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FUNCTION-THEORETIC DEGENERACY CRITERIA FOR RIEMANNIAN MANIFOLDS

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The modulus of a relatively compact set with border consisting of at least two components is a measure of its magnitude with regard to harmonic functions. A divergent modular sum associated with difference sets obtained from an exhaustion of a Riemannian manifold is characteristic of parabolicity. The existence of a divergent minimum modular sum implies that the manifold carries no nonconstant harmonic functions with finite Dirichlet integral.

The modular criteria presented in this paper are generalizations of criteria established for Riemann surfaces by Noshiro [6] and Sario [8], [9]. In the two-dimensional case function-theoretic degeneracy is completely determined by the conformal structure, whereas it has been shown by Nakai and Sario [5] that the type of a Riemannian manifold varies when the metric is replaced by a conformally equivalent one. The significance of our result stems from this fact.

For the completeness of the presentation it is shown that the various characterizations of parabolicity due to Ahlfors, Brelot, Nevanlinna and Ohtsuka remain equivalent in higher dimensions. This overlaps with the work of Itô [2] and Loeb [3] in different settings. A new proof for Riemannian manifolds of the relation, $O_{HD} = O_{HBD}$, established in [10] is also given.

1. Let R be an orientable noncompact Riemannian manifold. Let $A \subset R$ and denote by $H(A)$ the class of harmonic functions on A and by $H^c(A)$ the functions in $H(A)$ with continuous extensions to $\bar{A}$. For every parametric region V there exists a Green's function q_x^V with the property $-h(x) = \int_{\partial V} h^* dq_x^V$ for every $h \in H^c(V)$. It is well-known that the sheaf of harmonic functions over R satisfies the axioms of a harmonic space and we shall use the standard facts of the theory freely. These together with Green's formulas will serve as our main tools.

2. Consider a fixed parametric region $V \subset R$ and a point $a \in V$. Let F consist of the constant $+\infty$ and of all nonnegative superharmonic functions s on R such that $s \mid V - q_a^V$ is bounded. Clearly F is a Perron family on $R - a$ and its lower envelope is either $+\infty$ or a function g_a harmonic on $R - a$. If the function g_a exists it is called the *Green's function* for R.

Let $\{\Omega_n\}$ be an exhaustion of R by regular regions. Let $t_n \in H^c(\Omega_n - \bar{\Omega}_0)$ with $t_n \mid \partial\Omega_0 = 0$, $t_n \mid \partial\Omega_n = 1$. The functions t_n form a decreasing sequence and the harmonic function $t = \lim_n t_n$ on $R - \bar{\Omega}_0$ is called the *harmonic measure of the ideal boundary of R with respect to Ω_0*.

3. The above definitions are related as follows.

THEOREM. *Conditions* (a), (b) *and* (c) *are equivalent.*

(a) *There exists a nonconstant positive superharmonic function on R.*

(b) *t is not identically zero.*

(c) *R does not belong to the class O_G of manifolds for which g_a does not exist.*

It is obvious that either (b) or (c) implies (a). To show that (a) implies (b), suppose that $t \equiv 0$ and let s be a positive superharmonic function. Set $m = \min_{\partial\Omega_0} s = \min_{\bar{\Omega}_0} s$ and observe that $s \geqq m(1 - t_n)$ on $\bar{\Omega}_n - \bar{\Omega}_0$. On letting n tend to ∞ we conclude that $s \geqq m$ on R and hence it is a constant. Now suppose that (a) is true. Then (b) holds for all choices of Ω_0 and in particular for the choice

$$\Omega_0 = \{x \mid q_a^V(x) > 1\} \subset V .$$

Set $\alpha = (\min_{\partial V} t)^{-1}$. By virtue of the fact that $q_a^V \mid \partial V = 0$ we conclude that the function

$$q(x) = \begin{cases} q_a^V(x), & x \in \Omega_0 \\ 1 - 2\alpha t(x), & x \in R - \Omega_0 \end{cases}$$

is superharmonic on R and bounded from below by $1 - 2\alpha$. Thus F contains more than one element and consequently $R \notin O_G$.

The *modulus* τ_n of the region $\Omega_n - \bar{\Omega}_0$ is by definition $\left(\int_{\partial\Omega_0} {}^*dt_n\right)^{-1}$ and we can state the following immediate

COROLLARY. *The constants τ_n tend to $+\infty$ if and only if $R \in O_G$.*

4. In addition to the above notions we define the *modulus function* $\omega_i \in H^c(\Omega_i - \bar{\Omega}_{i-1})$ by the conditions $\omega_i \mid \partial\Omega_{i-1} = 0$, $\omega_i \mid \partial\Omega_i = \mu_i$ (const.) and $\int_{\partial\Omega_{i-1}} {}^*d\omega_i = 1$. The constant μ_i is called the *modulus* of $\Omega_i - \bar{\Omega}_{i-1}$. We now generalize the O_G modular criterion of Noshiro [6] and Sario [9] to Riemannian manifolds.

