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FIXED-POINT-FREE OPERATOR GROUPS OF ORDER 8

FLETCHER GROSS

Let A be a group of order 2^n which acts as a fixed-point-free group of operators on the finite solvable group G. If no additional assumptions are made concerning G, then "reasonable" upper bounds on the nilpotent length, $l(G)$, of G have been obtained only when A is cyclic [Gross] or elementary abelian [Shult]. As a small step in extending the class of 2-groups A for which such bounds exist, it is shown in the present paper that if $|A| = 8$, then $l(G) \leqq 3$ if A is elementary abelian or quaternion and $l(G) \leqq 4$ otherwise.

Unfortunately, the author was unable to generalize his methods of proof to a wider class of groups.

The notation used in this paper agrees with that of [1] with two additions: (1) If G is a linear group operating on V and U is a G-invariant subspace, then $\{G \mid U\}$ denotes the restriction of G to U; and (2) $F_0(G) = 1$ and $F_{n+1}(G)/F_n(G)$ is the greatest normal nilpotent subgroup of $G/F_n(G)$.

THEOREM 1. *Let $G = NQ$ be a finite solvable linear group over a field K whose characteristic is not 2 and does not divide $|F_1(N)|$. Assume that N is a normal 2-complement of G and Q is a group of order 8 containing an element x of order 4. If, in addition, $C_N(Q) = 1$ and $\sum_{g \in Q} g = 0$, then it must must follow that*

$$[x^2, F_2(N)/F_1(N)] = 1 .$$

Proof. According to the hypothesis Q can be any group of order 8 except an elementary abelian group. If Q is cyclic, this theorem is a special case of [4, Th. 1.2], and if Q is a quaternion group, then a stronger result is possible. Thus the main interest in the theorem is when Q is either dihedral or is the direct product of cyclic groups of orders 4 and 2.

To prove the theorem we first notice that extending K affects neither hypothesis or conclusion. Thus we may as well assume that K is algebraically closed. We now assure that G is a minimal counterexample to the theorem and let V be the space on which G operates.

Choose S to be a subgroup of $F_2(N)$ such that Q normalizes S, $[x^2, S] \not\leqq F_1(N)$, and S is minimal with respect to the above properties. S must be a p-group for some prime p. Now Q normalizes $[x^2, S]$, and $[x^2, [x^2, S]] = [x^2, S]$ [2]. Due to the minimality of S, this implies that $[x^2, S] = S$.

Now $C_S(O_{p'}(F_1(N))) = S \cap F_1(N)$. Thus there is an r-group R for some prime $r \neq p$ such that QS normalizes R, $R \leqq F_1(N)$, $[S, R] \neq 1$, and R is minimal with respect to the above properties. R must be a special r-group, and R/R' must be transformed irreducibly by QS.

Since the characteristic of K does not divide $|F_1(N)|$, V is a completely reducible $K-R$ module. From this and the fact that $[S, R] \lhd QSR$, it follows that V contains a maximal $K-QSR$ submodule M such that $[S, R]$ is not the identity on V/M. Now let H be the kernel of the representation of QSR afforded by V/M.

Since $\langle x \rangle$ must be faithfully represented on V/M, we have that either $Q \cap H = 1$ or $Q/Q \cap H$ is cyclic of order 4. But Q has no nonzero fixed vector in V and so certainly has none in V/M. Thus if $Q/Q \cap H$ is cyclic of order 4, then it follows from [4] that $[x^2, S, R] = 1$. Hence we must have $Q \cap H = 1$. This implies that QSR/H acting as a linear group on V/M satisfies the hypothesis but not the conclusion of the theorem. Therefore, in proving the theorem we may as well assume that $G = QSR$ and that V is an irreducible $K-G$ module.

Clifford's theorem now implies that V is a completely reducible $K-SR$ module and $V = V_1 \oplus V_2 \oplus \cdots \oplus V_t$ where the V_i are the homogeneous $K-SR$ modules. Q must permute the V_i transitively, and, since $[S, R] \lhd QSR$, it must be that $\{[S, R] \mid V_i\} \neq 1$ for all i.

We now proceed to prove that $t = 1$, or, in other words, that V is a homogeneous $K-SR$ module. For this purpose let

$$Q_i = \{g \mid g \in Q,\ V_i g = V_i\}$$

and

$$C_i = \{g \mid g \in Q_i,\ \{[g, SR] \mid V_i\} = 1\} .$$

Then Q_i and Q_j as well as C_i and C_j are conjugate in Q for all i and j. $[Q\colon Q_i] = t$, V_i is an irreducible $K-Q_iSR$ module, and $\{\sum_{g \in Q_i} g \mid V_i\} = 0$ for all i. The last fact implies that $Q_i \neq 1$. Since $\{[x^2, S] \mid V_i\} \neq 1$, x^2 cannot belong to C_i.

