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BOUNDED APPROXIMATION BY RATIONAL FUNCTIONS

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Let D be a bounded open subset of the complex plane $\mathbb{C}$ which is the interior of its closure, and let h be a bounded analytic function on D. The classical theorem of Runge implies that there is a sequence of rational functions with poles in the complement of the closure of D which converges to h uniformly on compact subsets of D. The question naturally arises as whether this sequence may be chosen so that the supremum norms over D of the rational functions remain uniformly bounded. Of course, if the boundary of D consists of a finite number of disjoint circles (that is, D is a circle domain), then it is a classical result that the approximating sequence may be chosen so that their norms do not exceed the norm of h. But suppose that the boundary of D is quite complicated or D has infinitely many components in its complement. This general question has been the subject of several recent papers and is the subject of this one.

In [4] Rubel and Shields showed that if the complement of the closure of D is connected, then there is a sequence $\{h_n\}$ of polynomials with $\| h_n \| \leqq \| h \|$ and $h_n(z) \rightarrow h(z)$ for each z in D. Ahern and Sarason extended this result in [2] and proved that if a bounded open set D is the interior of its closure and has only finitely many components in its complement, then such bounded pointwise approximation is always possible, where the approximating functions have poles in the (finitely many) components of the complement of the closure of D, and their norms on D do not exceed the norm of the limit function.

The chief results in this paper show that rather elementary techniques may be used to extend the theorems of Rubel-Shields and Ahern-Sarason to certain domains with infinitely many complementary components.

We first introduct some notation to be used throughout the remainder of the paper: U is the open unit disc, $\{z \mid |z| < 1\}$; Γ is the unit circle, $\{z \mid |z| = 1\}$; if D is an open set, $H^\infty(D)$ is the space of bounded analytic functions on D and $\bar{D}$ denotes the closure of D; if K is a compact set, then $R(K)$ is the uniform closure on K of the rational functions with poles off K; finally, ∂S denotes the boundary of S.

THEOREM 1. *Let $S_1, S_2, \cdots$ be a sequence of disjoint closed discs in the open unit disc U which are centered on the positive real axis* *and whose centers, c_n, and radii, r_n, decrease to 0. Suppose that there is a constant $\delta > 1$ such that $(c_n - r_n) \geqq \delta(c_{n+1} + r_{n+1})$ for all n. Let D be the domain $U - \{0\} \cup [\bigcup_{i=1}^{\infty} S_i]$. If $f \in H^{\infty}(D)$, then there is a sequence $\{f_n\}$ of rational functions with poles off $\bar{D}$ such that $\|f_n\| \leqq \{\delta/(\delta - 1)\} \|f\|$ and $f_n(z) \to f(z)$ for each z in D.*

Proof. Let γ_n be the circle of radius $c_n + r_n$ about 0 and let $D_n = U - S_1 \cup \cdots \cup S_{n-1}$. For z in D_n define

$$f_n(z) = \sum_{j=0}^{n-1} \frac{1}{2\pi i} \int_{\Gamma_j} \frac{f^*(w)}{w - z}\, dw$$

where Γ_0 is the unit circle, $\Gamma_i = \partial S_i$ for $1 \leqq i \leqq n-1$ and the f^* on ∂D_n is the usual boundary-value function of f. For z near ∂D_n we have

$$f(z) = f_n(z) + \frac{1}{2\pi i} \int_{\gamma_n} \frac{f(w)}{w - z}\, dw ,$$

by Cauchy's formula. Thus, for z near ∂D_n,

$$|f_n(z)| \leqq |f(z)| + \left| \frac{1}{2\pi i} \int_{\gamma_n} \frac{f(w)}{w - z}\, dw \right|$$

and hence

$$\|f_n\| \leqq \|f\| \left[1 + \frac{1}{2\pi} \frac{2\pi(r_n + c_n)}{(c_{n-1} - r_{n-1}) - (r_n + c_n)}\right] \leqq \|f\| \left[\frac{\delta}{\delta - 1}\right]$$

as desired. It is immediate that $f_n(z) \to f(z)$ for each z in D since $c_n + r_n \to 0$ as $n \to \infty$; thus $f_n \to f$ uniformly on compact subsets of D. Since D_n is a circle domain, the f_n's may be replaced by rational functions with poles off $\bar{D}$ without increasing the estimate on the norms and without affecting the uniform convergence on compact subsets.

