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CLASSES WITHOUT THE AMALGAMATION PROPERTY

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The contents of this paper belong to the general algebraic theory of those algebras which are studied in connection with algebraic logic. The main results, Theorems 1 and 2, give sufficient conditions for the amalgamation and the embedding property to fail in a class of Boolean algebras with operators. As a corollary, for $1 < \alpha < \omega$, the amalgamation property fails in the class of all (representable) cylindric algebras of dimension α and in the class of all (representable) polyadic (equality) algebras of dimension α. Thus, there are finitely axiomatizable equational classes of Boolean algebras with operators for which the amalgamation property fails.

The amalgamation and embedding properties have proved to be an extremely useful tool in model-theoretic investigations (references may be found in Jónsson [8]) and in the development of algebraic analogues to logical theorems (cf., Daigneault [2]). Recently, Don Pigozzi has shown that for certain classes of cylindric algebras the amalgamation property is equivalent to a certain algebraic form of Craig's Interpolation Theorem.

The first answer to Jónsson's question in [8] concerning whether or not there exist an equational class of Boolean algebras with operators for which the amalgamation property fails was given in R. McKenzie [12]. Using Lyndon algebras McKenzie showed that this property fails for the class RRA of all representable relation algebras and that the embedding property fails in a nontrivial way for the classes RRA and RA (the class of all relation algebras). We also obtain these results. The negative results obtained in Corollary 4 complement the work of A. Daigneault and J. Johnson. In [2] Daigneault shows that the amalgamation property holds for the class of all locally finite polyadic (equality) algebras of infinite dimension. Johnson [7] has extended Daigneault's work to show that the amalgamation property holds for the class of all polyadic and polyadic equality algebras of infinite dimension. Whether or not it holds for the class CA_α, $\alpha \geqq \omega$, in general, is not known.

It is shown in Łos [10] that a necessary condition for any two algebras of a class to have a common extension in the class is that all algebras of the class have isomorphic minimal subalgebras. An example, due to Białynicki-Birula, given in Łos [11] shows the condition is not sufficient even for equational classes. The classes listed in Corollary 5 provide further examples of this phenomenon. In § 3 we give a necessary and sufficient condition for two algebras of a class to have a common extension in the class. This condition is then used to show we cannot remove one of our restrictions on classes of algebras from the hypothesis of Theorem 2.

1. Throughout this paper we assume familiarity with the basic notions and results of mathematical logic, set theory and general algebra. In this section we will briefly introduce and discuss some of our terminology.

We assume that the ordinal numbers have been defined in such a way that each ordinal is the set of all smaller ordinals. The cardinality of a set X is denoted by $|X|$. The set of all functions from a set X into a set Y is denoted by ${}^X Y$; for a binary relation R on a set X and x in the domain of R, we denote $\{y \in X: (x, y) \in R\}$ by R^*x. If f is a function and X is a subset of the domain of f, we denote by $f | X$ the function with domain X which is equal to f on X. We use the usual arrow notation $\mathfrak{A} \rightarrow \mathfrak{B}$, $\mathfrak{A} \rightarrowtail \mathfrak{B}$, etc., to denote respectively, a homomorphism from an algebra $\mathfrak{A}$ into an algebra $\mathfrak{B}$, a monomorphism from $\mathfrak{A}$ into $\mathfrak{B}$, etc. For a homomorphism f on an algebra $\mathfrak{A}$ we denote the homomorphic image of $\mathfrak{A}$ by $f^*\mathfrak{A}$.

Let K be a class of similar algebras. We say that K has the *amalgamation property* if for ever $\mathfrak{A}$, $\mathfrak{B}_0$, $\mathfrak{B}_1$, $\in K$ and all monomorphisms $f_i: \mathfrak{A} \rightarrowtail \mathfrak{B}_i (i = 0, 1)$, there exist a algebra $\mathfrak{C} \in K$ and monomorphisms $g_i: \mathfrak{B}_i \rightarrowtail \mathfrak{B}$ such that the diagram

$$\begin{array}{ccc} \mathfrak{B}_0 & \xrightarrow{g_0} & \mathfrak{C} \\ {\scriptstyle f_0}\big\downarrow & & \big\downarrow{\scriptstyle g_1} \\ \mathfrak{A} & \xrightarrow{f_1} & \mathfrak{B}_1 \end{array}$$

commutes, i.e., $g_0 f_0 = g_1 f_1$.

