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$.

Suppose that $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$. Then $r_n \neq. 1$, and thus

$$\alpha_n/\delta_n =. 1/(1-a_{\delta(n-1)}/a_{n-1}) \to 1 \ .$$

From Theorem 2, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.

The next theorem helps to establish the significance of the quantities T_n when dealing with acceleration in general.

THEOREM 6. $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, $\alpha_n \sim T_n/r_n$, *and* $\alpha_n \sim 1 + T_{n+1}$ *are equivalent.*

Proof. From Theorem 1, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ if and only if $a_{n+1}\alpha_{n+1} \sim S - S_n \to 0$; and this is equivalent to $\alpha_{n+1} \sim (S - S_n)/a_{n+1} = T_{n+1}/r_{n+1}$. Moreover, $\alpha_n \sim T_n/r_n$ is equivalent to $\alpha_n \sim 1 + T_{n+1}$, since $T_n/r_n = 1 + T_{n+1}$.

We now establish a useful algebraic expression for $(S - S_{\delta(n-1)})/(S - S_{n-1})$ in terms of ΔT_n.

LEMMA 7. *If Σa_n is a convergent series and n is a positive integer such that $T_{n+1} - T_n \neq -1$, then*

$$(S - S_{\delta(n-1)})/(S - S_{n-1}) = (T_{n+1} - T_n)/(1 + T_{n+1} - T_n) .$$

Proof. From $(1 - r_n)(1 + T_{n+1}) = 1 + T_{n+1} - T_n \neq 0$, $T_{n+1} \neq -1$ and $r_n \neq 1$. Thus $S - S_{n-1} = a_n(1 + T_{n+1}) \neq 0$. We then have

$$\begin{aligned}(S - S_{\delta(n-1)})/(S - S_{n-1}) &= (S - S_{n-1} - a_n\delta_n)/(S - S_{n-1})\\ &= 1 - a_n\delta_n/(S - S_{n-1})\\ &= 1 - \frac{a_n}{S - S_{n-1}}\,\frac{1}{1 - r_n} = 1 - \frac{1}{T_n}\,\frac{r_n}{1 - r_n}\\ &= 1 - \frac{T_n/(1 + T_{n+1})}{1 - T_n/(1 + T_{n+1})}\,\frac{1}{T_n}\\ &= 1 - 1/(1 + T_{n+1} - T_n) = (T_{n+1} - T_n)/(1 + T_{n+1} - T_n) .\end{aligned}$$

We now establish necessary and sufficient conditions for the δ^2-process to accelerate the convergence of a convergent series Σa_n.

THEOREM 8. *$\Sigma a_{\delta n} \in MR(\Sigma a_n)$ if and only if $T_{n+1} - T_n \to 0$.*

1*st Proof.* From Theorem 6, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ if and only if $\delta_n \sim 1 + T_{n+1}$, and this is equivalent to $(1 + T_{n+1})(1 - r_n) \to 1$, since $\delta_n =. 1/(1 - r_n)$. Finally, $(1 + T_{n+1})(1 - r_n) \to 1$ if and only if $T_{n+1} - T_n \to 0$, since $T_{n+1} - T_n =. (1 + T_{n+1})(1 - r_n) - 1$.

2*nd Proof.* If $T_{n+1} - T_n \to 0$, then $T_{n+1} - T_n \neq. -1$. Thus, from Lemma 7, $(S - S_{\delta(n-1)})/(S - S_{n-1}) =. (T_{n+1} - T_n)/(1 + T_{n+1} - T_n) \to 0$. Conversely, suppose that $(S - S_{\delta(n-1)})/(S - S_{n-1}) \to 0$. Then $a_n \neq. 0$ and $r_n \neq. 1$, since $\delta_n \neq. 0$. We must have $1 + T_{n+1} - T_n \neq. 0$, since otherwise $(1 - r_n)(T_n/r_n) =. 1 + T_{n+1} - T_n =: 0$, and $S - S_{n-1} =: 0$; a contradiction. From Lemma 7, $(T_{n+1} - T_n)/(1 + T_{n+1} - T_n) =. (S - S_{\delta(n-1)})/(S - S_{n-1}) \to 0$, and thus $T_{n+1} - T_n \to 0$.