THEOREM 2. *Let I be an arbitrary closed subset of the interval $[-\frac{1}{2}, \frac{1}{2}]$ of zero arc length. Let $\{c_i\}$ be a countable set of distinct points in the unit disc with $\operatorname{Im} c_i \neq 0$ such that I is precisely the set of accumulation points of $\{c_i\}$. Let S_i be a closed disc in $U - I$ centered at c_i of radius r_i where the radii are chosen so small that* (a) *$S_i \cap S_j = \varnothing$ for $i \neq j$ and* (b) *$\sum_{i=1}^{\infty} r_i/d_i = C < \infty$ where d_i is the distance from the i^{th} disc to the nearest (other) disc. Let $D = U - I \cup \bigcup_{i=1}^{\infty} S_i$. If $f \in H^{\infty}(D)$, then there is a sequence $\{f_n\}$ of elements of $R(\bar{D})$ such that $\|f_n\| \leqq (C + 1)\|f\|$ and $f_n(z) \to f(z)$ for each z in D.*

Proof. Let $E_n = U - S_1 \cup \cdots \cup S_n$ and for z in E_n define

$$f_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f^*(w)}{w-z}\,dw + \sum_{j=1}^{n}\frac{1}{2\pi i}\int_{\partial S_j}\frac{f^*(w)}{w-z}\,dw$$

where the f^* in the integrand is the usual boundary-value function for f. Cauchy's formula holds in D in the following form

$$(*)\qquad f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f^*(w)}{w-z}\,dw + \sum_{j=1}^{\infty}\frac{1}{2\pi i}\int_{\partial S_j}\frac{f^*(w)}{w-z}\,dw$$

for each z in D. This is easily proved in the following manner. For each positive integer n choose a finite number of open discs whose union, U_n, contains I and the sum of whose radii is less than n^{-1}. This is possible since I has zero linear measure. Let $D_n = D - \bar{U}_n$. Then the boundary of D_n is rectifiable and consists of a finite number of piecewise smooth curves. If γ_n denotes the boundary of U_n, and X_n the intersection of ∂D_n with ∂D, we have by Cauchy's formula

$$f(z) = \frac{1}{2\pi i}\int_{X_n}\frac{f^*(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{\gamma_n}\frac{f(w)}{w-z}\,dw\ .$$

The second integral is bounded by $C\,||f||_\infty n^{-1}$ where C is a constant depending only on the distance between z and U_n. The first integral differs from the integral in $(*)$ by less than a constant times the tail of the convergent series $\sum_{j=1}^{\infty} r_j/d_j$. Letting n approach infinity we obtain the desired conclusion. From this form of Cauchy's formula estimates like those in Theorem 1 show that $||f_n|| \leqq (1+C)\,||f||$ and that $f_n(z) \to f(z)$ for each z in D. Use the fact that E_n is a circle domain to replace the f_n's by elements of $R(\bar{D})$.

A COUNTER-EXAMPLE. Bounded approximation of H^∞ functions by rational functions is not possible if the set I of accumulation points of the complementary components is "too" big, no matter how nice these components are.

Let I be an arbitrary closed subset of $[-\frac{1}{2}, \frac{1}{2}]$ of positive arc-length and let $\{S_i\}$ be a sequence of disjoint closed discs in $U - I$, the sum of whose radii is finite, which collect at each point of I and only there. Then there are bounded analytic functions on $D = U - I \cup \bigcup_{i=1}^{\infty} S_i$ which cannot be approximated pointwise on D by a *uniformly bounded* sequence of rational functions.

