PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 2, 1969

BANACH ALGEBRA BUNDLES

BERNARD R. GELBAUM

If $\mathscr{E}$ is a fibre bundle over a space X with fibre A, a Banach algebra, and group the group of isometric automorphisms of A then the set of sections of the fibre bundle can be endowed with the structure of a Banach algebra. If the fibre A is a so-called Q-uniform Banach algebra (e.g., a commutative Banach algebra) then the maximal ideal space of the Banach algebra of sections can be identified as a fibre bundle with base X, fibre the set of maximal ideals of the Banach algebra A and group the group of self-homeomorphisms of the space of maximal ideals of A. Similar results are obtained for certain epimorphism structures associated with the algebras described.

In discussing fibre bundles we shall operate in the following context: A fibre bundle $\mathscr{E}$ specified up to equivalence [3] by a bundle space, E, a base space X, a fibre A, a continuous projection $p: E \to X$, an open covering $\mathscr{U} = \{U\}$ of X, homeomorphisms $\varphi_U: U \times A \to p^{-1}(U)$ for $U \in \mathscr{U}$. The φ_U are fibre-preserving in that $\varphi_U(x \times A) = p^{-1}(x)$. Furthermore there is an effective topological group $\mathscr{A}$ of self-homeomorphisms (auteomorphisms) of the fibre A. The mappings φ_U and the fibre A are related as follows: For

$$x \in U \cap V,\ U,\ V \in \mathscr{U}\ , \quad \text{and} \quad a \in A, \text{let}\, \varphi_U^{-1}\varphi_V(x, a) = (x, g_{UV}(x)(a))\ .$$

Then $g_{UV}(x) \in \mathscr{A}$ and the map $g_{UV}: U \cap V \to \mathscr{A}$ is continuous. If

$$y \in p^{-1}(x),\ x \in U \in \mathscr{U}\ ,$$

we shall write $\varphi_U^{-1}(y) = (x, t_U(y))$.

In our discussions we assume that $\mathscr{A}$ is topologized via neighborhoods of which the following is typical:

$$N(T_0) = \{T: T(a_i) \in N(T_0(a_i)),\ i = 1, 2, \cdots, n\}$$

where T_0, $T \in \mathscr{A}$, $a_i \in A$ and $N(T_0(a_i))$ is a neighborhood of $T_0(a_i)$ in the topology of A. Thus $\mathscr{A}$ is topologized by pointwise convergence.

Let $g_{UV}(x_0) = T_0$ and let $N(T_0)$ be given as above. Note that $g_{UV}(x)(a)$ is continuous on $(U \cap V) \times A$ since it is the composition of $\varphi_U^{-1}\varphi_V$ and the continuous open projection $(U \cap V) \times A \to A$. Thus there is a neighborhood $N(x_0) \subset U \cap V$ such that, for

$$x \in N(x_0),\ g_{UV}(x)(a_i) \in N(g_{UV}(x_0)a_i),\ i = 1, 2, \cdots, n\ .$$

Thus $g_{UV}(N(x_0)) \subset N(T_0)$ and $g_{UV}: U \cap V \to \mathscr{A}$ is continuous.

For our special purposes, A will be a Banach algebra with identity e, X will be compact Hausdorff and $\mathscr{A}$ will be a group of isometric C-automorphisms of A. We shall then show how to identify $\Gamma(\mathscr{E})$, the set of continuous sections $\gamma: X \to E\,(p\gamma(x) = x)$ as a new Banach algebra D. For a class of Banach algebras (the so-called Q-uniform algebras), among which are the commutative Banach algebras, we shall relate various structure spaces for A and D and show how the fibre bundle structure of $\mathscr{E}$ imposes a structure on the structure spaces for D[1].

In consonance with the remarks made earlier, we topologize $\mathscr{A}$ by neighborhoods of which the following is typical

$$N(\alpha_0) = \{\alpha: \|\alpha(a_i) - \alpha_0(a_i)\|_A < \varepsilon,\ i = 1, 2, \cdots, n\}$$

where $\alpha_0, \alpha \in \mathscr{A}$ and $a_i \in A$. Thus the map $\mathscr{A} \times A \ni (\alpha, a) \to \alpha(a) \in A$ is continuous.

Other topologies are useful in special situations. However, we shall confine ourselves to that described above[2]. Direct calculation shows that in the given topology $\mathscr{A}$ is a topological group.

1. $\Gamma(\mathscr{E})$. In this section we show $\Gamma(\mathscr{E})$ may be given the structure of a Banach algebra D that is in fact a bimodule over the algebra $C(X)$ of C-valued continuous functions on X.

For $\gamma_1, \gamma_2 \in \Gamma(\mathscr{E}), f_1, f_2 \in C(X)$ and $x \in U \in \mathscr{U}$ let

$$\begin{aligned}
&f_1\cdot\gamma_1(x) = \gamma_1\cdot f_1(x) \equiv \varphi_U(x, f_1(x)t_U(\gamma_1(x))) \\
&[f_1\cdot\gamma_1 + f_2\cdot\gamma_2](x) \equiv \varphi_U(x, f_1(x)t_U(\gamma_1(x)) + f_2(x)t_U(\gamma_2(x))) \\
&\gamma_1\cdot\gamma_2(x) = \varphi_U(x, t_U(\gamma_1(x))t_U(\gamma_2(x)))\ .
\end{aligned}$$