The preceding theorem immediately yields the corollary, also proven in Tucker [2], that the convergence of $\{T_n\}$ imples $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.

LEMMA 9. *If Σa_n is a convergent series and n is a positive integer such that $a_{n-1}a_na_{n+1} \neq 0$, then*

$$r_{n+1} - r_n = (T_{n+2} - T_{n+1})(1 - r_n)(1 - r_{n+1}) - (T_{n+2} - T_{n+1})(1 - r_n) + (T_{n+1} - T_n)(1 - r_{n+1}) .$$

Proof. We have

$$(1 - r_n)(1 + T_{n+1}) = 1 - r_n + T_{n+1} - r_nT_{n+1} = 1 + T_{n+1} - r_n(1 + T_{n+1}) = 1 + T_{n+1} - T_n ,$$

so that

$$T_{n+1} - T_n = (1 - r_n)(1 + T_{n+1}) - 1 .$$

Similarly,

$$T_{n+2} - T_{n+1} = (1 - r_{n+1})(1 + T_{n+2}) - 1 .$$

Thus,

$$\begin{aligned}(T_{n+2} &- T_{n+1})(1 - r_n)(1 - r_{n+1}) - (T_{n+2} - T_{n+1})(1 - r_n) \\ &+ (T_{n+1} - T_n)(1 - r_{n+1}) = (T_{n+2} - T_{n+1})(1 - r_n)(1 - r_{n+1}) \\ &- (1 - r_n)[(1 - r_{n+1})(1 + T_{n+2}) - 1] \\ &+ (1 - r_{n+1})[(1 - r_n)(1 + T_{n+1}) - 1] \\ =& (T_{n+2} - T_{n+1})(1 - r_n)(1 - r_{n+1}) + (1 - r_n) \\ &- (1 - r_n)(1 - r_{n+1})(1 + T_{n+2}) - (1 - r_{n+1}) \\ &+ (1-r_n)(1-r_{n+1})(1+T_{n+1}) = (1-r_n)(1-r_{n+1})[(T_{n+2}-T_{n+1}) \\ &- (1 + T_{n+2}) + (1 + T_{n+1})] + r_{n+1} - r_n = r_{n+1} - r_n .\end{aligned}$$

LEMMA 10. *If Σa_n is a convergent series and n is a positive integer such that $(1 - r_n)(1 - r_{n+1})a_{n+1} \neq 0$, then $a_{\delta n}/a_n = (T_{n+2} - T_{n+1}) - (T_{n+2} - T_{n+1})/(1 - r_{n+1}) + (T_{n+1} - T_n)/(1 - r_n)$.*

Proof. We have $a_{n-1}a_na_{n+1} \neq 0$, and

$$a_{\delta n}/a_n = (r_{n+1} - r_n)/(1 - r_n)(1 - r_{n+1})$$

according to Lemma 3. We now apply Lemma 9.

LEMMA 11. *If $a_{\delta n} \in MR(\Sigma a_n)$ and $0 < B \leqq . |1 - r_n|$ for some number B, then $a_{\delta n}/a_n \to 0$.*

Poof. From Theorem 8, $T_{n+1} - T_n \to 0$. Using Lemma 10 and $0 < B \leqq . |1 - r_n|$, it is obvious that $a_{\delta n}/a_n \to 0$.

THEOREM 12. *Suppose that $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ and $0 < B \leqq . |1 - r_n|$.*

Then $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, *where* $\alpha_n = .\, (1 - r_{n+1})/(1 - 2r_{n+1} + r_n r_{n+1})$ *or* $\alpha_n = .\, (1 - r_{n-1})/(1 - 2r_n + r_{n-1} r_n)$.