To see this, we note that since I has positive length there is a function h which is bounded, nontrivial, and analytic on the complement of I relative to the sphere [1; p. 254]. Without loss of generality it may be assumed that

$$h'(\infty) = \frac{1}{2\pi i}\int_{\Gamma} h(z)dz \neq 0 .$$

Since h is analytic on a neighborhood of S_i, $\int_{\partial S_i} h(z)dz = 0$ for $i = 1, 2, \cdots$. Note that for φ in $R(\bar{D})$, we have $\int_X \varphi(z)dz = 0$ where $X = \partial D - I$ and X has the usual positive orientation.

Suppose $\{\varphi_n\}$ is a sequence of elements of $R(\bar{D})$ such that $\|\varphi_n\| \leqq M$ for some M and $\varphi_n(z) \rightarrow h(z)$ for each z in D. Then at least some subsequence of $\{\varphi_n\}$ converges to h in the weak-star topology of $L^\infty(X, dz)$. But then

$$0 = \int_X \varphi_n dz \rightarrow \int_X h dz = \int_\Gamma h dz = 2\pi i h'(\infty) \neq 0 ,$$

a contradiction.

THEOREM 3. *Let $\{S_i\}$ be a sequence of pairwise disjoint closed discs in U all of whose accumulation points form a closed set E in unit the circle of zero arc length. Let $D = U - \bigcup_{i=1}^\infty S_i$. Suppose that for each i there is a point p_i in S_i such that $\sum_{i=1}^\infty (1 - |p_i|) < \infty$.*

If $h \in H^\infty(D)$, then there is a sequence $\{h_n\}$ of elements of $R(\bar{D})$ such that $\|h_n\| \leqq \|h\|$ and $h_n(z) \rightarrow h(z)$ for each z in D.

The proof of the theorem will require the following lemma. The lemma involves harmonic measure, details about which may be found in [3].

LEMMA. *Let D be the domain of Theorem 3. If f is a bounded harmonic function on D, then there is a unique function F in $L^\infty(\partial D, \mu)$, where μ is harmonic measure for some point p of D, such that f is the harmonic extension to D of F and $\|f\|_\infty = \|F\|_\infty$. Further, $f \geqq 0$ implies that $F \geqq 0$.*

Proof of the lemma. We only sketch the proof here since it is a simple limiting argument. Let $\{D_n\}$ be a sequence of subdomains of D satisfying the following three conditions: (a) $D_n \subset D_{n+1}$ and $\cup D_n = D$; (b) D_n is the interior of its closure and the complement of D_n has only a finite number of components; (c) $\bigcup_{n=1}^\infty \partial D \cap \partial D_n = \partial D - E$. The restriction of f to D_n is a bounded harmonic function on D_n. Since the conclusions of the lemma are known for D_n (see [2; § 3]), there is a unique bounded function F_n in $L^\infty(\partial D_n, \mu_n)$ such that the harmonic extension of F_n to D_n equals f and

$$\|F_n\|_\infty = \|f \mid D_n\| \leqq \|f\| .$$

Hence, $F_n = F_m$ on $\partial D \cap \partial D_m$ for all $n \geqq m$. This common function on $\partial D - E$ is easily shown to be the desired element of $L^\infty(\partial D, \mu)$.

Now we turn to the proof of Theorem 3.

We suppose first that $1 \leqq |h(z)| \leqq 3$ for all z in D and hence that $h(z) = \exp[f(z) + if^*(z)]$ where f is a positive bounded harmonic function on D and f^* is the harmonic conjugate of f.

By the lemma there is a nonnegative function F in $L^\infty(\partial D, \mu)$ with $f(z) = \int_{\partial D} F d\mu_z$ for all z in D and $\|F\|_\infty = \|f\|_D \leqq 2$. Let $D_n = U - \bigcup_{i=1}^n S_i$ and let f_n be the harmonic extension to D_n of $F \mid \partial D_n$. Then $\|f_n\| \leqq 2$ and $f_n \rightarrow f$ uniformly on compact subsets of D.

