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VECTOR VALUED ORLICZ SPACES, II

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In this paper properties of linear spaces generated by GN-functions, which are called vector valued Orlicz spaces, are studied. The class of GN-functions were introduced and studied by the author in the paper Vector Valued Orlicz Spaces, I. This work extends the usual theory of Orlicz spaces generated by real valued N-functions of a real variable. In particular, GN-functions are a generalization of the variable N-functions used by Portnov and the nondecreasing N-functions by Wang.

This paper is divided into four sections. In § 2 the concept of an Orlicz class and its related Orlicz space will be introduced. Furthermore, a norm is defined. It will be shown that the Orlicz space is a Banach space relative to this norm. One of the main results of § 2 states that every Orlicz class is an Orlicz space if and only if the GN-function satisfies a generalized Δ-condition as defined in Part I [7].

The concept of modular convergence is introduced in § 3 and conditions when norm convergence is equivalent to modular convergence are given. We also give a characterization of the Δ-condition in terms of modulars. In § 4 we generalize some of the basic results involving conjugate functions. In particular, a generalized Hölder inequality is given and an equivalent norm to that introduced in § 2 is defined. Finally, we characterize all the continuous linear functions defined on the Orlicz space under investigation. These theorems generalize the corresponding results which can be found in [1, 3, 5].

2. Vector valued Orlicz classes and spaces.

Let us begin by establishing some notation that will be used throughout this paper. We denote by X the class of all measurable functions

$$x : x^i(t) \qquad (t \text{ in } T, i = 1, \cdots, n)$$

where $x^i(t)$ are real valued functions. We will represent the functions in X by the vector notation

$$x : x(t) \qquad (t \text{ in } T)$$

whenever it is convenient to do so. For example, if x, y are functions in X, and a, b are real numbers, the symbol $ax + by$ denotes the function

$$ax + by : ax(t) + by(t) \qquad (t \text{ in } T) .$$

Let us identify all functions x in X which are equal to zero for almost all t in T. Then we denote by the same symbol, X, the set of equivalence classes of functions defined by this identification.

Having established this notation, we now define an Orlicz class for GN-functions.

DEFINITION 2.1. Let $M(t, x)$ be a GN-function. By an *Orlicz class* L_M we mean the set of all functions x in X for which

$$(+) \qquad R_M(x) = \int_T M(t, x(t))dt < \infty .$$

It is easy to see that L_M is a convex set of functions. On the other hand, L_M need not be a vector space in general. The next two theorems give conditions when L_M is linear.

THEOREM 2.1. *L_M is a vector space if and only if L_M is closed under positive scalar multiplication.*

If L_M is a vector space, the closure statement is clear. To show the converse, we first show that if x is in L_M, then $-x$ is in L_M. By definition of a GN-function there are constants $K > 0$ and $d \geqq 0$ such that $M(t, x) \leqq KM(t, y)$ if $d \leqq |x| \leqq |y|$ (see, [7, Th. 2.2]). This means, since $|-x| = |x|$, that if $|x(t)| \geqq d$ and x is in L_M, then $-x$ is in L_M. Moreover, if $d > 0$, then we know that $\bar{M}(t, d)$ is integrable over T. Therefore, if $|x(t)| < d$ and x is in L_M, we also have $-x$ in L_M since $M(t, -x(t)) \leqq \bar{M}(t, d)$.

Suppose now that a, b are any nonzero real numbers and x, y are in L_M. If $ab > 0$, then for each t in T we have

$$(2.1.1) \qquad \frac{ax(t) + by(t)}{a + b} = \frac{|a|\, x(t)}{|a| + |b|} + \frac{|b|\, y(t)}{|a| + |b|} .$$

If $ab < 0$, say $a < 0 < b$, then

$$(2.1.2) \qquad \frac{by(t) + |a|\,(-x(t))}{b + |a|} = \frac{by(t)}{b + |a|} + \frac{|a|\,(-x(t))}{b + |a|} .$$

Since the sum of the coefficients of x and y on the right sides of (2.1.1) and (2.1.2) is one, the convexity of L_M and the fact that $-x$ is in L_M yields that the left sides of these equations are in L_M. However, by hypothesis and the fact that either $a + b > 0$ or $b + |a| > 0$, we obtain $ax + by$ in L_M proving the theorem.

THEOREM 2.2. *L_M is a vector space if and only if $M(t, x)$ satisfies a Δ-condition.*

Suppose $M(t, x)$ satisfies a Δ-condition (see, [7, Definition 3.1]) where $\delta(t) \geqq 0$. We show that L_M is linear. However, according to Theorem 2.1, it suffices to show that if x is in L_M, $2x$ is in L_M. Let x be in L_M and define

$$g(t) = \begin{cases} x(t) & \text{if } \quad |x(t)| < \delta(t) \\ 0 & \text{otherwise}\ , \end{cases}$$

$$h(t) = \begin{cases} x(t) & \text{if } \quad |x(t)| \geqq \delta(t) \\ 0 & \text{otherwise}\ . \end{cases}$$

This means g, h are in L_M and

$$M(t, 2x(t)) = M(t, 2g(t)) + M(t, 2h(t))\ . \tag{2.2.1}$$

Since $|h(t)| \geqq \delta(t)$ and $2|g(t)| < 2\delta(t)$, the Δ-condition implies that (2.2.1) reduces to

$$M(t, 2x(t)) \leqq \bar{M}(t, 2\delta(t)) + KM(t, h(t))\ . \tag{2.2.2}$$

The right side of (2.2.2) being integrable over T yields the integrability of $M(t, 2x(t))$. This means $2x$ is in L_M.

