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DIFFERENCE EQUATIONS FOR SOME ORTHOGONAL POLYNOMIALS

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It is well-known that every orthogonal polynomial set $\{P_n(x)\}$ satisfies a 3-term *recurrence* relation of the form

(1.1) $P_{n+1}(x) = (a_n x + b_n)P_n(x) + c_n P_{n-1}(x) \qquad (n = 1, 2, \cdots)$.

Some orthogonal sets (polynomials of Jacobi, Hermite and so on) are solutions of differential equations. It will be shown that there exist orthogonal polynomial sets that satisfy 3-term *difference* equations of the form

(1.2) $$A(x)y(x + \alpha) + B(x)y(x - \alpha) + C(x)y(x) = \lambda y(x)$$

where A, B, C are polynomials of degree $\leqq 2$ and λ is a parameter.

Consider the difference equation

(1.3) $$A(x)y(x + \alpha) + B(x)y(x + \beta) + C(x)y(x) = \lambda y(x)$$

where A, B, C are real polynomials, λ is a parameter, and $\alpha, \beta, 0$ are distinct and real. We examine two cases, according as A, B, C are of degree $\leqq 1$:

(a) $A(x) = a_1 x + a_0,\ B(x) = b_1 x + b_0,\ C(x) = c_1 x,$

or are of degree $\leqq 2$:

(b) $A(x) = a_2 x^2 + a_1 x + a_0,\ B(x) = b_2 x^2 + b_1 x + b_0,\ C(x) = c_2 x^2 + c_1 x$ (a_2, b_2, c_2 not all zero). We shall use the notation (1.3a), (1.3b) to denote equation (1.3) for the respective conditions (a), (b).

Equation (1.3) will be termed *admissible* if there exists a real sequence $\{\lambda_n\}$ $(n = 0, 1, \cdots)$ such that for $\lambda = \lambda_n$ there is a polynomial solution $y_n(x)$, unique to within a multiplicative constant, and $y_n(x)$ is of degree (exactly) n. It follows that admissibility implies that

(1.4) $$\lambda_m \neq \lambda_n (m \neq n) .$$

LEMMA 1.1. *Equation* (1.3a) *is admissible if and only if*

(1.5) $$a_1 + b_1 + c_1 = 0 , \qquad \delta \equiv a_1\alpha + b_1\beta \neq 0 .$$

And in this case we have

(1.6) $$\lambda_n = (a_0 + b_0) + n(a_1\alpha + b_1\beta) \qquad (n = 0, 1, \cdots) .$$

Proof. Let n be arbitrary. If we substitute the nth degree polynomial

$$(1.7)\qquad y(x) = x^n + \sum_{j=0}^{n-1} p_j x^j$$

into (1.3a), a necessary and sufficient condition that $y(x)$ be a solution is that coefficients of $x^{n+1}, x^n, \cdots$ agree on both sides. The coefficient of x^{n+1} yields the first of (1.5), that of x^n gives $\lambda = \lambda_n$ as in (1.6); and those of $x^{n-1}, \cdots, x^0$ give successive equations for $p_{n-1}, \cdots, p_0$. In these equations the coefficients of $p_{n-1}, \cdots, p_0$ are respectively $\lambda_n - \lambda_{n-1}, \lambda_n - \lambda_{n-2}, \cdots, \lambda_n - \lambda_0$, so there is one and only one choice of the p_j's if and only if $\lambda_n \neq \lambda_j$ $(j \leqq n-1)$. This condition is equivalent to the second part of (1.5); and the lemma is established.

LEMMA 1.2. *Equation* (1.3b) *is admissible if and only if*

$$(1.8)\qquad a_2 + b_2 + c_2 = 0\ ,\quad a_1 + b_1 + c_1 = 0\ ,\quad a_2\alpha + b_2\beta = 0\ ;$$

$$(1.9)\qquad 2(a_1\alpha + b_1\beta) + n(a_2\alpha^2 + b_2\beta^2) \neq 0 \qquad (n = 0, 1, \cdots)\ .$$

And in this case λ_n *is given by*

$$(1.10)\qquad \begin{array}{l} \lambda_n = (a_0 + b_0) + n(a_1\alpha + b_1\beta) + n(n-1)(a_2\alpha^2 + b_2\beta^2)/2 \\ (n = 0, 1, \cdots)\ . \end{array}$$

Proof. Substituting (1.7) into (1.3b) and equating like terms (as a necessary and sufficient condition for a solution) we find that the terms in x^{n+2}, x^{n+1} give (1.8), the x^n term gives $\lambda = \lambda_n$ as in (1.10), and $p_{n-1}, \cdots, p_0$ again are uniquely determined if and only if $\lambda_n \neq \lambda_j$ $(j \leqq n-1)$. Now the condition $\lambda_m \neq \lambda_n$ $(m \neq n)$ is seen to reduce to (1.9); so the lemma is proved.