Let f_n^* be the harmonic conjugate of f_n on D_n, and let p_{nj} be the period of f_n^* about S_j, $1 \leqq j \leqq n$. h is singlevalued so that the period of f^* about each S_j is some integer multiple of 2π. Since f_n converges uniformly on compact subsets of D to f, $p_{nj} \rightarrow 0 \pmod{2\pi}$ as $n \rightarrow \infty$ for each $j = 1, 2, \cdots$.

For fixed j, let r_{nj} be the unique number in $[\frac{1}{2}, \frac{3}{2})$ such that $2\pi r_{nj} = p_{nj} \pmod{2\pi}$. Since $p_{nj} \rightarrow 0 \pmod{2\pi}$ for fixed j, we have $r_{nj} \rightarrow 1$ for fixed j as $n \rightarrow \infty$. Define a sequence of negative harmonic functions as follows: let

$$a_n(z) = \sum_{j=1}^n r_{nj} \log \left| \frac{z - p_j}{1 - \bar{p}_j z} \right|$$

for z in D_n. Then because $\sum (1 - |p_j|) < \infty$, we have

$$a_n(z) \rightarrow \sum_{j=1}^\infty \log \left| \frac{z - p_j}{1 - \bar{p}_j z} \right|$$

for each z in D and the convergence is uniform on compact subsets. Let $g_n = \exp[f_n + a_n + i(f_n^* + a_n^*)]$. Then g_n is a single-valued analytic function on D_n and for z in D, $g_n(z) \rightarrow h(z)B(z)$ where B is the Blaschke product on U whose zeros are at the points $\{p_j\}$.

Now if $h \in H^\infty(D)$ and $\|h\| \leqq 1$, then $2 + h$ is in $H^\infty(D)$ and $1 \leqq |h(z) + 2| \leqq 3$. Hence, there is a sequence $\{g_n\}$ with $g_n \in H^\infty(D_n)$ such that $\|g_n\| \leqq \|h\| + 2$ and g_n converges pointwise on D to $(2 + h)B$. Hence, if $h_n = g_n - 2B$, then $h \in H^\infty(D_n)$, $\|h_n\| \leqq \|h\| + 4$ and $h_n(z) \rightarrow h(z)B(z)$ for all z in D. Thus, h_n converges to hB uniformly on compact subsets of D. Since D_n is a circle domain, we may conclude that for each h in $H^\infty(D)$ there is a uniformly bounded sequence $\{h_n\}$ of elements of $R(\bar{D})$ such that $h_n(z) \rightarrow h(z)B(z)$ for each z in D. It will be shown later that this implies that the h_n's may actually be chosen so that $\|h_n\|_D \leqq \|hB\|_D$. (See Theorem 4 and the observations that follow it.) Assuming this, we may complete the proof of the theorem.

Let

$$B_n(z) = \prod_{n+1}^{\infty} \frac{p_j - z}{1 - \bar{p}_j z} \left(\frac{\bar{p}_j}{p_j} \right)$$

for z in U and let

$$C_n = \prod_{1}^{n} \frac{p_j - z}{1 - \bar{p}_j z} \left(\frac{\bar{p}_j}{p_j} \right)$$

for z in U. Let h be in $H^\infty(D)$. Since the interior of S_j is dense in S_j we may assume without any loss of generality that p_j lies in the interior of S_j for each j. Then h/C_n is in $H^\infty(D)$. Note that hB_n converges uniformly on compact subsets of D to h as $n \to \infty$. Since each $hB_n = (h/C_n)(B)$ may be approximated uniformly on compact subsets of D by elements of $R(\bar{D})$ whose norms do not exceed $\| hB_n \| \leqq \| h \|$, the same is true for h.