We now show that if $M(t, x)$ does not satisfy the Δ-condition, L_M is not linear. If $M(t, x)$ does not satisfy a Δ-condition, there exists a sequence of points $\{x_k\}$ in E^n tending to infinity and a set T_0 of finite positive measure such that

$$M(t, 2x_k) > 2^k M(t, x_k) \tag{2.2.3}$$

for all t in T_0 and all $k = 1, 2, \cdots$. Moreover, we can assume by considering a subsequence of $\{x_k\}$ that $M(t, x_k) \geqq 1$ for all k and t in T_0. We will exhibit a function x in L_M for which $2x$ is not in L_M.

Let $\{e_k\}$ be any sequence of real numbers such that $0 < e_k \leqq 1/2^{2k}$. Moreover, we choose a nonoverlapping sequence $\{T_k\}$ of closed subsets of T_0 such that $|T_k| = |T_0|/2^k$. The notation $|T|$ denotes the measure of T. Since $M(t, x)$ is measurable in t for each x, given e_k we can uniformly approximate $M(t, x_k)$ on a subset S_k of T_k whose measure is $|T_k| - e_k$ by a simple function $N_k(t)$. That is, we can find

$$N_k(t) = \sum_i c_{ki} \chi_{T_k^i}$$

where

$$\sum_i |T_k^i| = |T_k| - e_k$$

and χ_E is a characteristic function of set E such that

$$| M(t, x_k) - N_k(t) | \leqq e_k$$

for all t in S_k. We now choose disjoint subsets V_k^i of T_k^i such that $| V_k^i | = | T_i^i |/c_{ki}$ and set $V_k = \bigcup_i V_k^i$.

Let us define the function x we need by

$$x(t) = \begin{cases} x_k & \text{if } t \text{ is in } V_k \quad (k = 1, 2, \cdots) \\ 0 & \text{otherwise} . \end{cases}$$

For each k we have

$$\begin{aligned} \int_{V_k} M(t, x_k)dt &\leqq \int_{V_k} [N_k(t) + e_k]dt \\ &\leqq \sum_i [c_{ki} + e_k] | V_k^i | \leqq \sum_i \frac{c_{ki} + e_k}{c_{ki}} | T_k^i | \leqq \frac{| T_0 |}{2^{k-1}} \end{aligned}$$

and

$$\begin{aligned} \int_{V_k} M(t, 2x_k)dt &> 2^k \int_{V_k} M(t, x_k)dt \geqq 2^k \sum_i [c_{ki} - e_k] | V_k^i | \\ &> 2^k \sum_i \frac{c_{ki} - e_k}{c_{ki}} | T_k^i | \geqq 2^k \sum_i | T_k^i | - 2^k \sum_i \frac{e_k | T_k^i |}{c_{ki}} \qquad (2.2.5) \\ &> | T_0 | - 2^k e_k - | T_0 | e_k . \end{aligned}$$

Therefore, summing (2.2.4) and (2.2.5) over all k yields

$$\int_T M(t, x(t))dt = \sum_{k=1}^{\infty} \int_{V_k} M(t, x_k)dt \leqq \sum_{k=1}^{\infty} \frac{| T_0 |}{2^{k-1}} < \infty$$

and

$$\begin{aligned} \int_T M(t, 2x(t))dt &= \sum_{k=1}^{\infty} \int_{V_k} M(t, 2x_k)dt \\ &> \sum_{k=1}^{\infty} | T_0 | - \sum_{k=1}^{\infty} \frac{| T_0 | + 1}{2^k} = \infty . \end{aligned}$$

This proves that x is in L_m while $2x$ is not in L_M completing the proof of the theorem.

Using the results given in the preceding theorems, we define the linear space we wish to consider in the remainder of this paper.

DEFINITION 2.2. Let $M(t, x)$ be a GN-function and let L_M be its associated Orlicz class. We call the closure of L_M under positive scalar multiplication a *vector valued Orlicz space*. It will be denoted by $\mathscr{L}_M$. By definition $\mathscr{L}_M$ is the set of functions x in X for which there is some positive constant c such that cx is in $\mathscr{L}_M$.

Let us observe that, by Theorem 2.2, $L_M = \mathscr{L}_M$ whenever the GN-function $M(t, x)$ satisfies a Δ-condition. That is, the Orlicz classes are linear spaces whenever the GN-function defining them has a restricted growth condition such as the Δ-condition. By the second part of Theorem 3.2, Part I [7] this means that $M(t, x)$ does not grow exponentially along lines which pass through the origin in E^n space. We now introduce a norm for the linear space $\mathscr{L}_M$. It will be defined in terms of a *quasi-norm* or *q-norm*. By a q-norm we mean a real valued function possessing all the usual properties of a norm except it is only a positive homogeneous function.

THEOREM 2.3. *Let $M(t, x)$ be a GN-function and let $\mathscr{L}_M$ be defined as in* 2.2. *Then*

$$\|x\| = \max(\|x\|^+, \|-x\|^+) \tag{2.3.1}$$

is a norm for $\mathscr{L}_M$ where

$$\|x\|^+ = \inf\left\{k > 0: \int_T M\left(t, \frac{x(t)}{k}\right)dt \leqq 1\right\}. \tag{2.3.2}$$

Before proving this theorem let us note that

$$\|x\| = \|x\|^+ = \|-x\|^+$$

if $M(t, x)$ is an even function of x. That is, if $M(t, x)$ is a real valued N-function, then our norm $\|x\|$ reduces to $\|x\|^+$ which is the standard Luxemburg norm. However, when we deal with GN-functions we no longer retain the property of symmetry relative to the origin as with real valued N-functions. Therefore, $\|x\|^+$ may not equal $\|-x\|^+$ and $\|x\|^+$ is only a positive homogeneous function.