The above definitions ostensibly make the results dependent on the choice of $U \in \mathscr{U}$. However, if $x \in V \in \mathscr{U}$, then, e.g., if we use V for definition,

$$\begin{aligned}
\gamma_1\cdot\gamma_2(x) &= \varphi_V(x, t_V(\gamma_1(x))t_V(\gamma_2(x))) \\
&= \varphi_V(x, g_{VU}(x)[t_U(\gamma_1(x)t_U(\gamma_2(x))]) \\
&= \varphi_U(x, t_U(\gamma_1(x))t_U(\gamma_2(x)))\ .
\end{aligned}$$

Thus $\gamma_1\cdot\gamma_2(x)$ is well-defined as are $f_1\cdot\gamma_1(x)$ and $[f_1\cdot\gamma_1 + f_2\cdot\gamma_2](x)$; clearly $\gamma_1\cdot\gamma_2, f_1\cdot\gamma, f_1\cdot\gamma_1 + f_2\cdot\gamma_2$ belong to $\Gamma(\mathscr{E})$.

We note the existence of the following special sections:

---

[1] All the morphisms of this paper are assumed to be continuous, i.e., in the category of Banach algebras and continuous homomorphisms among them.

[2] The author thanks the referee for numerous constructive comments on this and on other points.

$$\mathfrak{e}(x) \equiv \varphi_U(x, e) , \qquad x \in U \in \mathscr{U}$$
$$\mathfrak{o}(x) \equiv \varphi_U(x, 0) , \qquad x \in U \in \mathscr{U} .$$

Clearly these definitions are U-independent. Furthermore for all $\gamma \in \Gamma(\mathscr{E})$,

$$\gamma \cdot \mathfrak{e} = \mathfrak{e} \cdot \gamma = \gamma, \mathfrak{o} \cdot \gamma = \gamma \cdot \mathfrak{o} = \mathfrak{o}, \gamma + \mathfrak{o} = \mathfrak{o} + \gamma = \gamma .$$

The above definitions endow $\Gamma(\mathscr{E})$ with an algebraic structure.

If we set $|\gamma(x)| = \|t_U(\gamma(x))\|_A$ for $x \in U \in \mathscr{U}$, and then

$$\|\gamma\| = \sup_x |\gamma(x)| ,$$

we see that:

(i) since each $g_{UV}(x)$ is an isometry, $|\gamma(x)|$ is independent of the choice of $U \ni x$,

(ii) $|\gamma(x)|$ is continuous on X and

(iii) since X is compact, $\|\gamma\| < \infty$.

Direct verification shows that $\|\cdots\|$ is a norm on $\Gamma(\mathscr{E})$ and that with respect to this norm $\Gamma(\mathscr{E})$ is a Banach algebra D.

In the work below we shall need a lemma whose general character justifies its inclusion here.

LEMMA 1.1. *Let $U \in \mathscr{U}$ and let $f \in C(X, A)$, the set of A-valued continuous functions on X, where the support K of f is contained U. Then*

$$\begin{aligned} \gamma(x) &\equiv \varphi_U(x, f(x)) , && x \in U \\ &\equiv \mathfrak{o}(x) && x \notin U \end{aligned}$$

is in D and $t_U(\gamma(x)) = f(x)$.

Proof. For $x \in U, \gamma(x)$ is clearly continuous. If $x_0 \notin U$, then

$$\gamma(x_0) = \mathfrak{o}(x_0) .$$

The (compact) support K of f lies in U whence there is a neighborhood $N(x_0)$ not meeting K. Clearly, throughout this neighborhood $\gamma(x) = \mathfrak{o}(x)$. The equation $p\gamma(x) = x$ is valid by definition of γ.

REMARK. If $\{\psi_U\}$ is a C-valued partition of unity subordinate to $\mathscr{U}$ and $a \in A$ let

$$\begin{aligned} \gamma_U(x) &= \varphi_U(x, a\psi_U(x)) , && x \in U \\ &= \mathfrak{o}(x) , && x \notin U . \end{aligned}$$

If $\mathscr{U}$ is finite, let $\gamma = \sum_{\mathscr{U}} \gamma_U$. Then for $U_0 \in \mathscr{U}$

$$\begin{aligned} t_{U_0}(\gamma(x)) &= \sum t_{U_0}(\gamma_U(x)) \\ &= \sum g_{U_0U}(x)[t_U(\gamma_U(x))] \\ &= \sum g_{U_0U}(x)(a)\psi_U(x) . \end{aligned}$$

The last expression is not necessarily equal to a.

We note also that by the very definition of the structure of D the mapping $\theta_{U_x}: \gamma \to t_U(\gamma(x))$ for any $x \in U \in \mathscr{U}$ is a $\boldsymbol{C}$-epimorphism of D onto A.

The algebra D is clearly an analogy and an extension of the concept of $C(X, A)$. When $\mathscr{E}$ is the trivial bundle, then in fact $D = C(X, A)$. We note in passing that for a suitable tensor product norm, namely λ, the "least cross-norm" [2], we may identify $C(X, A)$ and $C(X) \otimes_\lambda A$. In another place, the author proposes to explore this suggestion since it appears to lead to an abstract and useful formulation of the algebras studied here.

2. **Uniform Banach algebras.** The Gelfand-Mazur theorem may be rephrased as follows:

If A is a commutative Banach algebra and if M is a regular maximal ideal of A then A/M is $\boldsymbol{C}$-isomorphic to $\boldsymbol{C}$.

