PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 2, 1969

CONTINUOUS DEPENDENCE FOR TWO-POINT BOUNDARY VALUE PROBLEMS

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Suppose the boundary value problem

$$y'' = f(t, y, y') \tag{1.1}$$

$$y(a) = \alpha , \qquad y(b) = \beta , \tag{1.2}$$

where $f(t, y, y')$ **is defined on** $D \equiv [a, b] \times R^2$, **has a unique solution** $y(t; \alpha, \beta)$ **which belongs to** $C^2[a, b]$, **for each** (α, β) **in some set** $S \subset R^2$. **This paper gives sufficient conditions for** $y(t; \alpha, \beta)$, $y'(t; \alpha, \beta)$, **and** $y''(t; \alpha, \beta)$ **to be continuous on** $[a, b] \times S$.

In § 2 it is shown that if $f(t, y, y')$ is continuous on D and $y(t; \alpha, \beta)$ is continuous on $[a, b] \times S$, then $y'(t; \alpha, \beta)$ and $y''(t; \alpha, \beta)$ are continuous on $[a, b] \times S$. In § 3 it is shown that $y(t; \alpha, \beta)$ is continuous under the assumption that solutions to boundary value problems for (1.1) exist and are unique in a certain strong sense. In § 4 the continuity of $y(t; \alpha, \beta)$ is established under the assumption that solutions to (1.1) satisfy a maximum principle.

Bebernes [1], Fountain and Jackson [3], and others have given sufficient conditions for the problem (1.1), (1.2) to have a unique solution, but the question of continuous dependence has not been given extensive attention.

2. Derivatives of convergent sequences. In this section we establish that if $f(t, y, y')$ is continuous on D, the continuity of $y(t; \alpha, \beta)$ is enough to guarantee the continuity of $y'(t; \alpha, \beta)$ and $y''(t; \alpha, \beta)$. The proof makes use of a lemma concerning derivatives of uniformly convergent sequences of solutions. First we prove a variation of a well-known result for initial value problems; e.g., see [4], p. 11.

LEMMA 2.1. *Let* $f(t, y, y')$ *be continuous on* D. *Given* $T > 0$ *there exists* $\alpha(T) > 0$ *such that if* $y(t)$ *is a solution to* (1.1) *on* $[a, b]$ *and* $|y(t_0)| + |y'(t_0)| \leqq T$ *for some* $t_0 \in [a, b]$, *then* $|y(t)| + |y'(t)| \leqq 2T$ *for*

$$t \in \Delta(t_0, \alpha(T)) \equiv [a, b] \cap [t_0 - \alpha(T), t_0 + \alpha(T)] .$$

Proof. Let

$$K \equiv \{(t, y, y') \in D : |y| + |y'| \leqq 2T\}$$

and let

$$C \equiv \max_K |f(t, y, y')| + 2T .$$

Choose $\alpha(T) < T/C$.

Suppose for contradiction that $y(t)$ is a solution to (1.1) on $[a, b]$ with $|y(t_0)| + |y'(t_0)| \leqq T$ and there exists $t_1 \in \Delta(t_0, \alpha(T))$ such that

$$|y(t_1)| + |y'(t_1)| > 2T .$$

For definiteness assume $t_1 > t_0$. There exists $t_2 \in \Delta(t_0, \alpha(T))$ such that $t_0 < t_2 < t_1$,

$$|y(t_2)| + |y'(t_2)| = 2T \tag{2.1}$$

and

$$|y(t)| + |y'(t)| \leqq 2T \tag{2.2}$$

for $t \in [t_0, t_2]$.