Suppose $x(t) = 0$ almost everywhere. Then, by definition of a GN-function, we have $\int_T M(t, x(t)/k)dt = 0$ for all $k > 0$, hence $\|x\|^+ = 0$. On the other hand, assume $\|x\|^+ = 0$. Then, for all $k > 0$, we have $R_M(x/k) \leqq 1$. However, if we let $k = 1/m$, $m = 1, 2, \cdots$ and use the convexity of $M(t, x)$ we arrive at $R_M(x) \leqq 1/m$ for all m. Therefore, $M(t, x(t)) = 0$ for almost all t in T. This means $x(t) = 0$ almost everywhere since $M(t, x)$ is a GN-function. It is clear that $\|x\|^+ \geqq 0$. The positive homogeneity of $\|x\|^+$ follows from the equation

$$\|ax\|^+ = a\inf\left\{\frac{k}{a} > 0: \int_T M\left(t, \frac{ax(t)}{k}\right)dt \leqq 1\right\} = a\|x\|^+$$

for $a > 0$.

We will complete the proof of Theorem 2.3 by showing that the triangle inequality is valid. Let us assume x, y are in $\mathscr{L}_M$ and $a =$

$\|x\|^+ > 0$, $b = \|y\|^+ > 0$. For, if $a = b = 0$, there is nothing to prove. Observe first that an application of Fatou's lemma yields

$$\int_T M\left(t, \frac{x(t)}{a}\right)dt \leqq 1\,, \qquad \int_T M\left(t, \frac{y(t)}{b}\right)dt \leqq 1\,. \tag{2.3.3}$$

From (2.3.3) we obtain, since $\|x\| \geqq a$, $\|y\| \geqq b$,

$$\int_T M\left(t, \frac{x(t)}{\|x\|}\right)dt \leqq 1\,, \qquad \int_T M\left(t, \frac{y(t)}{\|y\|}\right)dt \leqq 1\,. \tag{2.3.4}$$

Set $c = a + b$. Then, by convexity, we have for each t in T

$$\begin{aligned} M\left(t, \frac{x(t) + y((t)}{c}\right) &= M\left(t, \frac{a}{c}\frac{x(t)}{a} + \frac{b}{c}\frac{y(t)}{b}\right) \\ &\leqq \frac{a}{c} M\left(t, \frac{x(t)}{a}\right) + \frac{b}{c} M\left(t, \frac{y(t)}{b}\right)\,. \end{aligned} \tag{2.3.5}$$

If we integrate both sides of (2.3.5) over T, we attain using (2.3.3)

$$\int_T M\left(t, \frac{x(t) + y(t)}{c}\right)dt \leqq \frac{a}{c}\int_T M\left(t, \frac{x(t)}{a}\right)dt + \frac{b}{c}\int_T M\left(t, \frac{y(t)}{b}\right)dt \leqq 1\,.$$

That is, $\|x + y\|^+ \leqq c = \|x\|^+ + \|y\|^+$ proving the theorem.

We have just shown that equation (2.3.1) defines a norm for $\mathscr{L}_M$. We raise the question as to whether the norm is affected by altering the constant bounding the integral in equation (2.3.2). This is answered by the next theorem which states that all q-norms obtained by changing the constants are equivalent.

THEOREM 2.4. *Let $M(t, x)$ be a GN-function. Suppose for x in $\mathscr{L}_M$ we let $\|x\|_c^+ = \inf k$, k in K_c where c is a positive real number and*

$$K_c = \left\{k > 0 \colon \int_T M\left(t, \frac{x(t)}{k}\right)dt \leqq c\right\}\,.$$

Then, if $0 < c \leqq d$, we have

$$\|x\|_d^+ \leqq \|x\|_c^+ \leqq \frac{d}{c}\|x\|_d^+\,. \tag{2.4.1}$$

If $d \geqq c > 0$, then K_d contains K_c and the first inequality in (2.4.1) is valid. Moreover, using the convexity of $M(t, x)$ and the definition of the q-norm $\|x\|_c^+$, we obtain the inequalities

$$\begin{aligned}
\| x \|_d^+ &= \inf_{k \text{ in } K_d} k = \inf\left\{k \frac{c}{d} : \int_T M\left(t, \frac{dx(t)}{kc}\right)dt \leqq d\right\} \\
&\geqq \inf\left\{k \frac{c}{d} : \frac{d}{c}\int_T M\left(t, \frac{x(t)}{k}\right)dt \leqq d\right\} \qquad (2.4.2)\\
&\geqq \frac{c}{d} \inf\left\{k : \int_T M\left(t, \frac{x(t)}{k}\right)dt \leqq c\right\} \geqq \frac{c}{d} \| x \|_c^+ .
\end{aligned}$$

This proves the second inequality of (2.4.1) and the theorem.

3. Modular convergence.

We will now introduce a concept of modular convergence for Orlicz spaces generated by GN-functions.

DEFINITION 3.1. The functional $R_M(x)$ when defined on $\mathscr{L}_M$ is called *modular* if $R_M(x)$ is defined as in Definition 2.1(+). We say a sequence of functions $x_k(t)$ in $\mathscr{L}_M$ is *modular convergent* to $x_0(t)$ in $\mathscr{L}_M$ if

$$\lim_{K=\infty} R_M(x_K - x_0) = 0 .$$

The concept of modular convergence introduced here should not be confused with the same terminology used in the literature. For example, the same term is used by Musielak and Orlicz in [4, p. 50] but with a different meaning.