In the proofs of Lemmas 1.1, 1.2 it was seen that if a polynomial $y(x)$ of degree n satisfies (1.3a or b) then the corresponding value of λ is λ_n as given by (1.6) or (1.10); so we have the

COROLLARY. *If* (1.3a) *or* (1.3b) *is admissible then for each* $\lambda \neq \lambda_n$ $(n = 0, 1, \cdots)$ *the only polynomial solution is* $y(x) \equiv 0$.

Let (1.3a) or (1.3b) be admissible. In both cases the solution for $n = 1$ is

$$(1.11)\qquad y_1(x) = x + (a_0\alpha + b_0\beta)\delta^{-1}$$

where δ is given in (1.5). If we set

$$x + d = x^*,\ z(x^*) = y(x^* - d)$$

with

$$d = (a_0\alpha + b_0\beta)\delta^{-1}\ ,$$

the equation in $z(x^*)$ will also be admissible and will have the form (1.3a) or (1.3b) after the constant term in $C(x^*)$ has been absorbed into the λ. Moreover, for $n = 1$ we have

$$z_1(x^*) = x^* .$$

An admissible equation (1.3a) or (1.3b) in which for $n = 1$ the solution contains no constant term will be called *canonical*. It is no restriction to limit ourselves to canonical equations.

From (1.11) we obtain

LEMMA 1.3. *The admissible equation* (1.3a) *or* (1.3b) *is canonical if and only if*

$$a_0\alpha + b_0\beta = 0 . \tag{1.12}$$

2. Orthogonality for case (1.3a). We consider the problem of determining those canonical equations (1.3a) [(1.3b) in § 3] whose polynomial solutions form an orthogonal set. For all polynomials $y(x)$ we have

$$y(x + u) = \sum_{k=0}^{\infty} y^{(k)}(x)u^k/k\,! \tag{2.1}$$

so (1.3a) is equivalent, with respect to polynomial solutions, to the differential equation of infinite order

$$xy'(x) + \sum_{k=2}^{\infty} H_k(x)y^{(k)}(x)/k! = \sigma y(x) \tag{2.2}$$

where

$$H_k(x) = r_k + s_k x = (a_0\alpha^k + b_0\beta^k)\delta^{-1} + (a_1\alpha^k + b_1\beta^k)\delta^{-1}x \tag{2.3}$$

$(k = 1, 2, \cdots)$ with $\sigma = \{\lambda - (a_0 + b_0)\}\delta^{-1}$. Using (1.6) we find that the sequence $\{\sigma_n\}$ for which there are polynomial solutions is given by $\sigma_n = n$.

Equation (2.2) is identical with equation (3.1) of [1]. In Remark (i) ([1], p. 151) it is shown that if $r_2 = 0$ the polynomial solutions do not form an orthogonal set. We therefore assume $r_2 \neq 0$. In this case, Theorem 3.1 ([1], p. 151) states that the solutions of (our present) equation (2.2), hence of cononical equation (1.3a), form a weak orthogonal set if and only if

$$\begin{aligned} &r_{2p+1} = 0 , \qquad s_{2p+1} = s_3^p , \\ &r_{2p+2} = r_2 s_3^p , \quad s_{2p+2} = s_2 s_3^p \qquad (p = 0, 1, \cdots) . \end{aligned} \tag{2.4}$$

Moreover the weak orthogonal set is an orthogonal set when and only when one of the following two relations holds:

$$(2.5_1) \qquad s_2^2 - s_3 = 0 \; ;$$

$$(2.5_2) \qquad s_2^2 - s_3 \neq 0 \quad \text{and} \quad 2r_2(s_2^2 - s_3)^{-1} \neq 0, 1, 2, \cdots .$$

The condition $r_{2p+1} = 0$ is

$$(2.6) \qquad a_0\alpha^{2p+1} + b_0\beta^{2p+1} = 0 \qquad (p = 0, 1, \cdots) .$$

