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ON THE CONSTRUCTION OF LOWER RADICAL PROPERTIES

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The purpose of this paper is to give a simple construction of the lower radical properties for an arbitrary class of rings.

Let $\mathscr{S}$ be a class of rings. We shall say that the ring R is an $\mathscr{S}$-ring if R is in $\mathscr{S}$. An ideal J of R will be called an $\mathscr{S}$-ideal if J is an $\mathscr{S}$-ring. A ring which does not contain any nonzero $\mathscr{S}$-ideals will be called $\mathscr{S}$-semisimple. We shall call $\mathscr{S}$ a radical property if the following three conditions hold:

(A) homomorphic image of an $\mathscr{S}$-ring is an $\mathscr{S}$-ring,

(B) every ring R contains a largest $\mathscr{S}$-ideal S,

(C) the quotient ring R/S is $\mathscr{S}$-semi-simple.

The largest $\mathscr{S}$-ideal S of a ring R is called the $\mathscr{S}$-radical of R.

Given a class of rings $\mathscr{A}$, Kurosh has constructed a lower radical property $\mathscr{S}(\mathscr{A})$ determined by $\mathscr{A}$, [1], [2], i.e., $\mathscr{S}(\mathscr{A})$ is a radical property, $\mathscr{A} \subseteqq \mathscr{S}(\mathscr{A})$, and if $\mathscr{T}$ is any radical property and $\mathscr{A} \subseteqq \mathscr{T}$ then $\mathscr{S}(\mathscr{A}) \subseteqq \mathscr{T}$.

In this paper we are going to give a simpler construction.

The construction is similar to [3], where we take $\mathscr{A}$ to be the class of all nilpotent rings. It is proven in [3] that this construction is exactly the lower radical property determined by the class of nilpotent rings. We want to extend this construction to any class of rings.

Let $\mathscr{A}$ be a class of ring and let $\mathscr{A}_0$ be the class of all homomorphic images of rings in $\mathscr{A}$. For each ring R, let $D_1(R)$ be the set of all ideals of R, and by induction, we define $D_{n+1}(R)$ to be the family of all rings which are ideals of some ring in $D_n(R)$ and set

$$D(R) = \cup \{D_n(R) : n = 1, 2, 3, \cdots\} .$$

A ring R is called a $\mathscr{L}(\mathscr{A})$-ring if $D(R/I)$ contains a nonzero ring which is isomorphic to a ring in $\mathscr{A}_0$ for each ideal I of R and $I \neq R$. The following facts are clear.

LEMMA 1. $\mathscr{A} \subseteqq \mathscr{A}_0 \subseteqq \mathscr{L}(\mathscr{A})$.

LEMMA 2. *If I is an ideal of R then $D(I) \subseteqq D(R)$.*

LEMMA 3. *Every isomorphic image of an $\mathscr{L}(\mathscr{A})$-ring is an $\mathscr{L}(\mathscr{A})$-ring.*

LEMMA 4. *If A is isomorphic to B and $D(A)$ contains a ring which is isomorphic to a nonzero ring in $\mathscr{A}_0$ then so does $D(B)$.*

LEMMA 5. *If $\mathscr{A} \subseteqq \mathscr{B}$ then $\mathscr{L}(\mathscr{A}) \subseteqq \mathscr{L}(\mathscr{B})$.*

Also we need the following fact [1].

LEMMA 6. *A class of rings $\mathscr{S}$ is a radical property if and only if*

(A) *A homomorphic image of an $\mathscr{S}$-ring is an $\mathscr{S}$-ring.*

(D) *If every nonzero homomorphic image of a ring R contains a nonzero $\mathscr{S}$-ideal, then R is an $\mathscr{S}$-ring.*

LEMMA 7. *If $\mathscr{S}$ is a radical property, then for any ring R and any ideal I of R, the $\mathscr{S}$-radical of I is an ideal of R.*

THEOREM 1. *If $\mathscr{A}$ is a class of rings, then $\mathscr{L}(\mathscr{A})$, contructed above, is a radical property.*

Proof. If R is in $\mathscr{L}(\mathscr{A})$ and I is any ideal of R. Consider the quotient ring R/I and any proper ideal J/I of R/I, $R/I/J/I \cong R/J$.

By definition, $D(R/J)$ contains a ring which is isomorphic to a nonzero ring in $\mathscr{A}_0$ and therefore so does $D(R/I/J/I)$, and hence R/I is in $\mathscr{L}(\mathscr{A})$. Every homomorphic image of R is isomorphic with R/I for some I. Hence, by Lemma 3, (A) follows.

Suppose that every nonzero homomorphic image of R contains a nonzero $\mathscr{L}(\mathscr{A})$-ideal and let I be any ideal of R and $I \neq R$. Then R/I contains a nonzero $\mathscr{L}$-ideal J/I. Now $D(J/I) \subseteqq D(R/I)$, hence $D(R/I)$ contains a ring which is isomorphic to a nonzero ring in $\mathscr{A}_0$. By definition of $\mathscr{L}(\mathscr{A})$, R is in $\mathscr{L}(\mathscr{A})$. This proves (D). By Lemma 6, $\mathscr{L}(\mathscr{A})$ is a radical property.

THEOREM 2. *If $\mathscr{T}$ is a radical property then $\mathscr{L}(\mathscr{T}) = \mathscr{T}$.*

Proof. By Lemma 1, $\mathscr{T} \subseteqq \mathscr{L}(\mathscr{T})$.

If there is a ring R in $\mathscr{L}(\mathscr{T})$ but not in $\mathscr{T}$, let I be $\mathscr{T}$-radical of R. Then R/I is a nonzero ring in $\mathscr{L}(\mathscr{T})$ and is $\mathscr{T}$-semi-simple. Without loss of generality we may assume R is in $\mathscr{L}(\mathscr{T})$ but is $\mathscr{T}$-semi-simple. By definition $D(R)$ contains a ring $J \neq 0$ such that $J \in \mathscr{T}$. But if K is a nonzero ideal of R, i.e., $K \in D_1(R)$, then, by Lemma 7, the $\mathscr{T}$-radical of K is also an ideal of R. But R is $\mathscr{T}$-simi-simple. Hence K is also $\mathscr{T}$-semi-simple. By induction it is easy to see every ring in $D(R)$ is $\mathscr{T}$-semi-simple. This is a contradiction. Hence $\mathscr{T} = \mathscr{L}(\mathscr{T})$.

THEOREM 3. *If $\mathscr{A}$ is a class of rings then $\mathscr{L}(\mathscr{A})$ is the lower radical property determined by $\mathscr{A}$.*

Proof. Let $\mathscr{S}$ be any radical property such that $\mathscr{A} \subseteqq \mathscr{S}$. Then by Theorem 2 and Lemma 5 $\mathscr{S} = \mathscr{L}(\mathscr{S}) \supseteqq \mathscr{L}(\mathscr{A})$.

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