We are led to the following definition:

Let Q be a simple Banach algebra with identity e_Q and let A be a Banach algebra that is a Q-bimodule, i.e., for $q \in Q, a \in A$, qa and aq (possibly different) are defined, belong to A and

$$\begin{aligned} &\| qa \| \leqq \| q \| \| a \|, \| aq \| \leqq \| a \| \| q \|; e_Q a \\ &\quad = a e_Q = a, q(ab) = (qa)b, (ab)q = a(bq), (qa)q' \\ &\quad = q(aq'); (q, a) \to qa, (q, a) \to aq \end{aligned}$$

are bilinear. If, for every regular maximal ideal M of A, A/M is $\boldsymbol{C}$-isomorphic to Q, we say A is Q-uniform.

Whereas the set $\mathscr{M}_A$ of regular maximal ideals is of interest if A is commutative, the set $\mathrm{Epi}_{\boldsymbol{C}}(A, Q)$ of $\boldsymbol{C}$-epimorphisms η_A of A onto Q, where $\| \eta_A(a) \|_Q \leqq \| a \|_A$, is of interest when A is a not necessarily commutative Q-uniform Banach algebra. *Until further notice we assume A is a Q-uniform Banach algebra and that it has an identity e, as noted earlier.* Examples of such (noncommutative) A abound, e.g., $C(X, Q)$ where X is compact Hausdorff and Q is a simple Banach algebra, e.g., the set $\mathrm{End}_{\boldsymbol{C}}(\boldsymbol{C}^n)$ of endomorphisms of $\boldsymbol{C}^n$, $n > 1$. In a separate paper [1] the author will treat general Q-uniform algebras in detail.

We note here the surjection $k\colon \mathrm{Epi}_C(A, Q) \ni \eta \to \ker(\eta) \in \mathscr{M}_A$. If $\mathrm{Epi}_C(A, Q)$ is given the weak topology (a typical neighborhood is

$$N(\eta_0) = \{\eta\colon \|\eta(a_i) - \eta_0(a_i)\|_Q < \varepsilon, 1, 2, \cdots, n, a_i \in A\})$$

and if $\mathscr{M}_A$ is then given the strongest topology such that k is continuous, then $\mathrm{Epi}_C(A, Q)$ is a Hausdorff space and the map k is an open surjection.

We make one more observation in the form of

LEMMA 2.1. *Let $\tilde{A}$ be a Banach algebra that is a bimodule over a Banach algebra B. Then every regular ideal I of $\tilde{A}$ is also a B-ideal.*

Proof. If $u/I =$ identity of $\tilde{A}/I$, let $b \in B, x \in I$. Then $u(bx) - bx \in I$. However $(ub)x \in I$ whence $bx \in I$ and similarly $xb \in I$.

3. $\mathrm{Epi}_C(D, Q)$. When A is Q-uniform we assume $\mathscr{A}$ is the group of isometric Q-automorphisms of A. We prove first that D is Q-uniform and then we shall show that $\mathrm{Epi}_C(D, Q)$ is a new fibre bundle over X with fibre $\mathrm{Epi}_C(A, Q)$ and where the various maps and the group of the bundle are quite naturally related to the corresponding entities for $\mathscr{E}$. Note that the map $\theta_{U_x}\colon \gamma \to t_U(\gamma(x))$ for $x \in U \in \mathscr{U}$ is now a Q-epimorphism of D onto A.

LEMMA 3.1. *The algebra D is Q-uniform.*

Proof. We define actions of Q on D by:

$$q\cdot\gamma(x) = \varphi_U(x, qt_U(\gamma(x)))\ , \qquad x \in U$$
$$\gamma\cdot q(x) = \varphi_U(x, t_U(\gamma(x))q)\ , \qquad x \in U\ .$$

Because the g_{UV} are Q-automorphisms the above is a valid definition and makes D a Q-bimodule.

Next let M be a maximal ideal in D. We shall show that for some $x_0 \in X$, every $U \in \mathscr{U}$ such that $x_0 \in U$ and every $\gamma \in M$, $t_U(\gamma(x_0)) \neq e$. Indeed, otherwise, for each x there is some $U_x \ni x$ and a $\gamma_x \in M$ such that $t_{U_x}(\gamma_x(x)) = e$. In consequence $[t_{U_x}(\gamma_x(y))]^{-1}$ exists for all y in a neighborhood $N_x \subset U_x$. We may assume there are neighborhoods V_x, W_x satisfying $V_x \subset \bar{V}_x \subset W_x \subset \bar{W}_x \subset N_x$. Let x_i, $i = 1, 2, \cdots, n$ be such that $\bigcup_{i=1}^n V_{x_i} = X$. Let $f_i \in C(X)$ be such that $f_i = 1$ on $\bar{V}_{x_i}$, $f_i = 0$ off W_{x_i}, $0 \leqq f_i(x) \leqq 1$, all x. Then let $g_i(x) = f_i(x)\cdot[t_{U_{x_i}}(\gamma_{x_i}(x))]^{-1}$ for $x \in W_{x_i}$, $g_i(x) = 0$ otherwise. The support of g_i is contained in U_{x_i} and since inversion is continuous we find $g_i \in C(X, A)$. Thus there is a section γ_i such that $t_{U_{x_i}}(\gamma_i(x)) = g_i(x)$. Then $t_{U_{x_i}}(\gamma_i \cdot \gamma_{x_i}(x)) = e$ for $x \in V_{x_i}$. Note that $\tilde{\gamma}_i \equiv \gamma_i \cdot \gamma_{x_i} \in M$. Let $\{\psi_i\}$ be a partition of unity subordinate to the covering $\{V_{x_i}\}$. Then $\psi_i \cdot \tilde{\gamma}_i \in M$ (Lemma 2.1) whence $\tilde{\gamma} \equiv \sum_{i=1}^{n} \psi_i \tilde{\gamma}_i \in M$. However, for any x there is some $V_{x_i} \ni x$ and then

$$t_{U_{x_i}}(\tilde{\gamma}(x)) = \sum_{j=1}^{n} \psi_j(x) t_{U_{x_i}}(\tilde{\gamma}_j(x)) = \sum_{j=1}^{n} \psi_j(x) e = e \ .$$