If $a_0 = 0$ or $b_0 = 0$ then both are zero since $\alpha\beta \neq 0$. But then $r_2 = 0$, contrary to assumption. So $a_0b_0 \neq 0$. Taking $p = 0, 1$ in (2.6) we then get $\beta^2 = \alpha^2$. Since α, β are distinct, then $\beta = -\alpha$; and again from (2.6) with $p = 0$: $a_0 = b_0$. Thus, if $r_2 \neq 0$ then $r_{2p+1} = 0$ $(p = 0, 1, \cdots)$ if and only if

$$(2.7) \qquad \beta = -\alpha, a_0 = b_0 \neq 0 .$$

With (2.7) holding then

$$\delta = \alpha(a_1 - b_1) \neq 0 ,$$

so

$$(2.8) \qquad \begin{aligned} &r_{2p+1} = 0, s_{2p+1} = \alpha^{2p}, r_{2p+2} = 2a_0(a_1 - b_1)^{-1}\alpha^{2p+1}, \\ &s_{2p+2} = (a_1 + b_1)(a_1 - b_1)^{-1}\alpha^{2p+1} . \end{aligned}$$

Conditions (2.4) are seen to be satisfied. And (2.5_1), (2.5_2) become respectively:

$$(2.9_1) \qquad a_1b_1 = 0 \; ;$$

$$(2.9_2) \qquad a_1b_1 \neq 0, a_0(a_1 - b_1)(\alpha a_1b_1)^{-1} \neq 0, 1, \cdots .$$

To sum up:

THEOREM 2.1. *Let equation* (1.3a) *be canonical. Then its polynomial solutions from an orthogonal set if and only if* (2.7) *holds and one of* (2.9_1), (2.9_2) *holds.*

REMARKS. (i) If (1.3a) is canonical its polynomial solutions form an orthogonal set if and only if it is of the form

$$(2.10) \qquad \begin{aligned} (a_1x + a_0)y(x + \alpha) + (b_1x + a_0)y(x - \alpha) \\ - (a_1 + b_1)xy(x) = \lambda y(x) , \end{aligned}$$

with $a_0 \neq 0, a_1 \neq b_1, \alpha \neq 0$, and either (2.9_1) or (2.9_2) holding.

(ii) In (2.10) make the variable changes $x = \alpha x^*$, $z(x^*) = y(\alpha x^*)$. There results a similar difference equation in $z(x^*)$, in which α is replaced by 1. This equation has an orthogonal set of solutions when (2.10) does. It may be termed a *standard* canonical equation. After dividing by a_0 this equation has the form (dropping asterisks)

$$(c_1x+1)z(x+1)+(d_1x+1)z(x-1) \\ -(c_1+d_1)xz(x)=\mu z(x)\,, \tag{2.11}$$

with $c_1-d_1\neq 0$ and either $c_1d_1=0$ or

$$c_1d_1\neq 0,\ (c_1-d_1)(c_1d_1)^{-1}\neq 0,1,2,\cdots .$$

3. Orthogonality for case (1.3b). Let equation (1.3b) be canonical, so that (1.12) holds. Putting (2.1) into (1.3b) we get an infinite order differential equation with polynomial coefficients of degree $\leqq 2$, which is equivalent to (1.3b) at least for polynomial solutions:

$$xy'(x)+\sum_{k=2}^{\infty}T_k(x)y^{(k)}(x)/k!=\sigma y(x)\,, \tag{3.1}$$

where

$$T_k(x)=r_k+s_kx+t_kx^2=(a_0\alpha^k+b_0\beta^k)\delta^{-1}+(a_1\alpha^k+b_1\beta^k)\delta^{-1}x \\ +(a_2\alpha^k+b_2\beta^k)\delta^{-1}x^2 \qquad (k=2,3,\cdots) \tag{3.2}$$

and $\sigma=\{\lambda-(a_0+b_0)\}\delta^{-1}$ and δ is given by (1.5). From (1.10) we see that $\{\sigma_n\}$ is given by

$$\sigma_n=n+n(n-1)t_2/2\,.$$

Equations of the form (3.1), that is, with $\max_k\{\text{degree } T_k(x)\}=2$ were considered in [1], but the results obtained were not as complete as for the case where the coefficients are of degree $\leqq 1$. We must therefore proceed differently. We first show that if canonical equation (1.3b), hence also (3.1), has an orthogonal set of solutions then $\beta=-\alpha$.