By the Mean Value Theorem

$$\begin{aligned} &|y(t_2) - y(t_0)| + |y'(t_2) - y'(t_0)| \\ &\quad = (|y'(\zeta_1)| + |y''(\zeta_2)|)(t_2 - t_0) \end{aligned}$$

for some ζ_1 and ζ_2 in $[t_0, t_2]$. By (2.2), $|y'(\zeta_1)| \leqq 2T$. Moreover, (2.2) yields that $(\zeta, y(\zeta_2), y'(\zeta_2)) \in K$ and we have

$$|y''(\zeta_2)| = |f(\zeta, y(\zeta_2), y'(\zeta_2))| \leqq \max_K |f(t, y, y')| .$$

Thus

$$\begin{aligned} |y(t_2) - y(t_0)| + |y'(t_2) - y'(t_0)| &\leqq (2T + \max_K |f(t, y, y')|)(t_2 - t_0) \\ &\leqq C\alpha(T) \\ &< T; \end{aligned}$$

hence,

$$|y(t_2)| + |y'(t_2)| < T + |y(t_0)| + |y'(t_0)| \leqq 2T .$$

This contradicts (2.1).

LEMMA 2.2. *Let $f(t, y, y')$ be continuous on D and let $\{y_n(t)\}$ be a sequence of solutions to (1.1) on $[a, b]$ such that $y_n(t) \to y_0(t)$ uniformly on $[a, b]$ and $y_0(t)$ has a continuous derivative on $[a, b]$. If there exists a sequence $\{t_n\}$ in $[a, b]$ such that $t_n \to t_0$ and $y'_n(t_n) \to z_0$, then there exists a subsequence $\{y_{k(n)}(t)\}$ such that $y'_{k(n)}(t) \to y'_0(t)$ uniformly on $[a, b]$.*

Proof. Let $T_0 \equiv |y_0(t_0)| + |z_0| + 1$. There exists $N > 0$ such that

$$|y_n(t_n)| + |y'_n(t_n)| \leqq T_0$$

for $n \geqq N$. By Lemma 2.1, there exists $\alpha(T_0) > 0$ such that for $n \geqq N$

$$|y_n(t)| + |y'_n(t)| \leqq 2T_0$$

on $\Delta(t_n, \alpha(T_0))$. Since $t_n \to t_0$, there exists $N_0 \geqq N$ such that for $n \geqq N_0$

$$|y_n(t)| + |y'_n(t)| \leqq 2T_0$$

on $\Delta(t_0, \alpha(T_0)/2)$.

Let

$$K_0 \equiv \{(t, y, y') : t \in \Delta(t_0, \alpha(T_0)/2), |y| + |y'| \leqq 2T_0\} .$$

For $n \geqq N_0$ and $t \in \Delta(t_0, \alpha(T_0)/2)$ we have

$$|y''_n(t)| = |f(t, y_n(t), y'_n(t))| \leqq \max_{K_0} |f(t, y, y')| .$$

It follows from the Mean Value Theorem that $\{y'_n(t)\}$ is equicontinuous on $\Delta(t_0, \alpha(T_0)/2)$. Since $\{y'_n(t)\}$ is bounded by $2T_0$ on $\Delta(t_0, \alpha(T_0)/2)$ for $n \geqq N_0$, Ascoli's Theorem implies that $\{y'_n(t)\}$ has a subsequence $\{y'_{k_1(n)}\}$ which converges uniformly on $\Delta(t_0, \alpha(T_0)/2)$ to some $z_0(t)$. But since $y_{k_1(n)}(t) \to y_0(t)$ on $\Delta(t_0, \alpha(T_0)2)$ we must have $z_0(t) \equiv y'_0(t)$ on $\Delta(t_0, \alpha(T_0)/2)$.

If $\Delta(t_0, \alpha(T_0)/2) = [a, b]$ we are through. If not, at least one end point of $\Delta(t_0, \alpha(T_0)/2)$ is in (a, b). Denote such an end point by t_1.

Let $T \equiv \max_{[a,b]} (|y_0(t)| + |y'_0(t)|) + 1$.

There exists N_1 such that for $n \geqq N_1$

$$\begin{aligned} |y_{k_1(n)}(t_1)| + |y'_{k_1(n)}(t_1)| &\leqq |y_0(t_1)| + |y'_0(t_1)| + 1 \\ &\leqq T. \end{aligned}$$

By Lemma 2.1 there exists $\alpha(T)$ such that

$$|y_{k_1(n)}(t)| + |y'_{k_1(n)}(t)| \leqq 2T$$

for $t \in \Delta(t_1, \alpha(T))$ and $n \geqq N_1$. By the same arguments used for the interval $\Delta(t_0, \alpha(T_0)/2)$, $\{y_{k_1(n)}(t)\}$ has a subsequence $\{y_{k_2(n)}(t)\}$ such that $y'_{k_2(n)}(t) \to y'_0(t)$ uniformly on $\Delta(t_1, \alpha(T))$. Thus $y'_{k_2(n)}(t) \to y'_0(t)$ uniformly on $\Delta(t_1, \alpha(T)) \cup \Delta(t_0, \alpha(T_0)/2)$.