THEOREM 4. *Let D be the domain of Theorem 3 and let μ be harmonic measure for some point p in D. If f lies in the weak-star closure of $R(\bar{D})$ in $L^\infty(\partial D, \mu)$, then there is a sequence $\{f_n\}$ of elements of $R(\bar{D})$ such that $\| f_n \| \leqq \| f \|$ and $f_n \to f$ in the weak-star topology of L^∞.*

Proof. Note that if F is a closed subset of arc length zero in Γ or in ∂S_i for some i, then there is an element g of $R(\bar{D})$ such that $g = 1$ on F and $|g| < 1$ on $\partial D - F$. This follows immediately from the fact that closed sets of arc length zero on the unit circle are peak sets for closure of the polynomials on the closed unit disc. Note also that arc length and harmonic measure are mutually absolutely continuous on on ∂D. Thus if m is a measure on ∂D which annihilates $R(\bar{D})$, then m is absolutely continuous with respect to μ.

The remainder of the proof now parallels that of [2, Th. 3] and hence need not be repeated.

Theorem 4 implies the following: let D be the domain of Theorem 3 and let $f \in H^\infty(D)$. If there is some uniformly bounded sequence $\{f_n\}$ of elements of $R(\bar{D})$ such that $f_n(z) \to f(z)$ for all z in D, then the f_n may be chosen so that $\| f_n \| \leqq \| f \|$. This follows from the lemma and Theorem 4 as follows.

By the lemma there is a unique element F of $L^\infty(\partial D, \mu)$ such that $f(z) = \int_{\partial D} F d\mu_z$ for every z in D and $\| f \| = \| F \|$. The functions f_n are continuous on ∂D and uniformly bounded there. Hence, they have a weak-star convergent subsequence in L^∞, which we again denote by f_n. If g is the limit of this sequence, then $f(z) = \int_{\partial D} g d\mu_z$ because

$$f(z) = \lim f_n(z) = \lim \int_{\partial D} f_n d\mu_z = \int_{\partial D} g d\mu_z \, .$$

Hence, $g = F$ a.e. μ and thus F lies in the weak-star closure of $R(\bar{D})$ in L^∞. But now Theorem 4 implies there is a sequence g_n in $R(\bar{D})$ with $\|g_n\| \leqq \|F\| = \|f\|$ and $g_n \to F$ in the weak-star topology of L^∞. Whence,

$$g_n(z) = \int_{\partial D} g_n d\mu_z \to \int_{\partial D} F d\mu_z = f(z)$$

for each z in D because the harmonic measures are mutually absolutely continuous. This establishes the conclusion.

FINAL REMARKS. The conclusions of Theorems 1, 2, and 3 do not depend on the smoothness of the boundary. Thus, for example, instead of deleting closed discs from U to form D in Theorem 3, the deleted sets S_i may be any compact, pairwise disjoint subsets of U provided that (a) the interior of each S_i is connected and dense in S_i and the complement of each S_i is connected and (b) the set of accumulation points of the S_i forms a closed set E in Γ of zero arc length and there is a point p_i in each S_i such that $\sum_{i=1}^\infty (1 - |p_i|) < \infty$. The proof given is easily modified to include this more general case. Similar comments apply to the lemma and Theorm 4 (which are needed to prove Theorem 3) and to Theorems 1 and 2.

Finally we note that if D is a domain which satisfies the hypotheses of either Theorem 1 or Theorem 3, then each function continuous on $\bar{D}$ and analytic on D may be uniformly approximated on $\bar{D}$ by a sequence of rational functions with poles off $\bar{D}$. This follows readily from the fact that $H^\infty(D)$ may be considered to be the weak-star closure of $R(\bar{D})$ in $L^\infty(\partial D, m)$ where m is either harmonic measure in the case of the domain of Theorem 3 or m is harmonic measure plus a point mass at the origin in the case that D is the domain of Theorem 1, and the fact that any measure on ∂D which annihilates $R(\bar{D})$ is absolutely continuous with respect to m.

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