For suppose not. Then $|\alpha|\neq|\beta|$, since α,β are distinct. We may assume that $|\alpha|>|\beta|$. By Theorem 2.2 ([1], p. 148) there is a sequence of constants $\{\alpha_n\}$ (the moments of the weight function corresponding to the orthogonal set), with $\alpha_0\neq 0$, that satisfies the system of equations

$$d^p_{p+k}=0,\ D^p_{p+k}=0 \qquad (p,k=0,1,\cdots) \tag{3.3}$$

where (in our present case, as seen in [1], p. 153)

$$d^p_{p+k}=\sum_{i=k}^{2k+2}\alpha_i\left[\binom{k}{i-k}r_{2p+2k+1-i}+\binom{k}{i-k-1}s_{2p+2k+2-i}\right. \\ \left.+\binom{k}{i-k-2}t_{2p+2k+3-i}\right], \tag{3.4}$$

$$D^p_{p+k} = \sum_{i=k}^{2k+3} \alpha_i \left[\frac{i+1}{k+1} \binom{k+1}{i-k} r_{2p+2k+2-i} + \frac{i}{k+1} \binom{k+1}{i-k-1} s_{2p+2k+3-i} + \frac{i-1}{k+1} \binom{k+1}{i-k-2} t_{2p+2k+4-i} \right]. \tag{3.5}$$

Here the convention is made that $\binom{m}{q} = 0$ for $q < 0$, and $r_j = s_j = t_j = 0$ for $j \leqq 0$ and $r_1 = t_1 = 0$, $s_1 = 1$.

Putting the values of r_k, s_k, t_k from (3.2) into (3.3) we get

$$\begin{cases} \alpha^{2p+2k+1} U_k + \beta^{2p+2k+1} V_k = 0 \\ \alpha^{2p+2k+2} W_k + \beta^{2p+2k+2} X_k = 0 \end{cases} \qquad (p, k = 0, 1, \cdots) \tag{3.6}$$

where

$$\begin{aligned} U_k &= \sum_{i=k}^{2k+2} \alpha_i \left[\binom{k}{i-k} a_0 \alpha^{-i} + \binom{k}{i-k-1} a_1 \alpha^{-i+1} + \binom{k}{i-k-2} a_2 \alpha^{-i+2} \right], \\ W_k &= \sum_{i=k}^{2k+3} \alpha_i \left[\frac{i+1}{k+1} \binom{k+1}{i-k} a_0 \alpha^{-i} + \frac{i}{k+1} \binom{k+1}{i-k-1} a_1 \alpha^{-i+1} + \frac{i-1}{k+1} \binom{k+1}{i-k-2} a_2 \alpha^{-i+2} \right], \end{aligned} \tag{3.7}$$

and V_k, X_k are obtained from U_k, W_k be replacing

$$a_0, a_1, a_2, \alpha \quad \text{by} \quad b_0, b_1, b_2, \beta .$$

Let k be arbitrary but fixed. If we divide (3.6) by $\alpha^{2p+2k+1}$, $\alpha^{2p+2k+2}$ respectively and let $p \to \infty$, then since $|\beta/\alpha| < 1$ we get

$$U_k = 0, \; W_k = 0 \qquad (k = 0, 1, \cdots) . \tag{3.8}$$

And from (3.6) we then have

$$V_k = 0, \; X_k = 0 \qquad (k = 0, 1, \cdots) . \tag{3.9}$$

For $k = 0$, (3.8), (3.9) reduce to

$$\begin{aligned} \alpha_0 a_0 + \alpha_1 a_1 + \alpha_2 a_2 = 0, \; \alpha_1 a_0 + \alpha_2 a_1 + \alpha_3 a_2 = 0 , \\ \alpha_0 b_0 + \alpha_1 b_1 + \alpha_2 b_2 = 0, \; \alpha_1 b_0 + \alpha_2 b_1 + \alpha_3 b_2 = 0 . \end{aligned} \tag{3.10}$$

Now from (3.3) with $p = k = 0$ we have

$$\alpha_0 r_1 + \alpha_1 s_1 + \alpha_2 t_1 = 0 .$$

But $r_1 = t_1 = 0$, $s_1 = 1$; hence

$$\alpha_1 = 0 .$$

So (3.10) becomes

$$\begin{aligned} &\alpha_0 a_0 + \alpha_2 a_2 = 0, \alpha_2 a_1 + \alpha_3 a_2 = 0 , \\ &\alpha_0 b_0 + \alpha_2 b_2 = 0, \alpha_2 b_1 + \alpha_3 b_2 = 0 . \end{aligned} \tag{3.11}$$