The next result gives a characterization of the equivalence of norm and modular convergence in terms of the modular $R_M(x)$.

THEOREM 3.1. *A necessary and sufficient condition for norm convergence to be equivalent to modular convergence is that*

$$\lim_{k=\infty} R_M(x_k) = 0 \quad \textit{implies} \quad \lim_{k=\infty} R_M(ax_k) = 0 \qquad (3.1.1)$$

for all real a.

Suppose the sequence $\{x_k(t)\}$ is modular convergent to zero and modular convergence is equivalent to norm convergence. That is, $R_M(x_k) \to 0$ if and only if $\| x_k \| \to 0$ as $k \to \infty$. However, if $\lim_{k=\infty} \| x_k \| = 0$, then $\lim_{k=\infty} \| ax_k \| = 0$ for all real a. This, by assumption, means $\lim_{k=\infty} R_M(ax_k) = 0$ for all real a which proves (3.1.1).

Let us observe that norm convergence always implies modular convergence. For, suppose $\lim_{k=\infty} m_k = 0$ where $m_k = \| x_k \|$ and x_k is in $\mathscr{L}_M$. We can assume $m_k \leqq 1$ for all k. Using (2.3.4) and the convexity of $R_M(x)$, we attain

$$\frac{1}{m_k} R_M(x_k) \leqq R_M\left(\frac{x_k}{m_k}\right) \leqq 1 \, .$$

This means $R_M(x_k) \leqq m_k$ for all k proving the assertion.

Suppose condition (3.1.1) is valid, $\lim_{k=\infty} R_M(x_k) = 0$, and $\| x_k \| \geqq a > 0$ for all k sufficiently large. By (3.1.1) we must have $\lim_{k=\infty} R_M(x_k/a) = 0$. On the other hand, if $\| x_k \| \geqq a > 0$, then the definition of the norm yields $R_M(x_k/a) > 1$ for all sufficiently large k. This contradiction completes the proof of Theorem 3.1.

We note that condition (3.1.1) holds if and only if

$$\text{(3.1.2)} \qquad \lim_{k=\infty} R_M(x_k) = 0 \quad \text{implies} \quad \lim_{k=\infty} R_M(ax_k) = 0$$

holds for some real $a > 1$. This observation is easy to show (see [6, Th. 12.4]). Moreover, we might suspect that the Δ-condition and condition (3.1.1) or (3.1.2) are related. Indeed, this is the case as the next two theorems indicate.

THEOREM 3.2. *Suppose $M(t, x)$ is a GN-function satisfying a Δ-condition and $| T | < \infty$. Then $R_M(x)$ satisfies condition* (3.1.2).

A result of this type can be found in the paper of Musielak and Orlicz [4, Th. 2.32(b)] under slightly different conditions. However, with minor modifications the proof carries over to our assumptions which involve GN-functions, Δ-condition, and the specific modular $R_M(x)$ (see, [6, Th. 12.5]).

Musielak and Orlicz have stated that the converse of Theorem 3.2 does not hold in general when $R_M(x)$ is any modular in their sense. However, we observe that this is not the case when $R_M(x)$ is a modular as defined in 3.1. This is the content of the next theorem.

THEOREM 3.3. *If $R_M(x)$ is a modular as defined in* 3.1 + *which satisfies* (3.1.2), *then the GN-function $M(t, x)$ defining $R_M(x)$ satisfies a Δ-condition.*

We will assume that $M(t, x)$ does not satisfy the Δ-condition and exhibit a sequence of functions $\{x_k\}$ for which (3.1.2) does not hold. If the growth condition is not satisfied, then there exists a sequence of points $\{x_k\}$ in E^n tending to infinity and a set T_0 of finite positive measure such that $M(t, 2x_k) > 2^k M(t, x_k)$. Let us define the sequences $\{e_k\}$, $\{T_k\}$ as in the proof of Theorem 2.2. As in that theorem, given $e_k > 0$ we can uniformly approximate $M(t, x_k)$ on a subset S_k of T_k whose measure is $| T_k | - e_k$ by a simple function $N_k(t)$. That is, we can find

$$N_k(t) = \sum_i c_{ki}\chi_{T_k^i}$$

where

$$\sum_i |T_k^i| = |T_k| - e_k = |S_k|$$

such that

$$|M(t, x_k) - N_k(t)| \leqq e_k$$

for all t in S_k. Given any positive integer m, we choose for each $k = 1, 2, \cdots, m$ disjoint subsets $V_{k,m}^i$ of T_k^i such that

$$|V_{k,m}^i| = \frac{|T_k^i|}{mc_{ki}}$$

and set $V_k^m = \bigcup_i V_{k,m}^i$. Moreover, as in Theorem 2.2, we can assume by considering a subsequence of $\{x_k\}$ that $M(t, x_k) \geqq 1$ for all k and t in T_0.