If $\Delta(t_1, \alpha(T)) \cup \Delta(t_0, \alpha(T_0)/2) = [a, b]$ we are through. If not, there is an end point t_2 of $\Delta(t_1, \alpha(T)) \cup \Delta(t_0, \alpha(T_0)/2)$ in (a, b) and the above procedure may be repeated with T unchanged. Since $\alpha(T)$ is also unchanged, this process will terminate in a finite number of steps with a subsequence $\{y_{k_m(n)}(t)\}$ such that $y'_{k_m(n)}(t) \to y'_0(t)$ uniformly on

$$\Delta(t_0, \alpha(T_0)/2) \cup \left\{\bigcup_{i=1}^{m-1} \Delta(t_i, \alpha(T))\right\} = [a, b] .$$

LEMMA 2.3. *Let $f(t, y, y')$ be continuous on D. If $\{y_n(t)\}$ is a sequence of solutions to* (1.1) *on $[a, b]$ such that $y_n(t) \to y_0(t)$ uniformly on $[a, b]$ where $y_0(t)$ has a continuous derivative on $[a, b]$, then*

$y'_n(t) \to y'_0(t)$ uniformly on $[a, b]$.

Proof. By the Mean Value Theorem, for each n there exists $t_n \in [a, b]$ such that

$$|y_n(b) - y_n(a)| = |y'_n(t_n)|(b - a) .$$

Since there exists $B > 0$ such that $|y_n(t)| \leqq B$ on $[a, b]$ for all n, we have

$$|y'_n(t_n)| \leqq 2B/(b - a) .$$

Let $\{y_{k_0(n)}(t)\}$ denote an arbitrary subsequence of $\{y_n(t)\}$. Since $\{y'_{k_0(n)}(t_{k_0(n)})\}$ is bounded we may extract a further subsequence $\{y_{k_1(n)}(t)\}$ such that $y'_{k_1(n)}(t_{k_1(n)}) \to z_0$ and $t_{k_1(n)} \to t_0 \in [a, b]$. By Lemma 2.2, there exists a further subsequence $\{y_{k_2(n)}(t)\}$ such that $y'_{k_2(n)}(t) \to y'_0(t)$ uniformly on $[a, b]$.

Thus any subsequence of $\{y_n(t)\}$ has a further subsequence which has its derivatives converging uniformly to $y'_0(t)$ on $[a, b]$. It follows that $y'_n(t) \to y'_0(t)$ uniformly on $[a, b]$.

The conclusion of Lemma 2.3 does not hold if the hypothesis that $y_0(t)$ has a continuous derivative on $[a, b]$ is removed. The function $y_n(t) = \sqrt{b + 1/n - t}$ is a solution to $y'' = 2(y')^3$ on $[o, b]$ for each n. Moreover, $\{y_n(t)\}$ converges to $y_0(t) = \sqrt{b - t}$ uniformly on $[o, b]$. But $\{y'_n(t)\}$ does not converge to $y'_0(t)$ uniformly on $[o, b]$.

THEOREM 2.4. *Let $f(t, y, y')$ be continuous on D. Suppose* (1.1) *has a unique solution $y(t; \alpha, \beta)$ on $[a, b]$ satisfying* (1.2) *for $(\alpha, \beta) \in S \subset R^2$. If $y(t; \alpha, \beta)$ is continuous on $[a, b] \times S$, then $y'(t; \alpha, \beta)$ and $y''(t; \alpha, \beta)$ are continuous on $[a, b] \times S$.*