(To prove this, we begin with $x \in V_{x_i} \subset U_{x_i}$,

$$t_{U_{x_i}}(\tilde{\gamma}(x)) = \sum_{j=1}^{n} \psi_j(x) t_{U_{x_i}}(\tilde{\gamma}_j(x)) \ .$$

If $x \in U_{x_j}$ then

$$t_{U_{x_i}}(\tilde{\gamma}_j(x)) = g_{U_{x_i} U_{x_j}}(x) t_{U_{x_j}}(\tilde{\gamma}_j(x)) = e \ .$$

If $x \notin U_{x_j}$, then $x \notin V_{x_j}$, $\psi_j(x) = 0$ whence

$$\psi_j(x) t_{U_{x_i}}(\tilde{\gamma}_j(x)) = \psi_j(x) e \ .$$

In all cases then

$$\psi_j(x) t_{U_{x_i}}(\tilde{\gamma}_j(x)) = \psi_j(x) e) \ .$$

Thus $\tilde{\gamma}(x) = e(x)$ and M cannot be a proper ideal.

Note that the method of proof applies to the simplest case in which the bundle is trivial and $A = \boldsymbol{C}$ (or even $A = \boldsymbol{R}$, in which case A is not a complex but a "real" Banach algebra). The conclusion is that the identification of points and maximal ideals in function algebras is not completely dependent on their being complex or on their having an involution.

We thus choose some x_0 such that $t_U(\gamma(x_0)) \neq e$ for all $\gamma \in M$ and all $U \in \mathscr{U}$ such that $x_0 \in U$. Consequently $\theta_{U_{x_0}}(M) \neq A$, is an ideal and we show it is maximal. Otherwise there is a maximal ideal $\tilde{M} \supsetneqq \theta_{U_{x_0}}(M)$ and since $\theta_{U_{x_0}}$ is surjective, $\theta_{U_{x_0}}^{-1}(\tilde{M})$ is an ideal in D. However $\theta_{U_{x_0}}^{-1}(\tilde{M}) \supset M$ whence, $\theta_{U_{x_0}}^{-1}(\tilde{M}) = M$ and so $\tilde{M} = \theta_{U_{x_0}}(M)$, a contradiction. Set $\theta_{U_{x_0}}(M) = M_{A0}$ and let $\tau\colon A \to Q$ have kernel M_{A0}. Define $\eta_D\colon D \to Q$ by $\eta_D(\gamma) = \tau(t_U(\gamma(x_0))) \equiv \tau\theta_{U_{x_0}}(\gamma)$. Then $\eta_D \in \mathrm{Epi}_{\boldsymbol{C}}(D, Q)$ and thus $\ker(\eta_D)$ is a maximal ideal. Clearly, $M \subset \ker(\eta_D)$ and so $M = \ker(\eta_D)$. We conclude D/M is $\boldsymbol{C}$-isomorphic to Q and thus that D is Q-uniform.

In the argument just given we have used the following

LEMMA 3.2. *Let* $x \in U \in \mathscr{U}$, $\eta_A \in \mathrm{Epi}_{\boldsymbol{C}}(A, Q)$. Then

$$\eta_D \equiv \eta_A \theta_{U_x} \in \mathrm{Epi}_{\boldsymbol{C}}(D, Q) \ .$$

Proof. Since

$$\theta_{U_x} \in \mathrm{Epi}_Q(D, A) \subset \mathrm{Epi}_{\boldsymbol{C}}(D, A)$$

and since $\eta_A \in \mathrm{Epi}_{\boldsymbol{C}}(A, Q)$, the result is clear.

Our principal object now is to prove a form of converse to Lemma 3.2, namely

LEMMA 3.3. *If $\eta_D \in \mathrm{Epi}_{\boldsymbol{C}}(D, Q)$ there is a unique $x_0 \in X$ such that if $x_0 \in U \in \mathscr{U}$ there is an η_A satisfying: $\eta_D = \eta_A \theta_{U_{x_0}}$ (η_A may depend on U).*