Proof. From Lemma 11, $a_{\delta n}/a_n \to 0$. We now apply Theorem 4, if $\alpha_n = .\, (1 - r_{n+1})/(1-2r_{n+1}+r_n r_{n+1})$; or Theorem 5, if $\alpha_n = .\, (1-r_{n-1})/(1 - 2r_n + r_{n-1} r_n)$.

THEOREM 13. *If* $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ *and* $|r_n| \leqq .\, B$ *for some number* B, *then* $r_{n+1} - r_n \to 0$.

Proof. From Theorem 8, Lemma 9, and $|r_n| \leqq .\, B$, it is obvious that $r_{n+1} - r_n \to 0$.

The following theorem gives simple necessary and sufficient conditions for the δ^2-transform to accelerate convergence in the complex plane under the fairly general condition that $|r_n| \leqq .\, \rho < 1$. In addition, it generalizes the result on acceleration contained in Theorem 2 of Lubkin [1].

THEOREM 14. *Suppose that* $|r_n| \leqq .\, \rho < 1$ *for some number* ρ. *Then a necessary and sufficient condition that* $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ *is that* $r_{n+1} - r_n \to 0$.

Proof. Since $|r_n| \leqq .\, \rho < 1$, Σa_n converges. The necessity follows from Theorem 13. For the sufficiency, let $\varepsilon > 0$. Since $r_{n+1} - r_n \to 0$, $|r_{n+1} - r_n| \leqq .\, \varepsilon$. Consequently,

$$\begin{aligned} |T_{n+1} - T_n| &= .\, |(r_{n+1} - r_n) + r_{n+1}(r_{n+2} - r_n) + r_{n+1}r_{n+2}(r_{n+3} - r_n) \\ &\quad + \cdots + (r_{n+1} \cdots r_{n+k-1})(r_{n+k} - r_n) + \cdots | \leqq .\, |r_{n+1} - r_n| \\ &\quad + |r_{n+1}|\,|r_{n+2} - r_n| + \cdots + |r_{n+1} \cdots r_{n+k-1}|\,|r_{n+k} - r_n| \\ &\quad + \cdots \leqq .\, \varepsilon + 2\varepsilon |r_{n+1}| + \cdots + k\varepsilon |r_{n+1} \cdots r_{n+k-1}| \\ &\quad + \cdots \leqq .\, \varepsilon[1 + 2\rho + 3\rho^2 + \cdots + k\rho^{k-1} + \cdots] = \varepsilon/(1 - \rho^2)\,. \end{aligned}$$

Hence $T_{n+1} - T_n \to 0$, and thus, from Theorem 8, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.

The preceding theorem yields a simple proof of acceleration in a punctured disk in the complex place for certain power series as is now seen.

COROLLARY 15. *Suppose that* $|r_n| \leqq .\, \rho < 1$ *for some number* ρ, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ *and* $a'_n = a_n z^n$ *for every* n. *Then* $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$, *for each complex number* z *satisfying* $0 < |z| < 1/\rho$.

Proof. From Theorem 14, $r_{n+1} - r_n \to 0$. Let z be any complex number such that $0 < |z| < 1/\rho$. Then $|r'_n| =. |r_n z| \leqq. \rho |z| < 1$ and $r'_{n+1} - r'_n =. r_{n+1}z - r_n z =. z(r_{n+1} - r_n) \to 0$. Thus $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$, according to Theorem 14.

COROLLARY 16. *Suppose that* $|r_n| \leqq. \rho < 1$ *for some number* ρ, $r_{n+1} - r_n \to 0$ *and* $a'_n = a_n z^n$ *for every* n. *Then* $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$, *for each complex number* z *satisfying* $0 < |z| < 1/\rho$.

Proof. From Theorem 14, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. We now apply Corollary 15.

LEMMA 17. *If* $0 < A \leqq. |1 - r_n| \leqq. B$, *then* $a_{\delta n}/a_n =. (r_{n+1} - r_n)/(1 - r_n)(1 - r_{n+1})$, *and* $a_{\delta n}/a_n \to 0$ *if and only if* $r_{n+1} - r_n \to 0$.