Proof. Since $y'(t; \alpha, \beta)$ and $y''(t; \alpha, \beta)$ are continuous for fixed (α, β),

$$|y'(t; \alpha, \beta) - y'(t_0; \alpha_0, \beta_0)| \leqq |y'(t; \alpha, \beta) - y'(t; \alpha_0, \beta_0)| + |y'(t; \alpha_0, \beta_0) - y'(t_0; \alpha_0, \beta_0)| ,$$

and

$$|y''(t; \alpha, \beta) - y''(t_0; \alpha_0, \beta_0)| \leqq |y''(t; \alpha, \beta) - y''(t; \alpha_0, \beta_0)| + |y''(t; \alpha_0, \beta_0) - y''(t_0; \alpha_0, \beta_0)| ,$$

it is sufficient to show that $y'(t; \alpha, \beta)$ and $y''(t; \alpha, \beta)$ are continuous functions of α and β uniformly with respect to t.

Let (α_n, β_n) be a sequence in S such that $(\alpha_n, \beta_n) \to (\alpha_0, \beta_0) \in S$. Since $y(t; \alpha, \beta)$ is continuous on $[a, b] \times S$, $y(t, \alpha_n, \beta_n) \to y(t; \alpha_0, \beta_0)$ uniformly on $[a, b]$; hence, Lemma 2.3 yields the uniform convergence of $\{y'(t; \alpha_n, \beta_n)\}$ to $y'(t; \alpha_0, \beta_0)$. Since

$$y''(t; \alpha_n, \beta_n) = f(t, y(t; \alpha_n, \beta_n), y'(t; \alpha_n, \beta_n))$$

and $f(t; y, y')$ is continuous, it follows that $y''(t; \alpha_n, \beta_n) \to y''(t; \alpha_0, \beta_0)$ uniformly on $[a, b]$.

3. **Strong existence and uniqueness.** In this section we show that $y(t; \alpha, \beta)$ is continuous on $[a, b] \times R^2$ if solutions to (1.1) exist and are unique in the sense described in the following definitions.

DEFINITION. Solutions to boundary value problems for (1.1) will said to be *unique in the strong sense* on $[a, b]$ if for any solutions $\phi(t)$ and $\psi(t)$ to (1.1) on $[c, d] \subset [a, b]$, $\phi(c) = \psi(c)$ and $\phi(d) = \psi(d)$ imply that $\phi(t) \equiv \psi(t)$ on $[c, d]$.

DEFINITION. Solutions to boundary value problems for (1.1) are be said to *exist in the strong sense* on $[a, b]$ if for any real numbers α and β and any $[c, d] \subset [a, b]$ there is a solution $y(t)$ to (1.1) on an interval $I \supset [c, d]$ such that $y(c) = \alpha$, $y(d) = \beta$ and either

(i) $I = [a, b]$, or

(ii) $y(t)$ is unbounded.

THEOREM 3.1. *Suppose solutions to boundary value problems for* (1.1) *exist and are unique in the strong sense on* $[a, b]$. *If* $y(t; \alpha, \beta)$ *denotes the unique solution to* (1.1) *on* $[a, b]$ *satisfying* (1.2), *then* $y(t; \alpha, \beta)$ *is continuous on* $[a, b] \times R^2$.

Proof. Let $\varepsilon > 0$ be given.

Let $t_0 \in (a, b)$. Let $y_1(t)$ denote a solution to (1.1) on an interval $I_1 \supset [t_0, b]$ such that

$$y_1(t_0) = y(t_0; \alpha_0, \beta_0) + \varepsilon,\ y_1(b) = y(b; \alpha_0, \beta_0) = \beta_0 \tag{3.1}$$

and either $y_1(t)$ is unbounded or $I_1 = [a, b]$. Let $y_2(t)$ denote a solution on $I_2 \supset [a, t_0]$ such that

$$y_2(a) = \begin{cases} y_1(a), \text{ if } I_1 = [a, b] \\ y(a; \alpha_0, \beta_0) + \varepsilon, \text{ if } I_1 \subset (a, b], \end{cases} \qquad y_2(t_0) = y(t_0; \alpha_0, \beta_0) + 2\varepsilon \tag{3.2}$$

and either $y_2(t)$ is unbounded or $I_2 = [a, b]$.