We now define for each m

$$x_m(t) = \begin{cases} x_k \text{ if } t \text{ is in } V_k^m & (k = 1, \cdots, m) \\ 0 \quad \text{otherwise} \end{cases} .$$

For fixed m, we have

$$\sum_{k=1}^m \int_{V_k^m} M(t, x_k)dt \leqq \sum_{k=1}^m \sum_i \frac{c_{ki} + e_k}{c_{ki}} \frac{|T_k^i|}{m} \leqq \sum_{k=1}^m \frac{|T_0|}{m2^{k-1}} \tag{3.3.1}$$

and

$$\begin{aligned} \sum_{k=1}^\infty \int_{V_k^m} M(t, 2x_k)dt &> \sum_{k=1}^m 2^k \int_{V_k^m} M(t, x_k)dt \\ &\geqq \sum_{k=1}^m 2^k \sum_i \frac{c_{ki} - e_k}{c_{ki}m} |T_k^i| > \sum_{k=1}^m \frac{|T_0|}{m} \\ &- \sum_{k=1}^m \frac{2^k e_k}{m} - \sum_{k=1}^m \frac{|T_0|}{m2^k} . \end{aligned} \tag{3.3.2}$$

However, by definition of $x_m(t)$, we know that

$$R(x_m) = \int_T M(t, x_m(t))dt = \sum_{k=1}^m \int_{V_k^m} M(t, x_k)dt \tag{3.3.3}$$

and

$$R(2x_m) = \int_T M(t, 2x_m(t))dt = \sum_{k=1}^m \int_{V_k^m} M(t, 2x_k)dt . \tag{3.3.4}$$

If we combine inequalities (3.3.1) through (3.3.4) and take a limit as m tends to infinity, we find $\lim_{m=\infty} R(2x_m) > 0$ whereas $\lim_{m=\infty} R(x_m) = 0$.

This means, by Theorem 3.1, that condition (3.1.2) does not hold for $\{x_m\}$ proving Theorem 3.3.

Let us conclude this section by noting some important observations. Let B_M be the closure of the set of bounded functions in $\mathscr{L}_M$. The set of bounded functions is dense in the Orlicz class L_M in the sense of modular convergence if we assume $\bar{M}(t, c)$ is integrable in t for each c. It follows that the set of bounded functions is dense in $L_M = \mathscr{L}_M$ if $M(t, x)$ satisfies a Δ-condition and $|T|$ is finite. For, modular convergence and norm convergence are equivalent in this case. This means that $B_M = L_M = \mathscr{L}_M$.

4. Conjugate functions and linear functionals.

In developing the Orlicz spaces $\mathscr{L}_M$ generated by GN-functions $M(t, x)$ we have not made use of the concept of a conjugate GN-function. However, if we wish to investigate the linear functionals defined on $\mathscr{L}_M$, we will need to employ these functions. Conjugate GN-functions $M^*(t, x)$ were defined in Part I of this study and some basic properties were given.

It should be noted from Part I that if $M(t, x)$ is a GN-function, $M^*(t, x)$ may not be a GN-function. Therefore, unless we further restrict $M(t, x)$ we can not define the corresponding conjugate Orlicz space $\mathscr{L}_{M^*}$ generated by $M^*(t, x)$. It is for this reason we chose to use the development given in § 2.

Let us introduce some additional notation for this section. Given a conjugate GN-function $M^*(t, x)$ we can define an Orlicz class L_{M^*}, as was done in § 2, to be all x in X such that

$$R_{M^*}(x) = \int_T M^*(t, x(t))dt < \infty .$$

When we can define a linear space $\mathscr{L}_{M^*}$, as in § 2, $R_{M^*}(x)$ becomes a modular on $\mathscr{L}_{M^*}$. We denote the norm associated with $\mathscr{L}_{M^*}$ by $\|x\|_*$ and set $Q(x, y) = \int_T x(t)y(t)dt$.

Since property (iv) in the definition of a GN-function may not hold for $M^*(t, x)$, we will assume $M(t, x)$ is an even function of x. In this case $M^*(t, x)$ is an even function x as shown in Theorem 5.2 of Part I. If $M^*(t, x)$ is an even function, then we can define the linear space $\mathscr{L}_{M^*}$.

We now prove a theorem which yields a generalized Hölder inequality.

THEOREM 4.1. *The inequality* $|Q(x, z)| \leqq 2\|x\|\,\|z\|_*$ *holds for any pair of functions* x *in* $\mathscr{L}_M$ *and* z *in* $\mathscr{L}_{M^*}$.

If we let x be in $\mathscr{L}_M$ and z in $\mathscr{L}_{M^*}$ in § 5 inequality $(++)$ of

Part I and if we integrate both sides over T, we obtain

$$\sup_{R_{M^*}(z)\leqq 1} |\, Q(x, z)\,| \leqq R(x) + 1 \, .$$

Substituting $x = x(t)/\|\, x\,\|$ into this inequality yields

$$|\, Q(x, z)\,| \leqq \sup_{R_{M^*}(z)\leqq 1} |\, Q(x, z)\,| \leqq 2 \tag{4.1.1}$$

for all z such that $R_{M^*}(z) \leqq 1$. However, when $z = z(t)/\|\, z\,\|_*$ we have, by (2.3.4), that $R_{M^*}(z(t)/\|\, z\,\|_*) \leqq 1$. Substituting this value of z into (4.1.1) yields

$$|\, Q(x, z)\,| \leqq 2\,\|\, x\,\|\,\|\, z\,\|_* \, .$$

We can characterize the class of functions in $\mathscr{L}_M$ by introducing another norm. This norm is equivalent to the Orlicz norm introduced in § 2. In the next few theorems we define the norm and state some important properties.

THEOREM 4.2. *Suppose x is in $\mathscr{L}_M$. Then*

$$\sup_{R_{M^*}(y)\leqq 1} |\, Q(x, y)\,| < \infty \, .$$

This theorem is proved in Krasnoselskii and Rutickii (see [3; p. 68]). Although it is proven there for real variable N-functions, the proof carries over word for word to the class of functions here.

THEOREM 4.3. *Let*

$$\|\, x\,\|_0 = \sup_{R_{M^*}(y)\leqq 1} |\, Q(x, y)\,|$$

where x is in $\mathscr{L}_M$. Then $\|\, x\,\|_0$ is a norm.

The axioms defining a norm are clearly satisfied by $\|\, x\,\|_0$.