Proof. Let $\eta_D \in \mathrm{Epi}_{\boldsymbol{C}}(D, Q)$ be given and let $\ker(\eta_D) = M_D$. We note first that there is an $x_0 \in X$ such that if $x_0 \in U \in \mathscr{U}$ then $M_A \equiv \theta_{U_{x_0}}(M_D)$ is a maximal ideal in A. The argument for this fact was given in the proof of Lemma 3.1. If $\tilde{\eta}_A \in \mathrm{Epi}_{\boldsymbol{C}}(A, Q)$ is such that $\ker(\tilde{\eta}_A) = M_A$ then $\ker(\tilde{\eta}_A \theta_{U_{x_0}}) = \ker(\eta_D)$ and thus $\eta_D = \alpha \cdot \tilde{\eta}_A \theta_{U_{x_0}}$ where α is a $\boldsymbol{C}$-automorphism of Q. Setting $\eta_A = \alpha \tilde{\eta}_A$ we conclude

$$\eta_D = \eta_A \theta_{U_{x_0}} .$$

We show now that the x_0 is unique. Indeed, let x_1 be such that if $x_1 \in V \in \mathscr{U}$ there is an η'_A satisfying $\eta'_A \theta_{V_{x_1}} = \eta_A \theta_{U_{x_0}}$. We show that $x_0 = x_1$ by showing that if $x_0 \in W_0$, $x_1 \in W_1$, where W_0, W_1 are open, $W_0 \subset U_0 \in \mathscr{U}$, $W_1 \subset U_1 \in \mathscr{U}$, then $W_0 \cap W_1 \neq \varnothing$. Indeed, we can find $f \in C(X)$, $0 \leqq f(x) \leqq 1$, $f(x_1) = 1$ and $f = 0$ off W_1. Then $\gamma \equiv f \cdot e \in C(X, A)$ is such that $\theta_{U_{1x_1}}(\gamma) = e$ whereas $\theta_{U_{0x_0}}(\gamma) = 0$ (Lemma 1.1) and thus $\eta'_A \theta_{U_{1x_1}} \neq \eta_A \theta_{U_{0x_0}}$. Since X is Hausdorff we see $x_1 = x_0$, i.e., x_0 is unique.

We are now in a position to define a projection P: $\mathrm{Epi}_{\boldsymbol{C}}(D, Q) \to X$, namely by letting $P(\eta_D) = x_0$ in the notation above. Furthermore, for $U \in \mathscr{U}$ define $\Phi_U(U \times \mathrm{Epi}_{\boldsymbol{C}}(A, Q)) \to P^{-1}(U)$ by the formula

$$\Phi_U(x, \eta_A) = \eta_A \theta_{U_x} .$$

Finally, for each $x \in U \cap V$, where $U, V \in \mathscr{U}$, define $G_{UV}: U \cap V \to \mathrm{Perm}(\mathrm{Epi}_{\boldsymbol{C}}(A, Q)) \equiv$ the set of permutations of the set $\mathrm{Epi}_{\boldsymbol{C}}(A, Q)$ according to the following formula

$$G_{UV}(x)(\eta_A)(a) = \eta_A(g_{UV}(x)(a))$$

("$G_{UV} = g_{UV}{}^*$").

We shall show below that if the weak topologies (§ 2) are used for $\mathrm{Epi}_{\boldsymbol{C}}(D, Q)$ and $\mathrm{Epi}_{\boldsymbol{C}}(A, Q)$ then:

(i) P is a continuous surjection.

(ii) Each Φ_U is a homeomorphism onto $P^{-1}(U)$.

(iii) $\Phi_U^{-1}\Phi_V(x, \eta_A) = (x, G_{VU}(x)(\eta_A))$ if $x \in U \cap V$.

(iv) Each G_{UV} is a continuous map of $U \cap V$ into the set Auteo $(\mathrm{Epi}_C(A, Q))$ in the sense that each $G_{UV}(x) \in$ Auteo $(\mathrm{Epi}_C(A, Q))$ and $G_{UV}\colon U \cap V \rightarrow$ Auteo $(\mathrm{Epi}_C(A, Q))$ is continuous with respect to a suitable topology for Auteo $(\mathrm{Epi}_C(A, Q))$.

These statements imply

THEOREM 3.1. $\mathrm{Epi}_C(D, Q)$ *is a fibre bundle over* X *with fibre* $\mathrm{Epi}_C(A, Q)$, *projection* P, *maps* Φ_U *and group* Auteo $(\mathrm{Epi}_C(A, Q))$.

Ad (i). Let

$$\eta_{D\lambda} \rightarrow \eta_{D0},\ P(\eta_{D\lambda}) = x_\lambda,\ P(\eta_{D0}) = x_0 .$$

If $x_\lambda \nrightarrow x_0$, let $x_0 \in U \in \mathscr{U}$ and find open V, $W \ni x_0$ such that $\bar{W} \subset V \subset U$ and such that for a subnet $x_{\lambda'} \notin V$. Choose $f \in C(X)$ so that

$$0 \leqq f(x) \leqq 1,$$

$f = 1$ on $\bar{W}$, $f = 0$ off V and let $\gamma \in D$ be such that $t_U(\gamma(x)) = f(x)e$ (Lemma 1.1). (Alternatively, let $\gamma = f \cdot e$.) Then

$$\eta_{D\lambda'}(\gamma) \rightarrow \eta_{D0}(\gamma)$$

whereas

$$\eta_{D\lambda'}(\gamma) = \eta_{A\lambda'}\theta_{Ux_\lambda}(\gamma) = 0$$

and

$$\eta_{D0}(\gamma) = \eta_{A0}\theta_{Ux_0}(\gamma) = e_Q ,$$

a contradiction. Thus P is continuous. It is clearly surjective.