Now $a_2 b_2 \neq 0$. For if a_2 or b_2 is zero then from $a_2\alpha + b_2\beta = 0$ (in (1.8)) and $\alpha\beta \neq 0$ we get $a_2 = b_2 = 0$. Hence (again from (1.8)) $c_2 = 0$; so all coefficients in (1.3b) are of degree < 2, contrary to assumption. Again, $a_0 b_0 \neq 0$. For if a_0 or b_0 is zero then (3.11) implies that $\alpha_2 = 0$. Since we already have $\alpha_1 = 0$, then $\Delta_1 = \begin{vmatrix} \alpha_0 \alpha_1 \\ \alpha_1 \alpha_2 \end{vmatrix} = 0$. But for the moments $\{\alpha_n\}$ corresponding to an orthogonal set it is known [2] that

$$\Delta_n \equiv \begin{vmatrix} \alpha_0 \alpha_1 \cdots \alpha_n \\ \alpha_1 \alpha_2 \cdots \alpha_{n+1} \\ \cdots\cdots\cdots\cdots \\ \alpha_n \alpha_{n+1} \cdots \alpha_{2n} \end{vmatrix} \neq 0 \qquad (n = 0, 1, \cdots) ;$$

so we have a contradiction. Thus,

$$a_2 b_2 \neq 0, a_0 b_0 \neq 0, \alpha_2 \neq 0 . \tag{3.12}$$

The right hand equations in (3.11) give us

$$-b_1 a_2 + a_1 b_2 = 0 .$$

This with

$$\alpha a_2 + \beta b_2 = 0$$

from (1.8) implies

$$\alpha a_1 + \beta b_1 = 0 ,$$

contrary to (1.9) for $n = 0$. So the assumption $\beta \neq -\alpha$ leads to a contradiction, and we have

$$\beta = -\alpha . \tag{3.13}$$

Then from (1.12):

$$a_0 = b_0 . \tag{3.14}$$

In (3.2) we now have

$$\begin{cases} r_{2p} = 2a_0 \delta^{-1} \alpha^{2p}, \ s_{2p} = (a_1 + b_1)\delta^{-1}\alpha^{2p}, \ t_{2p} = (a_2 + b_2)\delta^{-1}\alpha^{2p} \\ r_{2p+1} = 0, \ s_{2p+1} = (a_1 - b_1)\delta^{-1}\alpha^{2p+1}, \ t_{2p+1} = (a_2 - b_2)\delta^{-1}\alpha^{2p+1} \end{cases} \tag{3.15}$$

$(p = 1, 2, \cdots)$, with $r_1 = t_1 = 0$, $s_1 = 1$. (3.12) and (3.15) show that

$$r_2 \neq 0 .$$

Let

$$u_p = s_{2p+1},\ v_p = t_{2p+1},\ w_p = t_{2p+2} . \tag{3.16}$$

From $a_2\alpha + b_2\beta = 0$, $\beta = -\alpha \neq 0$ we get

$$a_2 = b_2 . \tag{3.17}$$

It is then readily seen that

$$v_p = 0,\ w_p - t_2 u_p = 0 \qquad (p = 0, 1, 2, \cdots) \tag{3.18}$$

Choose r_2, s_2, t_2, s_3 to satisfy the conditions

$$r_2 \neq 0,\ 2 + kt_2 \neq 0 (k = 0, 1, \cdots),\ \Delta_2 \neq 0 , \tag{3.19}$$

where $\alpha_1 = 0$, $\alpha_2, \alpha_3, \alpha_4$ are obtained from the equations

$$D_0^0 = 0,\ d_1^0 = 0,\ D_1^0 = 0 . \tag{3.20}$$

(3.18)–(3.20) make Theorem 4.2 ([1], p. 158) applicable, so that the solutions of (1.3b) form a weak orthogonal set if and only if

$$\begin{cases} s_{2p+1} = s_3^p,\ t_{2p+1} = 0,\ r_{2p+1} = 0 \\ s_{2p+2} = s_2 s_3^p,\ t_{2p+2} = t_2 s_3^p,\ r_{2p+2} = r_2 s_3^p \end{cases} \qquad (p = 0, 1, \cdots) . \tag{3.21}$$

Now these conditions do hold in view of (3.14).