Proof. Since $0 < A \leqq. |1 - r_n| \leqq. B$, $0 < A^2 \leqq. |(1 - r_n)(1 - r_{n+1})| \leqq. B^2$. Hence from Lemma 3, $a_{\delta n}/a_n =. (r_{n+1} - r_n)/(1 - r_n)(1 - r_{n+1})$. Thus from $0 < A^2 \leqq. |(1 - r_n)(1 - r_{n+1})| \leqq. B^2$, $a_{\delta n}/a_n \to 0$ if and only if $r_{n+1} - r_n \to 0$.

LEMMA 18. *If* $|r_n| \leqq. \rho < 1$, *then*

$$a_{\delta n}/a_n =. (r_{n+1} - r_n)/(1 - r_n)(1 - r_{n+1}) ,$$

and $a_{\delta n}/a_n \to 0$ *if and only if* $r_{n+1} - r_n \to 0$.

Proof. From $|r_n| \leqq. \rho < 1$, $0 < 1 - \rho \leqq. |1 - r_n| \leqq. 2$. We now apply Lemma 17.

THEOREM 19. *Suppose that* $|r_n| \leqq. \rho < 1$. *Then* $a_{\delta n} \in MR(\Sigma a_n)$ *if and only if* $a_{\delta n}/a_n \to 0$.

Proof. From Lemma 18, $a_{\delta n}/a_n \to 0$ if and only if $r_{n+1} - r_n \to 0$. From Theorem 14, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ if and only if $r_{n+1} - r_n \to 0$. Consequently, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ if and only if $a_{\delta n}/a_n \to 0$.

THEOREM 20. *If* $|r_n| \leqq. \rho < 1$ *and* $a_{\delta n}/a_n \to 0$, *then* $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, *where* $\alpha_n =. (1 - r_{n+1})/(1 - 2r_{n+1} + r_n r_{n+1})$ *or* $\alpha_n =. (1 - r_{n-1})/(1 - 2r_n + r_{n-1}r_n)$.

Proof. From Theorem 19, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. From Theorem 4, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ if $\alpha_n =. (1 - r_{n+1})/(1 - 2r_{n+1} + r_n r_{n+1})$. If

$$\alpha_n =. (1 - r_{n-1})/(1 - 2r_n + r_{n-1}r_n) ,$$

we may apply Theorem 5 to obtain $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$.

THEOREM 21. *If*

$$|r_n| \leqq. \rho < 1 \quad \textit{and} \quad r_{n+1} - r_n \to 0 ,$$

then $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, *where* $\alpha_n =. (1 - r_{n+1})/(1 - 2r_{n+1} + r_n r_{n+1})$ *or* $\alpha_n =. (1 - r_{n-1})/(1 - 2r_n + r_{n-1}r_n)$.

Proof. From Lemma 18, $a_{\delta n}/a_n \to 0$. We now apply Theorem 20.

In Tucker [2] it was proven in Theorem 3.7 that if $a'_n/a_n \to 0$, $|r_n| \leqq. \rho_1 < 1/2$ and $|r'_n| \leqq. \rho_2 < 1$, then $\Sigma a'_n$ converges more rapidly than Σa_n. Furthermore, it was shown there in Counterexample 3.8 that the replacement of "1/2" by any larger number produced in-valid result. We now turn to our next theorem which shows that "1/2" may be replaced by "1" under the additional hypothesis that $\Delta r_n \to 0$.

THEOREM 22. *If*

$$a'_n/a_n \to 0, \; |r_n| \leqq. \rho_1 < 1, \; |r'_n| \leqq. \rho_2 < 1$$

and $\Delta r_n \to 0$, *then* $\Sigma a'_n$ *converges more rapidly than* Σa_n.

Proof. From Theorems 8 and 14, $\Delta T_n \to 0$. Also $|1 + T'_{n+1}| \leqq. 1/(1 - \rho_2)$. Thus,

$$\frac{|S' - S'_{n-1}|}{|S - S_{n-1}|} =. \frac{|a'_n|}{|a_n|} \frac{|T'_n/r'_n|}{|T_n/r_n|} =. \frac{|a'_n|}{|a_n|} \frac{|1 + T'_n|}{|(1 + \Delta T_n)/(1 - r_n)|} \to 0 .$$

Our final two theorems are on infinite products.