Ad (ii). We show first that Φ_U is continuous. Let

$$(x_\lambda, \eta_{A\lambda}) \rightarrow (x_0\ \eta_{A0}) .$$

Assume $\eta_{A\lambda}\theta_{Ux_\lambda} \equiv \eta_{D\lambda} \nrightarrow \eta_{D0} = (x_0, \eta_{A0})$.
Then there is a neighborhood

$$N(\eta_{D0}) \equiv \{\eta_D\colon \| \eta_D(\gamma_i) - \eta_{D0}(\gamma_i) \|_Q < \varepsilon, \qquad i = 1, 2, \cdots, n\}$$

and a subnet $\eta_{D\lambda'} \notin N(\eta_{D0})$. We know that eventually

$$\| t_U(\gamma_i(x_{\lambda'})) - t_U(\gamma_i(x_0)) \|_A < \varepsilon/2 , \qquad i = 1, 2, \cdots, n$$

and thus eventually

$$\| (\eta_A(t_U(\gamma_i(x_{\lambda'}))) - \eta_A(t_U(\gamma_i(x_0))) \|_Q < \varepsilon/2$$

(since $\|\eta_A(a)\|_Q \leqq \|a\|_A$), $i = 1, 2, \cdots, n$. Hence

$$\begin{aligned}\|\eta_{A\lambda'}(t_U(\gamma_i(x_{\lambda'}))) - \eta_{A0}(t_U(\gamma_i(x_0)))\|_Q \\ \leqq \|\eta_{A\lambda'}(t_U(\gamma_i(x_{\lambda'}))) - \eta_{A\lambda'}(t_U(\gamma_i(x_0)))\|_Q \\ + \|\eta_{A\lambda'}(t_U(\gamma_i(x_0))) - \eta_{A0}(t_U(\gamma_i(x_0)))\|_Q .\end{aligned}$$

Eventually the first term falls below $\varepsilon/2$ uniformly in $\eta_{A\lambda'}$, $i = 1, 2, \cdots, n$, and then eventually the second term falls below $\varepsilon/2$, whence eventually $\eta_{D\lambda'} \in N(\eta_{D0})$, a contradiction.

Next we show Φ_U is a bijection between $U \times \mathrm{Epi}_C(A, Q)$ and $P^{-1}(U)$. If $(x, \eta_A) \neq (x', \eta'_A)$ then either $x \neq x'$ or $x = x'$ and $\eta_A \neq \eta'_A$. If $x \neq x'$ we may find $f \in C(X, A)$, as in earlier arguments, so that the support of f is inside U, $f(x) = e$, $f(x') = 0$ and for some $\gamma \in D$, $t_U(\gamma(x)) = f(x)$. Then $\theta_{U_x}(\gamma) = e$, $\theta_{U_x}(\gamma) = 0$ and so

$$\Phi_U(x, \eta_A) \neq \Phi_U(x', \eta'_A) .$$

If $x = x'$, let $\eta_A(a_0) \neq \eta'_A(a_0)$, and let $\theta_{U_x}(\gamma_0) = a_0$. Again, we see $\Phi_U(x, \eta_A) \neq \Phi_U(x', \eta'_A)$. Furthermore we have seen that each η_D may be written in the form $\eta_A\theta_{U_x}$. Thus Φ_U is a bijection of $U \times \mathrm{Epi}_C(A, Q)$ on $P^{-1}(U)$.

Now we show Φ_U^{-1} is continuous. Thus let $\eta_{D\lambda} \to \eta_{D0}$. Since P is continuous, $P(\eta_{D\lambda}) \equiv x_\lambda \to x_0 \equiv P(\eta_{D0})$. If $\eta_{D\lambda} = \eta_{A\lambda}\theta_{U_{x_\lambda}}$ and $\eta_{D0} = \eta_{A0}\theta_{U_{x_0}}$ we wish to show $\eta_{A\lambda} \to \eta_{A0}$. Otherwise, there is a neighborhood

$$N(\eta_{A0}) \equiv \{\eta_A: \|\eta_A(a_i) - \eta_{A0}(a_i)\|_Q < \varepsilon, i = 1, 2, \cdots, n\}$$

and a subnet $\eta_{A\lambda'} \notin N(\eta_{A0})$. Furthermore we may assume that for some open W and V there obtains: $x_0 \in W \subset \bar{W} \subset V \subset U$ and that $x_{\lambda'} \in W$ for all λ'. Again we find $f_i \in C(X, A)$ such that $f_i(x) \equiv a_i$ on $\bar{W}$, $f_i(x) = 0$ off V. If $t_U(\gamma_i(x)) = f_i(x)$, let

$$N(\eta_{D0}) = \{\eta_D; \|\eta_D(\gamma_i) - \eta_{D0}(\gamma_i)\|_Q < \varepsilon, i = 1, 2, \cdots, n\} .$$

Thus $\|\eta_{D\lambda'}(\gamma_i) - \eta_{D0}(\gamma_i)\|_Q = \|\eta_{A\lambda'}(a_i) - \eta_{A0}(a_i)\|_Q \geqq \varepsilon$, and we arrive at a contradiction of: $\eta_{D\lambda'} \to \eta_{D0}$.

Ad (iii).

$$\Phi_V(x, \eta_A) = \eta_A\theta_{V_x}, \Phi_U^{-1}(\eta_A\theta_{V_x}) = (x, \eta'_A) ,$$

where

$$\eta_A\theta_{V_x} = \eta'_A\theta_{U_x} = \eta'_A g_{UV}(x)\theta_{V_x} = G_{UV}(x)(\eta'_A)\theta_{V_x} .$$

Thus $\eta_A = G_{UV}(x)(\eta'_A)$, $\eta'_A = G_{VU}(x)(\eta_A)$, and so

$$\Phi_U^{-1}\Phi_V(x, \eta_A) = (x, G_{VU}(x)(\eta_A)) .$$

Ad (iv). To discuss the mappings $G_{UV}(x)$ we first prove

LEMMA 3.4. *If T is a C-automorphism of a Q-uniform Banach algebra B, then* $T^*\colon \mathrm{Epi}_C(B, Q) \to \mathrm{Epi}_C(B, Q)$ *defined by*

$$T^*(\eta)(b) = \eta(T(b))$$

is an automorphism of $\mathrm{Epi}_C(B, Q)$.