LEMMA. *$C_i = 1$ for all i.*

Proof. Suppose $C_i \neq 1$. Since $\langle x \rangle \cap C_i = 1$, it follows that C_i is cyclic of order 2 generated by an element y_i. Now $C_{RS}(x)$ is normalized by Q. It follows from this and the fact that conjugation by x transitively permutes the y_i that $[u, y_i] = [u, y_j]$ for all i and j and all $u \in C_{RS}(x)$. Since $[u, y_i]$ is represented by the identity on V_i, this all implies that $[C_{RS}(x), y_i] = 1$ for all i. Since x and y_i generate Q, we obtain that $C_{RS}(x) = C_{RS}(Q) = 1$. Hence x acts as a fixed-point-free automorphism on RS. From this follows $[x^2, S, R] = 1$ [3] which is a contradiction.

LEMMA. *$Q_i = Q$ and $t = 1$.*

Proof. If Q_i is elementary abelian, it follows from [7, Th. 4.1] that $C_i \neq 1$. Thus, since $Q_i \neq 1$, we must have either $Q_i = Q$ or Q_i is cyclic of order 4 generated by an element y. If Q_i is cyclic of order 4 we must have $y^2 = x^2$ because Q only has 8 elements. Now Q_i can have no nonzero fixed vector in V_i. Theorem 1.2 of [4] now yields that $[x^2, S, R]$ is represented by the identity on V_i. Since this is impossible, Q_i must be Q. Then $t = [Q: Q_i] = 1$ and so V is a homogeneous $K - SR$ module.

COROLLARY. $Z(SR) = R' = 1$.

Proof. $Z(SR)$ is represented by scalar matrices and so Q must centralize $Z(SR)$. Thus $Z(SR) \leqq C_{RS}(Q) = 1$. Now R' is normalized by QS and so, due to the minimality of R, we must have $[S, R'] = 1$. Therefore $R' \leqq Z(SR)$.

Now let $V = U_1 \oplus U_2 \oplus \cdots \oplus U_s$ where the U_i are the homogeneous $K - R$ submodules of V. Let $H_i = \{g \mid g \in QS, U_i g = U_i\}$ and $S_i = H_i \cap S$. Now SQ must permute the U_i transitively since V is an irreducible $K - QSR$ module. Thus $s = [QS: H_i]$ for all i. But V is a homogeneous $K - SR$ module. This implies that $(U_i S)Q = U_i S$. Hence $U_i S = V$ for all i. Therefore $s = [S: S_i] = [QS: H_i]$ which means that H_i must contain a Sylow 2-subgroup of SQ. Since the H_i are all conjugate in QS, this implies that $Q \leqq H_i$ for some i, $i = 1$ say. Then Q fixes U_1. Let R_1 be the kernel of the representation of R afforded by U_1. Clearly R_1 is normalized by Q. But R is abelian and so R is represented by scalar matrices on U_1. It now follows that $[R/R_1, Q] = 1$. Since $C_R(Q) = 1$, this implies that $R_1 = R$. But, since V is an irreducible $K - QSR$ module and $R \lhd QSR$, this is impossible. This contradiction proves the theorem.

THEOREM 2. *Let $G = NQ$ be a finite solvable linear group over a field K whose characteristic does not divide $|F_1(N)|$. Assume that N is a normal 2-complement of G and Q is an ordinary quaternion group. If, in addition, $C_N(Q) = 1$ and $\sum_{g \in Q} g = 0$, then it must follow that $[Q', F_1(N)] = 1$.*

Proof. Extending K affects neither hypothesis nor conclusion. Thus we assume that K is algebraically closed. If $[Q', F_1(N)] \neq 1$, then there is a subgroup P of $F_1(N)$ such that Q normalizes P, Q' does not centralize P, and P is minimal with respect to the above properties. Then P is a special p-group for some prime p and P/P' is transformed faithfully and irreducibly by Q. This implies that

$|P/P'| = p^2$, and so P is either elementary abelian of order p^2 or extra-special of order p^3 and exponent p.

If V is the vector space on which G operates, then

$$V = V_1 \oplus V_2 \oplus \cdots$$

where the V_i are the homogeneous $K - P$ modules. By renumbering, we may assume that $[Q', P]$ is not the identity on V_1. Now if Q, as a permutation group on the V_i, had an orbit of length 8, then $\sum_{g \in Q} g$ would not be 0. This implies that Q' must fix V_1.