The first two conditions of (3.19) become

$$a_0 \neq 0;\ (a_1 - b_1) + k\alpha a_2 \neq 0 \qquad (k = 0, 1, 2, \cdots) . \tag{3.22}$$

Finally, for weak orthogonality to imply orthogonality it is necessary and sufficient ([1], pp. 161–162) that $t_2 \notin S(r_2, s_2, s_3)$ where $S(r_2, s_2, s_3)$ is the set of all real values of t_2 for which $\pi_n(r_2, s_2, s_3, t_2) = 0$ for some $n > 1$. The expression for π_n is lengthy, and we do not reproduce it here. We merely observe that for given r_2, s_2, s_3 the set $S(r_2, s_2, s_3)$ is at most denumerable.

To sum up:

THEOREM 3.1. *Let the admissible equation* (1.3b) *be canonical. Its solutions form an orthogonal polynomial set if and only if*:

(i) (3.12), (3.13), (3.14), (3.17), (3.19) *hold.*

(ii) $t_2 \notin S(r_2, s_2, s_3)$.

REMARKS. (a) If the canonical equation (1.3b) has an orthogonal polynomial set of solutions then it has the form

$$(3.23)\quad \begin{aligned}(a_2x^2 + a_1x + a_0)y(x+\alpha) + (a_2x^2 + b_1x + a_0)y(x-\alpha)\\ - [2a_2x^2 + (a_1 + b_1)x]y(x) = \lambda y(x)\ ,\end{aligned}$$

with

$$(3.24)\quad a_0a_2(a_1 - b_1)\alpha \neq 0;\ (a_1 - b_1) + k\alpha a_2 \neq 0 \qquad (k = 0, 1, \cdots)\ .$$

(b) As in § 2 the transformation $x = \alpha x^*$, $z(x^*) = y(\alpha x^*)$ carries (3.24) into a similar equation with α replaced by 1.

4. Two examples. If an orthogonal polynomial set $\{P_n(x)\}$ satisfies (2.10) with $\lambda = \lambda_n$ for $y = P_n(x)$ then from (1.6) we have

$$(4.1)\qquad \lambda_n = 2a_0 + n\alpha(a_1 - b_1) \qquad (n = 0, 1, \cdots)\ .$$

Let $\{P_n(x)\}$, $\{Q_n(x)\}$ be polynomial sets defined by the respective generating functions

$$(4.2)\qquad e^{ct}(1-t)^{x+c} = \sum_{n=0}^{\infty} P_n(x)t^n \qquad (c \neq 0)\ ,$$

$$(4.3)\qquad (1-t)^{x-bd}\cdot(1-bt)^{-x+d} = \sum_{n=0}^{\infty} Q_n(x)t^n \qquad (b \neq 0, 1)\ .$$

We shall show that these sets are orthogonal and satisfy an equation of the form (2.10).

Denote the left side of (2.10) by $L[y]$. If $G(x, t)$ is the generating function in (4.2) then

$$(4.4)\quad L[G] = G\{(a_1x + a_0)(1-t)^{\alpha} + (b_1x + a_0)(1-t)^{-\alpha} - (a_1 + b_1)x\}\ .$$

Also,

$$(4.5)\qquad \begin{aligned}\sum_{n=0}^{\infty} \lambda_n P_n(x)t^n &= 2a_0G + \alpha(a_1 - b_1)t\,\partial G/\partial t\\ &= G\{2a_0 + \alpha(a_1 - b_1)t[c - (x + c)(1-t)^{-1}]\}\ .\end{aligned}$$

$\{P_n(x)\}$ will satisfy (2.10) if (4.4) and (4.5) are identical. It is a straightforward computation to show that they are identical if

$$(4.6)\qquad \alpha = 1;\ a_1 = 0;\ b_1 = a_0/c\ .$$

Hence $\{P_n(x)\}$ is an orthogonal set which satisfies the equation

$$(4.7)\quad P_n(x+1) + (x+c)P_n(x-1) - xP_n(x) = (2c - n)P_n(x)\ .$$

In the same way it is found that $\{Q_n(x)\}$ is an orthogonal set that is a solution of (2.10) for

$$(4.8)\qquad \alpha = 1;\ a_1 = bb_1;\ a_0 = -bdb_1\ .$$

The equation reduces to

$$b(x-d)Q_n(x+1)+(x-bd)Q_n(x-1)-(b+1)xQ_n(x) = \{-2bd+n(b-1)\}Q_n(x)\,. \tag{4.9}$$

In the case of (4.9) the condition (2.9_2) is to hold. It reduces to

$$-d(b-1) \neq 0, 1, 2, \cdots\,. \tag{4.10}$$

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