The next theorem states that the gradient of $M(t, x)$,

$$y^i(t) = M|\, M'(t, x(t); e_i)\, , \qquad (i = 1, 2, \cdots, n)$$

belongs to L_{M^*} and $R_{M^*}(y) \leqq 1$.

THEOREM 4.4. *Suppose $M(t, x)$ is an even GN-function for which $\bar{M}(t, c)$ is integrable in t for all c and for which $M'(t, x; y)$ is linear in y. Then if x is in $\mathscr{L}_M$ and $\|\, x\,\|_0 \leqq 1$, y is in L_{M^*} and $R_{M^*}(y) \leqq 1$ where $y^i(t) = M'(t, x(t); e_i)$ and e_i is a basis vector for E^n, $i = 1, \cdots, n$.*

Observe first that

$$(4.4.1)\qquad |Q(x, z)| \leqq \begin{cases} \|x\|_0 & \text{if } R_{M^*}(z) \leqq 1 \\ \|x\|_0 R_{M^*}(z) & \text{if } R_{M^*}(z) > 1 \,. \end{cases}$$

The first inequality in (4.4.1) follows by definition of $\|x\|_0$. If $R_{M^*}(z) > 1$, then

$$M^*\left(t, \frac{z(t)}{R_{M^*}(z)}\right) \leqq \frac{M^*(t, z(t))}{R_{M^*}(z)}$$

by convexity of $M^*(t, x)$ in x. Therefore, it follows that

$$R_{M^*}\left(\frac{z}{R_{M^*}(z)}\right) \leqq 1 \,.$$

Substituting $z = z(t)/R_{M^*}(z)$ into the first inequality of (4.4.1) produces the second inequality.

Moreover, let us further observe that if x in $\mathscr{L}_M$ is a bounded function, then y is in L_{M^*} where

$$y^i(t) = M'(t, x(t); e_i) \,, \qquad (i = 1, 2, \cdots, n) \,.$$

For, by convexity, we know that if $|x(t)| < d$, then

$$\begin{aligned} M'(t, x(t); e_i) &\leqq M(t, x(t) + e_i) - M(t, x(t)) \\ &\leqq M(t, x(t) + e_i) \leqq \bar{M}(t, d + 1) \end{aligned}$$

for all $i = 1, \cdots, n$. This means that there is a constant K such that $|y(t)| \leqq K\bar{M}(t, d + 1)$ from which it follows that

$$(4.4.2)\qquad \int_T |y(t)|\,dt \leqq K\int_T \bar{M}(t, d + 1)dt < \infty \,.$$

Hence, we conclude, using Theorem 5.1, Part I [7] and (4.4.2), that

$$\begin{aligned} R_{M^*}(y) &= \int_T x(t)y(t)dt - \int_T M(t, x(t))dt \\ &\leqq \int_T x(t)y(t)dt \leqq d\int_T |y(t)|\,dt < \infty \,. \end{aligned}$$

This means y is in L_{M^*}.

Suppose now $\|x\| \leqq 1$. We set

$$x_m(t) = \begin{cases} x(t) & \text{if } |x(t)| \leqq m \,, \\ 0 & \text{if } |x(t)| > m \,. \end{cases}$$

Since $x_m(t)$ are bounded functions, by what we have just shown above, we know the vector $y_m(t)$ whose components are $M'(t, x_m(t), e_i)$ is in L_{M^*}. Suppose the conclusion of the theorem is false. Then there is m_0 such that $R_{M^*}(y_{m_0}) > 1$. However, by Theorem 5.1, Part I, we obtain

$$M^*(t, y_{m_0}(t)) < M^*(t, y_{m_0}(t)) + M(t, x_{m_0}(t)) = x_{m_0}(t)y_{m_0}(t) .$$

This means that $R_{M^*}(y_{m_0}) < Q(x_{m_0}, y_{m_0})$ from which it follows, using (4.4.1), that

$$1 < R_{M^*}(y_{m_0}) < Q(x_{m_0}, y_{m_0}) \leqq \| x_{m_0} \|_0 R_{M^*}(y_{m_0})$$

or

$$1 < \| x_{m_0} \|_0 \leqq \| x \|_0 \leqq 1 .$$

This contradiction proves the theorem.

We will now show that the norm $\| x \|_0$ is equivalent to $\| x \|$ and that it characterizes the space $\mathscr{L}_M$.

THEOREM 4.5. *Under the same assumptions on $M(t, x)$ as in Theorem 4.4 we know*

(i) $\| x \| \leqq \| x \|_0 \leqq 2 \| x \|$,

(ii) *x is in $\mathscr{L}_M$ if and only if $\| x \|_0 < \infty$.*

Let $y^i(t) = M'(t, x(t); e_i)$, $i = 1, \cdots, n$. Then by Theorem 4.4, we know $R_{M^*}(y) \leqq 1$. Since, according to Theorem 5.1, Part I, it is true that

$$y(t)x(t) = M(t, x(t)) + M^*(t, y(t)) ,$$

we have

$$R_M(x) \leqq R_M(x) + R_{M^*}(y) = Q(x, y) \leqq \| x \|_0 . \tag{4.5.1}$$

This means, when $x = x(t)/\| x \|$, that $\| x \| \leqq \| x \|_0$. On the other hand, by Theorem 4.1, we obtain

$$| Q(x, z) | \leqq 2 \| x \| \| z \|_*$$

from which we get, whenever $x = x(t)/\| x \|$, that $\| x \|_0 \leqq 2 \| x \|$ proving statement (i). If x is in $\mathscr{L}_M$, then by Theorem 4.2 we have $\| x \|_0 < \infty$. Conversely, if $\| x \|_0 < \infty$, then using (4.5.1), we arrive at $R_M(x/\| x \|) \leqq 1$. That is, x is in $\mathscr{L}_M$ which proves (ii) and the theorem.