Let $y_3(t)$ and $y_4(t)$ denote similar solutions on $I_3 \supset [t_0, b]$ and $I_4 \supset [a, t_0]$ with

$$y_3(t_0) = y(t_0; \alpha_0, \beta_0) - \varepsilon, \qquad y_3(b) = y(b; \alpha_0, \beta_0) = \beta_0 \tag{3.3}$$

(3.4) $y_4(a) = \begin{cases} y_3(a), & \text{if } I_3 = [a, b] \\ y(a; \alpha_0, \beta_0) - \varepsilon, & \text{if } I_3 \subset (a, b], \end{cases} \qquad y_4(t_0) = y(t_0; \alpha_0, \beta_0) - 2\varepsilon .$

Assume for definiteness that $I_1 = [a, b]$, $I_2 \subset [a, b)$, $I_3 \subset (a, b]$, and $I_4 = [a, b]$. The other cases may be treated with arguments similar to those below.

Uniqueness in the strong sense and (3.1) imply that

$$y_1(a) = y_2(a) > \alpha_0 = y(a; \alpha_0, \beta_0) .$$

Since $y_2(t_0) > y_1(t_0)$ follows from (3.2), uniqueness in the strong sense also implies that $y_2(t) \geqq y_1(t)$ on I_2; hence, $y_2(t)$ must become unbounded positively to the right of t_0.

Uniqueness in the strong sense and (3.3) imply that $y_3(t) \leqq y(t; \alpha_0, \beta_0)$ on I_3; hence, $y_3(t)$ must become unbounded negatively to the left of t_0. Since $y_4(t_0) < y_3(t_0)$, there exists $a < t_1 < t_0$ such that $y_4(t_1) = y_3(t_1)$; hence, uniqueness in the strong sense implies that $y_4(t) < y_3(t)$ on $[t_0, b]$. In particular, $y_4(b) < y_3(b) = \beta_0 = y(b; \alpha_0, \beta_0)$.

Let $\delta_1(t_0) \equiv \min\, [y_2(a) - \alpha_0, \alpha_0 - y_4(a), \beta_0 - y_4(b)]$. If $|\alpha - \alpha_0| + |\beta - \beta_0| < \delta_1(t_0)$, then $y_4(a) < \alpha < y_2(a)$ and $\beta > y_4(b)$. By uniqueness in the strong sense we must have $y_4(t) \leqq y(t; \alpha, \beta)$ on $[a, b]$ and since $y_2(t)$ becomes unbounded positively to the right of t_0 we must also have $y(t; \alpha, \beta) \leqq y_2(t)$ on I_2.

There exists $\delta_2(t_0)$ such that for

$$|t - t_0| < \delta_2(t_0),\ t \in I_2,\ |y_2(t) - y_2(t_0)| < \varepsilon ,$$

and $|y_4(t) - y_4(t_0)| < \varepsilon$. Thus, for $|\alpha - \alpha_0| + |\beta - \beta_0| < \delta_1(t_0)$ and $|t - t_0| < \delta_2(t_0)$ we have

$$\begin{aligned} |y(t; \alpha, \beta) - y(t; \alpha_0, \beta_0)| &\leqq y_2(t) - y_4(t) \\ &\leqq |y_2(t) - y_2(t_0)| + y_2(t_0) - y_4(t_0) \\ &\quad + |y_4(t) - y_4(t_0)| \\ &\leqq \varepsilon + 4\varepsilon + \varepsilon = 6\varepsilon . \end{aligned}$$

Uniqueness in the strong sense implies that for

$$\begin{gathered} |\alpha - \alpha_0| + |\beta - \beta_0| < \delta_1(a) = \delta_1(b) \equiv \varepsilon , \\ y(t; \alpha_0 - \varepsilon, \beta_0 - \varepsilon) \leqq y(t; \alpha, \beta) \leqq y(t; \alpha_0 + \varepsilon, \beta_0 + \varepsilon) \end{gathered}$$

on $[a, b]$. There exists $\delta_2(a)$ such that for $|t - a| < \delta_2(a)$

$$|y(t; \alpha_0 + \varepsilon, \beta_0 + \varepsilon) - y(a; \alpha_0 + \varepsilon, \beta_0 + \varepsilon)| < \varepsilon$$

and

$$|y(t; \alpha_0 - \varepsilon, \beta_0 - \varepsilon) - y(a; \alpha_0 - \varepsilon, \beta_0 - \varepsilon)| < \varepsilon .$$