We say that K has the *embedding* or *common extension property* if for every $\mathfrak{A}_0, \mathfrak{A}_1 \in K$ there exist a $\mathfrak{B} \in K$ and monomorphisms $f_0: A_0 \rightarrowtail \mathfrak{B}$ and $f_1: \mathfrak{A}_1 \rightarrowtail \mathfrak{B}$.

For an algebra $\mathfrak{A}$ we denote by $\overset{n-\mathfrak{A}}{\tau}$ the n-ary operation on $\mathfrak{A}$ induced by the term τ in the language of the similarity type of $\mathfrak{A}$ and with variables included among $v_0, \cdots, v_{n-1}$. Now suppose K and L are classes of algebras of similarity type μ and μ' respectively. Suppose τ is a function from dom μ' into the set of terms of the language of μ with the property that τ_i has variables included among $v_0, \cdots, v_{\mu'(i)-1}$ for all $i \in \operatorname{dom} \mu'$. We say that τ is an *equational definitional embedding* (e.d.e.) of K into L if the algebra

$$\tau^*\mathfrak{A} = \langle A, \overset{\mu'(i)-\mathfrak{A}}{\tau_i} \rangle_{i \in \operatorname{dom}\mu'}$$

belongs to L for all algebras $\mathfrak{A}$ in K. It should be observed that if K and L are considered as a categories in the natural way then the correspondence of $\mathfrak{A} \in K$ to $\tau^*\mathfrak{A} \in L$ associated with an e.d.e. τ is the object map of functor from K into L.

We now give a few basic definitions from the theories of cylindrification and cylindric algebras. For additional information see Henkin and Tarski [6].

A *cylindric algebra of dimensional* $\alpha(aCA_\alpha)$ is a system

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$$

where α is an ordinal, $0, 1, d_{\kappa\lambda} \in A$, $-, c_\kappa \in {}^AA$, $+, \cdot \in {}^{A\times A}A$ and the following conditions hold for all $x, y \in A$, $\kappa, \lambda, \mu < \alpha$:

(C_0) the system $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra;
(C_1) $C_\kappa 0 = 0$;
(C_2) $x \leqq c_\kappa x$;
(C_3) $c_\kappa(x \cdot c_\kappa y) = c_\kappa x \cdot c_\kappa y$;
(C_4) $c_\kappa c_\lambda x = c_\lambda c_\kappa x$;
(C_5) $d_{\kappa\kappa} = 1$;
(C_6) if $k \neq \lambda, \mu$, then $d_{\lambda\mu} = c_\kappa(d_{\lambda\kappa} \cdot d_{\kappa\mu})$;
(C_7) if $\kappa \neq \lambda$, then $c_\kappa(d_{\kappa\lambda} \cdot x) \cdot c_\kappa(d_{\kappa\lambda} \cdot -x) = 0$.

For a CA_α $\mathfrak{A}$ and a finite subset Γ of α, say $\Gamma = \{\lambda_0, \cdots, \lambda_{n-1}\}$ we let $c_{(\Gamma)}x = c_{\lambda_0}c_{\lambda_1} \cdots c_{\lambda_{n-1}}x$ for all $x \in A$. We call $\prod_{\kappa,\lambda<\alpha} d_{\kappa\lambda}$ the *main diagonal* of $\mathfrak{A}$ when it exists and occasionary denote it by $d^{\mathfrak{A}}$ or just d. We shall discuss neither the elementary arithmetic for cylindric algebras nor the familiar algebraic concepts of subalgebras, homomorphisms, subdirect and direct products of cylindric algebras, nor the notion of a simple CA_α. For information on these concepts see Henkin and Tarski [6].