THEOREM. *There exists an exhaustion $\{\Omega_n\}$ of R such that $\sum \mu_n = +\infty$ if and only if $R \in O_G$.*

Suppose that such an exhaustion exists. The Dirichlet integral of $w_n = \tau_n t_n$ over $\Omega_n - \bar{\Omega}_0$ is

$$D_n(w_n) = \int_{\Omega_n - \bar{\Omega}_0} dw_n \wedge {}^*dw_n = \int_{\partial\Omega_n - \partial\Omega_0} w_n {}^*dw_n = \tau_n \ .$$

On the other hand, $D_n(w_n)$ is equal to the sum $\sum_{i=1}^{n} D_i(w_n)$ of Dirichlet integrals over $\Omega_i - \bar{\Omega}_{i-1}$. Using the Schwarz inequality we obtain

$$D_i(w_n) \geqq \frac{D_i^2(\omega_i, w_n)}{D_i(\omega_i)} = \frac{\left(\int_{\partial\Omega_i - \partial\Omega_{i-1}} \omega_i {}^*dw_n\right)^2}{\int_{\partial\Omega_i - \partial\Omega_{i-1}} \omega_i {}^*d\omega_i} = \frac{\mu_i^2}{\mu_i} \ .$$

Thus $\tau_n \geqq \sum_{i=1}^{n} \mu_i$, and consequently $\tau_n \to +\infty$. This implies, by Corollary 3, that $R \in O_G$.

Conversely, suppose that $R \in O_G$ and $\{\Omega_n\}$ is any exhaustion of R. Then there exists an n_1 such that $\tau_{n_1} > 1$. Then $\{\tau'_n\}$ constructed for the exhaustion $\{\Omega_n\}_{n_1}^{\infty}$ also tend to $+\infty$ and a fortiori there is an n_2 with $\tau'_{n_2} > 1$. Proceeding in this fashion we obtain an exhaustion $\{\Omega_{n_i}\}$ such that the modular sum

$$\sum_{i=1}^{\infty} \mu_i = \sum_{i=1}^{\infty} \tau_{n_i}^{(i-1)} = +\infty \ .$$

The above condition is not necessary in the sense that every manifold has an exhaustion with arbitrarily modular sum (cf. [1]).

5. We now turn to a criterion for membership in the class O_{HD} of manifolds on which every member of the class $HD(R)$ of harmonic functions with finite Dirichlet integral over R is constant. The Dirichlet integral of an $f \in C^1(R)$ over R is, by definition,

$$D(f) = \lim_{n\to\infty} D_n(f) \ ,$$

where $D_n(f)$ is taken over Ω_n and $\{\Omega_n\}$ is an exhaution of R. Clearly $D(f)$ is independent of the choice of the exhaustion $\{\Omega_n\}$ and

$$D(f) < +\infty, D(g) < +\infty$$

imply the existence of the limit $D(f, g) = \lim_{n\to\infty} D_n(f, g)$.

First we prove the

THEOREM. *There exists a nonconstant element of $HD(R)$ if and only if there exists one which is also bounded. That is, $O_{HD} = O_{HBD}$.*

Let u be a nonconstant element of $HD(R)$ and set

$$u_m = \min(m, \max(u, -m)) \ .$$

Denote by v_{mn} the continuous function on R such that $v_{mn} = u_m$ on $R - \Omega_n$ and v_{mn} is harmonic on Ω_n. Since $\{v_{mn}\}_{n=0}^{\infty}$ is uniformly bounded it has a subsequence also denoted by $\{v_{mn}\}_{n=0}^{\infty}$ which converges to a harmonic function v_m uniformly on compact subsets. By Green's formula we obtain $D(u_m - v_{mn}) = D(u_m) - D(v_{mn}) \geqq 0$ and

$$D(v_{mn} - v_{m,n+p}) = D(v_{mn}) - D(v_{m,n+p}) \geqq 0 .$$

From this we see that $d = \lim_n D(v_{mn})$ exists. By Fatou's lemma we also have $D(v_{mn} - v_m) \leqq D(v_{mn}) - d$ and consequently $\lim_n D(v_{mn} - v_m) = 0$.