In the next result a class of linear functionals are defined for $\mathscr{L}_M$ and are shown to form a total set. This means, according to a theorem in Dunford and Schwartz [2; p. 421], that the linear functionals defined on $\mathscr{L}_M$ which are continuous in the weak topology generated by the total set of functionals are precisely the functionals in the total set. We state the theorem now.

THEOREM 4.6. *Let $M(t, x)$ be a GN-function for which $M'(t, x; y)$ is a linear function of y. If we set*

$$l_x(y) = \int_T M'(t, x(t); y(t))dt ,$$

then for each x in $\mathscr{L}_M$, $l_x(y)$ is a linear functional. Moreover, the set of linear functionals l_x form a total set.

It is clear that $l_x(y)$ is linear in y. Let us assume the set of functionals l_x do not form a total set. Then there is a $y \neq 0$ such that $l_x(y) = 0$ for all x. By convexity and the fact that $l_x(y) = 0$ for all x we have that

$$R_M(x + y) \geqq R_M(x) \tag{4.6.1}$$

for all x. Hence, letting $x = x - y$ in (4.6.1) yields

$$R_M(x) \geqq R_M(x - y) . \tag{4.6.2}$$

On the other hand, since $l_x(y) = l_x(- y) = 0$, inequalities (4.6.1) and (4.6.2) are valid when $y = - y$. This means

$$R_M(x - y) \geqq R_M(x) \geqq R_M(x + y)$$

and, by (4.6.1) and (4.6.2),

$$R_M(x + y) \geqq R_M(x) \geqq R_M(x - y)$$

from which it follows that

$$R_M(x + y) = R_M(x) = R_M(x - y) \tag{4.6.3}$$

for all x. Since $l_x(ay) = al_x(y) = 0$ for all real a, equation (4.6.3) holds replacing y by ay. This means

$$M(t, x(t) + ay(t)) = M(t, x(t)) = M(t, x(t) - ay(t))$$

for almost all t and for all real a. Therefore, $M(t, x(t))$ is constant in the direction $ay(t)$ from $x(t)$ which contradicts condition (iii) of Definition 2.1 for GN-functions in Part I. This completes the proof of the theorem.

Before turning to the characterization of the continuous linear functionals we wish to establish some notation. When we refer to the vector valued characteristic function $\chi_E(t)$ we will means that set function which assumes the vector

$$\left(\frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}}\right)$$

if t is in E and zero otherwise. That is, we denote

$$\chi_E(t) = (\chi_E^1(t), \cdots, \chi_E^n(t))$$

where

$$\chi_E^i(t) = \begin{cases} 1/\sqrt{n} & \text{if } t \text{ is in } E \\ 0 & \text{otherwise .} \end{cases}$$

This means $|\chi_E(t)| = 1$.

We will prove that under certain restrictions on the GN-function $M(t, x)$ the form of the continuous linear functionals on $\mathscr{L}_M$ is $Q(x, y)$ where x is in $\mathscr{L}_M$ and y is in $\mathscr{L}_{M^*}$. Let us also denote $Q(x, y)$ by $l_y(x)$. If it is clear that the functional is determined by y we will sometimes write $l(x)$ in place of $l_y(x)$.

By definition of the norm of a linear functional we have

$$\|l\| = \sup_{\|x\| \leqq 1} |Q(x, y)| . \tag{4.6.4}$$

Moreover, the Hölder inequality in Theorem 4.1 yields

$$\|l\| \leqq 2\|y\|_* . \tag{4.6.5}$$

On the other hand, by Theorem 4.5(i) and (4.6.4), we obtain

$$2\|y\|_* \leqq 2\|y\|_0 \leqq 4 \sup_{\|x\| < 1} |Q(x, y)| \leqq 4\|l\| . \tag{4.6.6}$$

If we combine (4.6.5) and (4.6.6), we achieve the relationship

$$\|l\| \leqq 2\|y\|_* \leqq 4\|l\| . \tag{4.6.7}$$

The inequalities (4.6.7) relate the conjugate space $\mathscr{L}_{M^*}$ and the space of continuous linear functionals defined by $l_y(x) = l(x) = Q(x, y)$.

Let us now state and prove the representation theorem for continuous linear functionals defined on $L_M = \mathscr{L}_M$.

THEOREM 4.7. *Suppose $M(t, x)$ is an even GN-function satisfying a Δ-condition such that $M(t, c)$ is integrable in t for each c. Moreover, suppose $M^{*\prime}(t, x; y)$ is linear in y. Then $Q(x, y)$ is the general form of the continuous linear functionals defined on $\mathscr{L}_M$ where x is in $\mathscr{L}_M$ and y is in $\mathscr{L}_{M^*}$.*

Let us assume $|T| < \infty$ and that $l(x)$ is any continuous linear functional on L_M. It suffices to consider only L_M since $M(t, x)$ satisfies a Δ-condition. We can define on the set of all measurable subsets E of T the set function $l(\chi_E)$. Let us note that χ_E is in L_M.

The set function $l(\chi_E)$ is a countably additive set function. For, if $\{E_i\}$ is a disjoint sequence of measurable subsets of E and

$$E = \bigcup_{i=1}^{\infty} E_i ,$$

it follows that $\sum_{i=1}^{\infty} \chi^j_{E_i}(t) = \chi^j_E(t)$ for each $j = 1, \cdots, n$. Therefore, $\sum_{i=1}^{\infty} \chi_{E_i}(t) = \chi_E(t)$. The countable additivity now follows from the linearity of $l(x)$.