If $\{P \mid V_1\}$ is abelian, then P is represented by scalar matrices on V_1 and so we would have $\{[Q', P] \mid V_1\} = 1$. Thus $\{P \mid V_1\}$ is not abelian. This implies that $P = \{P \mid V_1\}$ = an extra-special p-group of order p^3 and exponent p.

Now let $H = \{g \mid g \in Q, V_1 g = V_1\}$. In order that $\sum_{g \in Q} g = 0$, we must have $\{\sum_{g \in H} g \mid V_1\} = 0$. Now a faithful irreducible K-representation of P is uniquely determined by the representation of P' [6]. It follows from this that $H = C_Q(P')$. Since $C_P(Q) = 1, H \neq P$. But the automorphism group of P' is cyclic. Thus Q/H is cyclic. This implies that H is cyclic of order 4. Let x generate H and let y be an element of Q not contained in H.

Case 1. $p \equiv 1 \pmod 4$.

Suppose first that char $(K) \neq 2$. Then Theorem 3.1 of [7] implies that $\{[x^2, P] \mid V_1\} = 1$, which is a contradiction. If char $(K) = 2$, then Theorem B of [6] leads to $\{x^3 + x^2 + x + 1 \mid V_1\} \neq 0$, also a contradiction.

Case 2. $p \equiv 3 \pmod 4$.

In this case $GF(p)$ does not contain a primitive 4th root of unity. Since Q faithfully transforms P/P', it follows that there elements a, b generating P such that

$$a^y \equiv b,\ b^y \equiv a^{-1} \pmod{P'}\ .$$

But this implies that $[a, b]^y = [b, a^{-1}] = [a, b]$, contrary to $y \in C_Q(P')$.

THEOREM 3. *Let Q be a group of order* 8 *which acts as a fixed-point-free group of automorphisms of the finite group G. Then G is solvable and $l(G) \leqq 3$ if Q is either elementary abelian or a quaternion group and $l(G) \leqq 4$ otherwise. The upper bound in the case when Q is elementary abelian or a quaternion group is best-possible.*

Proof. If G admits a 2-group as a fixed-point-free operator group, then G must have odd order and so G must be solvable from the Feit-Thompson Theorem [1]. If Q is elementary abelian, the result follows from Thorem 4.3 of [7]. Therefore assume that Q has an elemen x of order 4. We now use induction on $|G|$.

If H_1, H_2 are distinct minimal Q-admissible normal subgroups, then $l(G) \leqq l[(G/H) \times (G/H_2)] = \text{Max}\,\{l(G/H_1), l(G/H_2)\}$. Thus in proving the theorem we may assume that G has only one minimal Q-admissible normal subgroup. Hence $F_1(G)$ is a p-group for some prime p. Now let $N = G/F_1(G)$ and consider NQ as a linear group acting on V where V is $F_1(G)/D(F_1(G))$ written additively. Theorems 1 and 2 imply that $[x^2, F_k(N)/F_{k-1}(N)] = 1$ where $k = 1$ if Q is a quaternion group and $k = 2$ otherwise. It follows from this that $[x^2, N/F_{k-1}(N)] = 1$. But then $N/F_{k-1}(N)$ admits a fixed-point-free operator group of order 4. This implies that $l(N/F_{k-1}(N)) \leqq 2$. We now have that

$$l(G) = 1 + l(N) = 1 + (k - 1) + l(N/F_{k-1}(N)) \leqq k + 2 .$$

Finally, the claim of best-possible in the statement of the theorem is justified by [5].

REFERENCES

1. W. Feit and J. Thompson, *Solvability of groups of odd order*, Pacific J. Math. **13** (1963), 775-1029.

2. G. Glauberman, *Fixed points in groups with operator groups*, Math. Z. **84** (1964), 120-125.

3. D. Gorenstein and I. Herstein, *Finite groups admitting a fixed-point-free automorphism of order* 4, Amer. J. Math. **83** (1961), 71-78.

4. F. Gross, *Groups admitting a fixed-point-free automorphism of order* 2^n, Pacific J. Math. (in press)

5. ———, *A note on fixed-point-free solvable operator groups*, Proc. Amer. Math. Soc. (in press).

6. P. Hall and G. Higman, *On the p-length of p-soluble groups and reduction theorems for Burnside's problem*, Proc. London Math. Soc. (3) **6** (1956), 1-42.

7. E. Shult, *On groups admitting fixed-point-free abelian groups*, Illinois J. Math. **9** (1965), 701-720.

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UNIVERSITY OF UTAH
SALT LAKE CITY, UTAH