Thus for $|\alpha - \alpha_0| + |\beta - \beta_0| < \delta_1(a)$ and $|t - a| < \delta_2(a)$

$$|y(t; \alpha, \beta) - y(t; \alpha_0, \beta_0)| \leqq 6\varepsilon .$$

A $\delta_2(b)$ may be defined in a similar manner.

By the Heine-Borel Theorem there exist $t_1, t_2, \cdots, t_k$ such that the intervals defined by $|t - t_i| < \delta_2(t_i)$ cover $[a, b]$. Let $\delta \equiv \min \delta_1(t_i)$. Then for any $t \in [a, b] |\alpha - \alpha_0| + |\beta - \beta_0| < \delta$ implies that

$$|y(t; \alpha, \beta) - y(t; \alpha_0, \beta_0)| \leqq 6\varepsilon .$$

Since $y(t; \alpha, \beta)$ is continuous for fixed α and β, it follows that $y(t; \alpha, \beta)$ is continuous on $[a, b] \times R^2$.

It is of interest to note that in the proof of Theorem 3.1, the fact that the functions involved were solutions to (1.1) was used only to assert that the functions were continuous. The arguments may be applied to any family of continuous functions satisfying the uniqueness and existence requirements. Theorem 3.1 is a variation of a result of Beckenbach ([2], p. 365) concerning two-parameter families of continuous functions.

As an immediate consequence of Theorems 2.4 and 3.1 we have

COROLLARY 3.2. *Let $f(t, y, y')$ be continuous on D and suppose solutions to boundary value problems for* (1.1) *exist and are unique in the strong sense on $[a, b]$. If $y(t; \alpha, \beta)$ denotes the unique solution to* (1.1) *on $[a, b]$ satisfying* (1.2), *then $y(t; \alpha, \beta)$, $y'(t; \alpha, \beta)$ and $y''(t; \alpha, \beta)$ are continuous on $[a, b] \times R^2$.*

4. The maximum principle.

In this section we consider the function $y(t; \alpha, \beta)$ in the presence of a maximum principle.

DEFINITION. Solutions to (1.1) will be said to satisfy the *maximum principle on* $[c, d] \subset [a, b]$ if for any solutions $\phi(t)$ and $\psi(t)$ to (1.1) on $[c, d]$, $|\phi(t) - \psi(t)|$ assumes its maximum on $[c, d]$ at either c or d.

Note that if solutions to (1.1) satisfy the maximum principle on $[a, b]$, then solutions to (1.1) on $[a, b]$ satisfying (1.2) are unique.

THEOREM 4.1. *Suppose solutions to* (1.1) *satisfy the maximum principle on $[a, b]$. If* (1.1) *has a (unique) solution $y(t; \alpha, \beta)$ on $[a, b]$ satisfying* (1.2) *for $(\alpha, \beta) \in S \subset R^2$, then $y(t; \alpha, \beta)$ is continuous on $[a, b] \times S$.*

Proof. By the maximum principle on $[a, b]$, for (α, β) and (α_0, β_0) in S we have

$$|y(t; \alpha, \beta) - y(t; \alpha_0, \beta_0)| \leqq \max[|\alpha - \alpha_0|, |\beta - \beta_0|] .$$

Since $y(t; \alpha, \beta)$ is continuous on $[a, b]$ for fixed (α, β), it follows that $y(t; \alpha, \beta)$ is continuous on $[a, b] \times S$.