Proof. We note that T^* is injective since $T^*(\eta_1) = T^*(\eta_2)$ if and only if $\eta_1(Tb) = \eta_2(Tb)$, and since T is an automorphism of B, we find $\eta_1 = \eta_2$. Furthermore, if η is given, direct calculation shows that $\eta = T^*((T^{-1})^*\eta)$, i.e., $(T^*)^{-1} = (T^{-1})^*$. Let $T(\eta_0) = \zeta_0$, and let

$$N(\zeta_0) = \{\zeta; \|\zeta(b_i) - \zeta_0(b_i)\|_B < \varepsilon, i = 1, 2, \cdots, n\} .$$

Then let

$$N(\eta_0) = \{\eta: \|\eta(Tb_i) - \eta_0(Tb_i)\|_B < \varepsilon, i = 1, 2, \cdots, n\} .$$

Clearly $T(N(\eta_0)) \subset N(\zeta_0)$. The same kind of argument shows $(T^*)^{-1} \equiv (T^{-1})^*$ is continuous.

From Lemma 3.4 we see that each $G_{UV}(x) \in \mathrm{Auteo}\,\mathrm{Epi}_C(A, Q)$.

By virtue of (iii) we see, in a manner analogous to that given in the introduction that $G_{UV}(x)(\eta_A)$ is continuous on $(U \cap V) \times \mathrm{Epi}_C(A, Q)$.

For our purposes, the relevant part of Auteo $(\mathrm{End}_C(A, Q))$ consists of all auteomorphisms of the form T^* where $T \in \mathscr{A}$. We denote this relevant part of Auteo $(\mathrm{End}_C(A, Q))$ by $\mathscr{A}^*$. Since

$$\mathscr{A} \ni T \to T^* \in \mathscr{A}^*$$

is bijective, we see that $\mathscr{A}^*$ may be regarded as an anti-isomorphic copy of $\mathscr{A}$. (If $T_1 \to T_1^*$, $T_2 \to T_2^*$ then $T_1T_2 \to T_2^*T_1^*$.) We topologize $\mathscr{A}^*$ by giving it the topology of $\mathscr{A}^*$ i.e., a set $S^* \subset \mathscr{A}^*$ is open if and only if the preimage S is open in $\mathscr{A}$. In this way $\mathscr{A}^*$ becomes a topological group.

Since the maps $g_{UV}; U \cap V \to \mathscr{A}$ are continuous we see that the maps $G_{UV}\colon U \cap V \to \mathscr{A}^*$ are also continuous.

4. $\mathscr{M}_D$.

The spaces $\mathscr{M}_D$ and $\mathscr{M}_A$, in the topologies they derive from $\mathrm{Epi}_C(D, Q)$ and $\mathrm{Epi}_C(A, Q)$ respectively, are related via a fibre bundle.

THEOREM 4.1. *$\mathscr{M}_D$ is a fibre bundle over X with fibre $\mathscr{M}_A$. The projection $\pi\colon \mathscr{M}_D \to X$ is defined by $\pi(M_D) = P(\eta_D)$ where* $\ker(\eta_D) = M_D$. *This definition is independent of the choice of η_D and π is continuous. The group is the group of auteomorphisms of $\mathscr{M}_A$. Furthermore, if $U \in \mathscr{U}$ then $\pi^{-1}(U)$ is homeomorphic to $U \times \mathscr{M}_A$. The map $\Psi_U\colon U \times \mathscr{M}_A \to \pi^{-1}(U)$ implements the homeomorphism according* *to the formula*:

$$\Psi_U(x, M_A) = \{\gamma: t_U(\gamma(x)) \in M_A\} \equiv M_D \in \mathscr{M}_D .$$

The maps $\mathscr{G}_{UV}: U \cap V \rightarrow$ Auteo ($\mathscr{M}_A$) *are defined by*

$$\mathscr{G}_{UV}(x)(M_A) = \{g_{UV}(x)(a): a \in M_A\} .$$

The function $\mathscr{G}_{UV}(x)(M_A)$ *is continuous in the pair* (x, M_A) *by virtue of the formula*:

$$\Psi_U^{-1}\Psi_V(x, M_A) = (x, \mathscr{G}_{VU}(x)(M_A)) .$$

The map $\mathscr{G}_{UV}: U \cap V \rightarrow$ Auteo ($\mathscr{M}_A$) *is continuous in a suitable topology for* Auteo ($\mathscr{M}_A$).

We omit the proof of Theorem 4.1 since the arguments and constructions of the proof closely parallel those given in § 3.

5. **Complements.** Some of the foregoing may be carried out in a more general context where no assumptions about the Banach algebra A are made except that it has an identity. The constructions of P and π, of Φ_U and Ψ_U, of $G_{UV}(x)$ and $\mathscr{G}_{UV}(x)$ can be carried out without recourse to hypotheses about Q-uniformity of A. However, some of the continuity proofs cannot be repeated in the "natural" topological context.