THEOREM 23. *If the sequence* $\{1/a_n - 1/a_{n-1}\}$ *is bounded, then the complex product* $\Pi_0^\infty (1 + a_n)$ *diverges.*

Proof. Assume that $\Pi_0^\infty (1 + a_n)$ converges. Then $a_n \to 0$ and there is an $m \geqq 0$ such that for $k \geqq 0$, the quantities

$$S'_k = (1 + a_m)(1 + a_{m+1}) \cdots (1 + a_{m+k})$$

satisfy the limiting relation $S'_k \to S'$ for some $S' \neq 0$. We may assume that $m = 0$ so that $S'_n = \Pi_0^n (1 + a_i)$ for $n \geqq 0$. Since the sequence $\{(1 - r_n)/a_n\} = \{1/a_n - 1/a_{n-1}\}$ is bounded and $a_n \to 0$, we have $r_n \to 1$. Let $a'_0 = S'_0 = (1 + a_0)$ and $a'_n = S'_n - S'_{n-1} = \Pi_0^n (1 + a_i) - \Pi_0^{n-1} (1 + a_i) = [\Pi_0^{n-1} (1 + a_i)][(1 + a_n) - 1] = a_n \Pi_0^{n-1} (1 + a_i)$ for $n \geqq 1$. Then $1/a'_{n+1} - 1/a'_n = [1/[a_{n+1}(1 + a_n)] - 1/a_n]/\Pi_0^{n-1} (1 + a_i) = [(1/a_{n+1} - 1/a_n) - 1/[r_{n+1}(1 + a_n)]]/\Pi_0^{n-1} (1 + a_i)$. Hence, since $r_n \to 1$, $a_n \to 0$, $\{1/a_n - 1/a_{n-1}\}$ is bounded and $\Pi_0^\infty (1 + a_n) = S' \neq 0$, we see that $\{1/a'_{n+1} - 1/a'_n\}$ is bounded. From Tucker [2], $\Sigma a'_n$ diverges, i.e., $\Pi_0^\infty (1 + a_n)$ diverges.

THEOREM 24. *Suppose that* $|r_n| \leqq. \rho < 1$ *and* $a_n \neq -1$ *for all* n.

Then a necessary and sufficient condition for the δ^2-transform to accelerate the convergence of the infinite product $\Pi_0^\infty (1 + a_n)$ *is that* $\Delta r_n \to 0$.

Proof. Set $S'_n = \Pi_0^n (1 + a_i)$ for $n \geqq 0$, $a'_0 = S'_0$ and $a'_n = S'_n - S'_{n-1}$ for $n \geqq 1$. Since $| r_n | \leqq . \rho < 1$, we successively obtain the convergence of $\Sigma | a_n |$, $\Pi_0^\infty (1 + | a_i |)$ and $\Pi_0^\infty (1 + a_i) = S' = \Sigma a'_n \neq 0$. Also, $a_n \to 0$ and $r'_n = . r_n + a_n$ yield $| r'_n | \leqq . \rho' = (\rho + 1)/2 < 1$ and the equivalence of the conditions $\Delta r_n \to 0$ and $\Delta r'_n \to 0$. From Tucker [2], $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$ if and only if $\Delta r'_n \to 0$. Hence, $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$ if and only if $\Delta r_n \to 0$.

REFERENCES

1. Samuel Lubkin, *A method of summing infinite series*, J. Res. Nat. Bur. Standards **48** (1952), 228-254.

2. Richard R. Tucker, *The δ^2-process and related topics*, Pacific J. Math. **22** (1967), 349-359.

Received March 27, 1967. Except for Theorem 22, 23, and 24, the material in this paper was taken from the author's Doctorial Dissertation, submitted to Oregon State University, Corvallis, Oregon, under the guidance of Professor A. T. Lonseth.

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