For every m, $v_m \in HBD(R)$ and we shall complete proof by showing that the assumption v_m is a constant for every m leads to a contradiction. Set $g_{mn} = u_m - v_{mn}$, $g_n = \lim_n g_{mn} = u_m - v_m$ and note that $\lim_n D(g_{mn} - u_m) = 0$. Since $\operatorname{supp} g_{mn} \subset \bar{\Omega}_n$, we have

$$D(g_{mn}, u) = \int_{\partial\Omega_n} g_{mn} {*du} = 0 .$$

Thus $D(u_m, u) = \lim_n D(g_{mn}, u) = 0$ and $D(u) = \lim_m D(u_m, u) = 0$, which contradicts the choice of u. Our proof stems in spirit from Royden [7].

6. Sario [8] introduced a sufficient condition for a Riemann surface not to carry nonconstant analytic functions with finite Dirichlet integral. Since analytic, as such, has no meaning in Riemannian manifolds we replace Sario's result with an O_{HD} criterion.

In general the open sets $\Omega_n - \bar{\Omega}_{n-1}$ will consist of components E_{in}, $i = 1, \cdots, i(n)$. Let ω_{in} denote the modulus function of E_{in}, i.e., $\omega_{in} \in H^c(E_{in})$, $\omega_{in} \mid \partial E_{in} \cap \partial\Omega_{n-1} = 0$, $\omega_{in} \mid \partial E_{in} \cap \partial\Omega_n = \mu_{in}$ (const.) such that $\int {*d\omega_{in}} = 1$. Set $\nu_n = \min_i \mu_{in}$.

THEOREM. *If $\{\Omega_n\}$ is an exhaustion of R with a divergent minimum modular sum $\sum \nu_n = +\infty$, then $R \in O_{HD}$.*

Since Euclidean n-space ($n \geqq 3$) is not in O_G but in O_{HD}, Theorem 4 shows that the divergence of the minimum modular sum is not a necessary condition for a manifold to be in O_{HD}.

For the proof of the theorem let u be a nonconstant harmonic function on R with $D(u) < +\infty$; by Theorem 5 we may assume that $|u| \leqq M$. We further normalize u by adding a constant so that

$$\int_{\partial\Omega_0} u{*du} = 0 .$$

By a theorem of Morse (cf. Milnor [4]) ω_n and its gradient can be uniformly approximated by C^∞ functions h and their gradients, where the h's have the same boundary values as ω_n and grad $h = 0$ only at a finite number of points. We choose such an h with the property that

$$\int_{\beta_{n\lambda}} |\operatorname{grad} h| \, dS \leqq 2 ,$$

where $\beta_{n\lambda}$ is the level surface $h = \lambda$, $0 < \lambda < \mu_n$.

Denote by $D_{n\lambda}(u)$ the Dirichlet integral of u over the open set bounded by $\partial\Omega_0$ and $\beta_{n\lambda}$. By the definition of gradient and the Schwarz inequality we obtain

$$\frac{d}{d\lambda} D_{n\lambda}(u) = \int_{\beta_{n\lambda}} \frac{|\operatorname{grad} u|^2}{|\operatorname{grad} h|} ds \geqq \frac{\left(\int_{\beta_{n\lambda}} |\operatorname{grad} u| \, dS\right)^2}{\int_{\beta_{n\lambda}} |\operatorname{grad} h| \, dS} \geqq \frac{1}{2}\left(\int_{\beta_{n\lambda}} |*du|\right)^2 ,$$

except for a finite number of values of λ. On the other hand, Green's formula gives

$$D_{n\lambda}(u) = \int_{\beta_{n\lambda}} u*du \leqq M \int_{\beta_{n\lambda}} |*du| .$$

We conclude that

$$\frac{d}{d\lambda} D_{n\lambda}(u) \geqq \frac{1}{2M^2} D^2_{n\lambda}(u) .$$

We now integrate this inequality for $n \geqq 2$:

$$\int_0^{\nu_n} \frac{\frac{d}{d\lambda} D_{n\lambda}(u)}{D_{n\lambda}(u)} d\lambda \geqq \frac{1}{2M^2} \int_0^{\nu_n} D_{n\lambda}(u) d\lambda \geqq \nu_n a ,$$

where $a = (2M^2)^{-1} D_1(u)$. This implies that

$$D_n(u) \geqq D_{n,\nu_n}(u) \geqq D_{n-1}(u) \exp(\nu_n a)$$

and iteration gives $D_n(u) \geqq D_1(u) \exp(a \sum_2^a \nu_n)$. Since $D(u) < +\infty$, we conclude that $\sum_1^\infty \nu_n < +\infty$.

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