First $\mathrm{Epi}_C(A, Q)$ and $\mathrm{Epi}_C(D, Q)$ must be replaced by $\mathrm{Epi}_C(A)$ and $\mathrm{Epi}_C(D)$ (the respective sets of all epimorphisms of A and D onto simple quotients). Next $\mathscr{M}_A$ and $\mathscr{M}_D$ should be given their hull-kernel (hk) topologies and then $\mathrm{Epi}_C(A)$ and $\mathrm{Epi}_C(D)$ are given the weakest topologies that make the mappings $\eta_A \rightarrow \ker(\eta_A)$ and $\eta_D \rightarrow \ker(\eta_D)$ continuous. It is of interest to note that P and π remain continuous but that Ψ_U need not be continuous. We show π is continuous and since $\mathrm{Epi}_C(D) \rightarrow \mathscr{M}_D$ is continuous, the continuity of P follows.

Indeed, if $M_{D0} \supset \bigcap_{M_D \in S} M_D$ and if $\pi(M_{D0}) = x_0 \notin \overline{\pi(S)}$ let $x_0 \in W$ where $W \cap \pi(S) = \emptyset$. As in earlier proofs, let $x_0 \in W_1 \subset \bar{W}_1 \subset W$. We may assume also that $W \subset U \in \mathscr{U}$. Then if $f \in C(X, A)$ has support in U and if $f(x) \equiv e$ on $\bar{W}_1$, $f(x) = 0$ off W_1, let $\gamma \in D$ be such that $t_U(\gamma(x)) = f(x)$. Then $t_U(\gamma(x_0)) = e$ and so $\gamma \notin M_{D0}$ whereas $t_U(\gamma(x)) = 0$, $x \notin W$ and so $\gamma \in \bigcap_{M_D \in S} M_D$. We arrive at a contradiction.

On the other hand the following example shows that Ψ_U need not be continuous.

EXAMPLE. Let A be the commutative Banach algebra of functions $f(z)$ analytic for $|z| < 1$ and continuous for $|z| \leqq 1$, $z \in C$; let $X = [0, 1]$

and let $\mathscr{E} = A \times X$. Then $D = \Gamma(\mathscr{E})$ is the set of all continuous maps from X to A. The bundle $\mathscr{E}$ is trivial and we wish to consider the singularly covering $\{X\}$ and to show that Ψ_X is not continuous. We shall exhibit a section $\gamma \in D$, a pair (X_0, M_0) and a set of pairs (x_λ, M_λ) such that

$$x_\lambda \longrightarrow x_0\ , \qquad M_0 \supset \bigcap M_\lambda$$
$$\gamma(x_\lambda) \in M_\lambda\ , \qquad \gamma(x_0) \notin M\ .$$

Thus, if $M_{D\lambda} = \Psi_U(x_\lambda, M_\lambda)$, then $\gamma \in \bigcap M_{D\lambda}$ and $\gamma \notin M_D \equiv \Psi_U(x_0, M_0)$ whence $\Psi_U\{(x_\lambda, M_\lambda)\} \not\subset \Psi_U\{(x_0, M_0)\}$. Specifically, let

$$x_\lambda = \lambda,\ M_\lambda \sim \frac{\lambda}{2},\ 0 \leqq \lambda \leqq 1\ .$$

Let

$$\gamma(x) \equiv f_x(z) = z^2 - \frac{x^2}{2(x-2)}\, z + \frac{x^2}{2(x-2)}\ .$$

Then $f_x(1) = 1$,

$$f_x\left(\frac{x}{2}\right) = \frac{x^2}{4} + \frac{x^2}{2(x-2)}\left(1 - \frac{x}{2}\right) = 0\ .$$

Furthermore

$$\|\gamma(x_1) - \gamma(x_2)\| \leqq \left|\frac{x_1^2}{(x_1-2)} - \frac{x_2^2}{(x_2-2)}\right|$$

whence $\gamma \in D$. As $\lambda \rightarrow 0$, $\gamma(x_\lambda) \rightarrow \gamma(x_0) = \gamma(0) = z^2$. On the other hand, $\bigcap_\lambda M_\lambda = \{0\} \subset M = \{f\colon f(1) = 0\}$. Thus

$$\gamma(x_\lambda) = \gamma(\lambda) = f_\lambda\left(\frac{\lambda}{2}\right) = 0\ ,$$

whence $\gamma(x_2) \in M_\lambda$. But $\gamma(x_0) = z^2 \notin M$. We conclude that Ψ_X is not continuous in this example.

The referee has suggested an alternative approach for the more general situation: For each simple Q and *any* A, the argument of § 3 shows that $\mathrm{Epi}_C(D, Q)$ has a fibre bundle structure with fibre $\mathrm{Epi}_C(A, Q)$, although $\mathrm{Epi}_C(D, Q)$ and $\mathrm{Epi}_C(A, Q)$ may be empty. There now arises the problem of patching "all" $\mathrm{Epi}_C(D, Q)$ and correspondingly "all" $\mathrm{Epi}_C(A, Q)$ together and thereafter relating the resulting structures.

BIBLIOGRAPHY

1. B. R. Gelbaum, *Q-uniform Banach algebras* (to appear)
2. R. Schatten, *A theory of cross-spaces*, Princeton, 1950.
3. N. Steenrod, *The topology of fibre bundles*, Princeton, (3d printing) 1960.

Recived January 19, 1968. This research was supported in part by Grant NSF GP-5436 for which the author expresses his appreciation.

UNIVERSITY OF CALIFORNIA, IRVINE