Let us observe also that $l(\chi_E)$ is an absolutely continuous set function. This follows since

$$|\, l(\chi_E) \,| \leqq \|\, l \,\| \, \|\, \chi_E \,\| \tag{4.7.1}$$

and

$$\lim_{|E|=0} \|\, \chi_E \,\| = 0 \,. \tag{4.7.2}$$

Inequality (4.7.1) is obtained from (4.6.4). To see equation (4.7.2) suppose it is not true. Then there is a constant $d > 0$ and a sequence of sets $\{E_i\}$ such that $\|\, \chi_{E_i} \,\| \geqq d$ for all i and $\lim_i |\, E_i \,| = 0$. However, by definition of the norm, if $\|\, \chi_{E_i} \,\| \geqq d$, then

$$\int_T M\Big(t, \frac{\chi_{E_i}(t)}{d}\Big) dt > 1 \tag{4.7.3}$$

for all i. Moreover, it follows from (4.7.3) that

$$1 < \int_{E_i} \bar{M}\Big(t, \frac{1}{d}\Big) dt \qquad \text{for all } i. \tag{4.7.4}$$

Since $\bar{M}(t, c)$ is integrable in t for each c, the integral in (4.7.4) is an absolutely continuous set function. This means the right side of (4.7.4) tends to zero as the measure of the sets E_i tend to zero. This contradiction proves (4.7.2).

Let us write $\chi_E(t) = \sum_{i=1}^{n} \chi_{E,i}(t)$ where $\chi_{E,i}(t) = (0, \cdots, \chi^i_E(t), \cdots, 0)$. It is clear from the above that $l(\chi_{E,i})$ is an absolutely continuous countably additive set function. By an application of the Radon-Nikodym theorem there is a real valued function $y^i(t)$ in L_1 such that

$$l(\chi_{E,i}) = \int_E y^i(t) dt = \int_T y^i(t) \chi^i_E(t) dt \tag{4.7.5}$$

for each $i = 1, \cdots, n$. It follows from (4.7.5) that

$$l(\chi_E) = \sum_{i=1}^{n} l(\chi_{E,i}) = \int_T y(t) \chi_E(t) dt \tag{4.7.6}$$

where $y(t) = (y^1(t), \cdots, y^n(t))$. Moreover, if

$$x(t) = \sum_{i=1}^{m} c_i \chi_{E_i}(t) \tag{4.7.7}$$

where $\{E_i\}$ are disjoint measurable subsets of T, then it is easy to see that (4.7.6) holds if we replace $\chi_E(t)$ by $x(t)$. That is, we have for any simple function $x(t)$ given by (4.7.7)

$$l(x) = \int_T y(t)x(t)dt \ . \tag{4.7.8}$$

Suppose x is any function in L_M. Then by the remarks at the end of § 3, we know there exists a sequence $\{x_m(t)\}$ of bounded functions which converges to $x(t)$ almost everywhere such that $|x_m(t)| \leqq |x(t)|$ for each m and $R_M(x - x_m)$ tends to zero as m approaches infinity. Moreover, $\|x_m\| \leqq \|x\|$ for each m and the sequence $\{|x_m(t)y(t)|\}$ converges to $|x(t)y(t)|$ almost everywhere. Applying Fatou's lemma and (4.7.8), we achieve

$$\begin{aligned} |Q(x, y)| &\leqq \liminf_m \int |x_m(t)y(t)| \, dt \\ &\leqq \liminf_m \|l\| \, \|x_m\| \leqq \|l\| \, \|x\| < \infty \ , \end{aligned} \tag{4.7.9}$$

for any x in L_M. This means, if we apply the argument given in Theorem 5.1, Part I, to $\mathscr{L}_{M^*}$, that y is in $\mathscr{L}_{M^*}$. For, if

$$x^i(t) = M^{*\prime}(t, y(t); e_i) \ , \qquad i = 1, \cdots, n,$$

then

$$R_{M^*}(y) \leqq R_{M^*}(y) + R(x) = Q(x, y) < \infty \ .$$

This proves that (4.7.8) holds for all x in $L_M = \mathscr{L}_M$ and y in $\mathscr{L}_{M^*}$.

Suppose now that $T = \bigcup_m T_m$, $T_m \subseteqq T_{m+1}$, and $|T_m| < \infty$ for each m. Using (4.7.8) restricted to each T_m we obtain a sequence of functions $\{y_m(t)\}$ in $\mathscr{L}_{M^*}$ such that, by (4.7.9), $\|y_m\|_* \leqq \|l\|$, $y_m(t) = y_{m+1}(t)$ if t is in T_m and

$$l(x_m) = \int_{T_m} x_m(t)y_m(t)dt \tag{4.7.10}$$

for every function x_m in L_M which vanishes outside T_m. Moreover, using (4.6.7), we have that $\|l_m\| \leqq \|l\|$ and

$$\|y_m\|_* \leqq 2\,\|l_m\| \leqq 2 \sup_{T_m \subset T} \|l_m\| \leqq 2\,\|l\|$$

for each m where l_m is the functional defined by (4.7.10).

Let $y(t) = \lim_{m=\infty} y_m(t)$. Then $y(t)$ is defined almost everywhere, $\|y\|_* \leqq 2\,\|l\|$, and by (4.7.10) we have in the limit

$$l(x) = \int_T x(t)y(t)dt$$

where $x(t) = \lim_{m=\infty} x_m(t)$ is in $\mathscr{L}_M$ and y is in $\mathscr{L}_{M^*}$. This proves the theorem.

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