COROLLARY 4.2. *Let $f(t, y, y')$ be continuous on D. Suppose solutions to* (1.1) *satisfy the maximum principle on* $[a, b]$. *If* (1.1) *has a (unique) solution $y(t; \alpha, \beta)$ on $[a, b]$ satisfying* (1.2) *for $(\alpha, \beta) \in S \subset R^2$, then $y(t; \alpha, \beta)$, $y'(t; \alpha, \beta)$ and $y''(t; \alpha, \beta)$ are continuous on $[a, b] \times S$.*

Various sets of hypotheses on $f(t, y, y')$ imply that solutions to (1.1) satisfy the maximum principle. As an example we state

THEOREM 4.3. *If $f(t, y, y')$ is continuous on D, $f(t, y, y')$ is nondecreasing in y on D, and for any compact subset $C \subset D$ there is a positive constant $K(C)$ such that*

$$|f(t, y, y_1') - f(t, y, y_2')| \leqq K(C)\,|y_1' - y_2'|$$

for any (t, y, y_1') and (t, y, y_2') in C, then solutions to (1.1) *satisfy the maximum principle on any $[c, d] \subset [a, b]$.*

Proof. This is an immediate consequence of Theorem 2.2 in [1].

As a partial converse to Theorem 4.3, we have

THEOREM 4.4. *If $f(t, y, y')$ is continuous on D and solutions to* (1.1) *satisfy the maximum principle on every $[c, d] \subset [a, b]$, then $f(t, y, y')$ is nondecreasing in y on D.*

Proof. Suppose for contradiction there exist (t_0, y_1, y_0') and (t_0, y_2, y_0') in D such that $y_1 > y_2$ and $f(t_0, y_1, y_0') < f(t_0, y_2, y_0')$. By continuity we may assume $t_0 \in (a, b)$.

Since $f(t, y, y')$ is continuous on D, there exists an interval $[c_0, d_0] \subset [a, b]$ with $t_0 \in (c_0, d_0)$ such that (1.1) has solutions $y_1(t)$ and $y_2(t)$ on $[c_0, d_0]$ with $y_1(t_0) = y_1$, $y_1'(t_0) = y_0'$, $y_2(t_0) = y_2$, and $y_2'(t_0) = y_0'$. Since

$$y_1''(t_0) - y_2''(t_0) = f(t_0, y_1, y_0') - f(t_0, y_2, y_0') < 0 \;,$$

$y_1'(t_0) - y_2'(t_0) = 0$, and $y_1(t_0) - y_2(t_0) > 0$, there exists $[c, d] \subset [c_0, d_0]$ such that $t_0 \in (c, d)$ and $y_1(t_0) - y_2(t_0) > y_1(t) - y_2(t) \geqq 0$ for any $t \neq t_0$ in $[c, d]$. In particular,

$$y_1(t_0) - y_2(t_0) > \max\,[|\,y_1(c) - y_2(c)\,|, |\,y_1(d) - y_2(d)\,|]$$

which contradicts the maximum principle being satisfied on $[c, d]$.

5. Continuous dependence without uniqueness in the strong sense. Though the hypotheses of Theorem 4.1 only required that solutions satisfy the maximum principle on $[a, b]$, a more "natural" assumption is that solutions satisfy the maximum principle on every subinterval $[c, d]$. If this stronger assumption is made, solutions to (1.1) are unique in the strong sense as was assumed in the hypothesis of Theorem 3.1.

A simple example shows that uniqueness in the strong sense is not a necessary condition for continuous dependence. Consider the equation

$$y'' = -y . \tag{5.1}$$

The unique solution to (5.1) on $[a, b] \equiv [0, 3\pi/2]$ satisfying

$$y(0) = \alpha,\ y(3\pi/2) = \beta \tag{5.2}$$

is

$$y(t; \alpha, \beta) = \alpha \cos t - \beta \sin t$$

which is clearly a continuous function of α, β and t. However, $y_1(t) \equiv 0$ and $y_2(t) \equiv \sin t$ are both solutions to (5.1) on $[0, \pi]$ with $y_1(0) = y_2(0) = 0$ and $y_1(\pi) = y_2(\pi) = 0$.

When strong uniqueness is not present and in other situations, the following theorem suggested to the author by A. M. Fink is sometimes useful.

THEOREM 5.1. *Let $f(t, y, y')$ be continuous on D. Suppose* (1.1) *has a unique solution $y(t; \alpha, \beta)$ on $[a, b]$ satisfying* (1.2) *for* (α, β) *in a subset $S \subset R^2$. If there exists $B > 0$ such that $|y(t; \alpha, \beta)| \leqq B$ and $|y'(t; \alpha, \beta)| \leqq B$ for (t, α, β) in $[a, b] \times S$, then $y(t, \alpha, \beta)$, $y'(t; \alpha, \beta)$, and $y''(t; \alpha, \beta)$ are continuous on $[a, b] \times S$.*

Proof. Since $f(t, y, y')$ is continuous and

$$y''(t; \alpha, \beta) = f(t, y(t; \alpha, \beta), y'(t; \alpha, \beta)) \tag{5.3}$$

there exists $M > 0$ such that $|y''(t; \alpha, \beta)| \leqq M$ for $(t, \alpha, \beta) \in [a, b] \times S$.

Let $\{(\alpha_n, \beta_n)\}$ be a sequence in S such that $(\alpha_n, \beta_n) \rightarrow (\alpha_0, \beta_0) \in S$. Let $\{y(t; \alpha_{k(n)}, \beta_{k(n)})\}$ denote an arbitrary subsequence of $\{y(t; \alpha_n, \beta_n)\}$. Since $|y'(t; \alpha_{k(n)}, \beta_{k(n)})| \leqq B$ and $|y''(t; \alpha_{k(n)}, \beta_{k(n)})| \leqq M$, the Mean Value Theorem implies that $\{y(t; \alpha_{k(n)}, \beta_{k(n)})\}$ and $\{y'(t; \alpha_{k(n)}, \beta_{k(n)})\}$ are equicontinuous. Since both sequence are also uniformly bounded, using Ascoli's Thorem we may choose a further subsequence $\{y(t; \alpha_{k_1(n)}, \beta_{k_1(n)})\}$

such that $y(t; \alpha_{k_1(n)}, \beta_{k_1(n)}) \to y_0(t)$ and $y'(t; \alpha_{k_1(n)}, \beta_{k_1(n)}) \to y_0'(t)$ uniformly on $[a, b]$ for some $y_0(t)$. Since $f(t, y, y')$ is continuous, it follows from (5.3) that $\{y''(t; \alpha_{k_1(n)}, \beta_{k_1(n)})\}$ converges to $y_0''(t)$ and $y_0(t)$ is a solution to (1.1) on $[a, b]$. Since $y_0(a) = \alpha$ and $y_0(b) = \beta$, uniqueness implies that $y_0(t) \equiv y(t; \alpha_0, \beta_0)$.

It follows that the original sequences must converge to the same limits; i.e.,

$$y(t; \alpha_n, \beta_n) \to y(t; \alpha_0, \beta_0) , \qquad y'(t; \alpha_n, \beta_n) \to y'(t; \alpha_0, \beta_0) ,$$

and $y''(t; \alpha_n, \beta_n) \to y''(t; \alpha_0, \beta_0)$ uniformly on $[a, b]$; hence, $y(t; \alpha, \beta)$, $y'(t; \alpha, \beta)$ and $y''(t; \alpha, \beta)$ are continuous on $[a, b] \times S$.

If a Nagumo growth condition of the type introduced in [5] and employed to obtain existence in [1] and [3] is imposed, then bounds on derivatives may be obtained whenever the solutions themselves are bounded.

REFERENCES

1. J. W. Bebernes, *A subfunction approach to a boundary value problem for ordinary differential equations,* Pacific J. Math. **13** (1963), 1053-1066.

2. E. F. Beckenbach, *Generalized convex functions,* Bull. Amer. Math. Soc. **43** (1937), 363-371.

3. L. Fountain and L. Jackson, *A generalized solution of the boundary value problem for* $y'' = f(x, y, y')$, Pacific J. Math. **12** (1962), 1251-1272.

4. P. Hartman, *Ordinary differential equations,* John Wiley and Sons, New York, (1964).

5. V. M. Nagumo, *Uber die Differentialgleichung* $y''=f(x, y, y')$, Proc. Physics-Math. Soc. Japan (3) **19** (1937) 861-866.

Received September 8, 1967. Portions of this paper are part of a doctoral thesis written under the supervision of Professor Jerrold Bebernes at the University of Colorado and supported by a NASA Traineeship.

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