For a set $U \neq 0$ and an ordinal α, consider the system

$$\langle \mathrm{S}({}^\alpha U), \cup, \cap, \sim, 0, {}^\alpha U, C_\kappa, D_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$$

where $S({}^\alpha U)$ is the set of all subsets of ${}^\alpha U$, $\cup$, $\cap$, and $\sim$ are the usual set-theoretic operations, and for $\kappa, \lambda < \alpha$ and $X \subseteqq {}^\alpha U$, $D_{\kappa\lambda}$ and $C_\kappa X$ are defined by

$$D_{\kappa\lambda} = \{y \in {}^\alpha U \colon y_\kappa = y_\lambda\}\ , \quad \text{and}$$
$$C_\kappa X = \{y \in {}^\alpha U \colon y \mid (\alpha \sim \{\kappa\}) = z \mid (\alpha \sim \{\kappa\}) \quad \text{for some} \quad z \in X\}\ .$$

The above system is called the *full cylindric set algebra of dimension α and base U* and is denoted by $\mathfrak{A}(\alpha, U)$. A $CA_\alpha\mathfrak{A}$ is *representable* (an RCA_α) if $\mathfrak{A}$ is isomorphic to a subdirect product of subalgebras of algebras $\mathfrak{A}(\alpha, U)$.

An algebra $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\kappa\rangle_{\kappa<\alpha}$ which satisfies (C_0) – (C_4) is called a *cylindrification algebra of dimension* $\alpha(aCy_\alpha)$. The notation and definitions given above extend in the obvious way to cylindrification algebras. In particular we denote the full cylindrification set algebra of dimension α and base U by $\mathfrak{A}_c(\alpha, U)$ and the class of all representable Cy_α's by RCy_α.

If a $CA_\alpha\mathfrak{A}$ has a simple minimal subalgebra, we say that $\mathfrak{A}$ has characteristic 0 if

$$0 \neq c_{(\lambda)} \prod_{i,j \in \lambda, i \neq j} - d_{ij}$$

for all $\lambda < (\alpha + 1) \cap \omega$. We may alternatively describe this class of algebras as the class of CA_α's whose minimal subalgebra is isomorphic to the minimal subalgebra of $\mathfrak{A}(\alpha, \alpha)$. We can define the class of polyadic equality algebras of characteristic 0 in a similar manner.

We now introduce conditions (A) and (B) on classes K of Boolean algebras with operators. Suppose τ is an e.d.e. of K into the class Cy_α.

Condition (A). There exist sets $U_j \neq 0$, algebras $\mathfrak{A}(U_j)$, $\mathfrak{M} \in K$ and monomorphisms $g: \mathfrak{M} \rightarrowtail \mathfrak{A}(U_j)$ for $j = 0, 1$ such that

(i) $\alpha \leqq |U_0| < |U_1|$ and $|U_0| < \omega$;

(ii) for $j = 0, 1$, $\tau^*\mathfrak{A}(U_j) = \mathfrak{A}_c(\alpha, U_j)$, the full set Cy_α with base U_j;

(iii) $D_j \in A(U_j)$ for $j = 0, 1$, and $g_1 \circ g_0^{-1}$ is an isomorphism from $g_0^*\mathfrak{M}$ onto $g_1^*\mathfrak{M}$ such that $g_1 g_0^{-1} D_0 = D_1$, where D_j is the main diagonal of $\mathfrak{A}(\alpha, U_j)$ for $j = 0, 1$.

Now suppose that τ is an e.d.e. of K into the class CA_α.

Condition (B). There exist sets $U_j \neq 0$ and algebras $\mathfrak{A}(U_j) \in K$ for $j = 0, 1$ such that

(i) $\alpha \leqq |U_0| < |U_1|$ and $|U_0| < \omega$;

(ii) for $j = 0, 1$ $\tau^*\mathfrak{A}(U_j) = \mathfrak{A}(\alpha, U_j)$, the full set CA_α with base U_j.

2. We now state the main results of this paper.

THEOREM 1. *The amalgamation property fails for any class K which satisfies Condition (A) for some* e.d.e. *τ of K into the class Cy_α for some α where $1 < \alpha < \omega$.*

THEOREM 2. *The embedding property fails for any class K which satisfies condition* (B) *for some* e.d.e. *τ of K into the class CA_α for some α where $1 < \alpha < \omega$.*

The proofs depend on the following lemma.

LEMMA 3. *Suppose $1 < \alpha \leqq |U_0| < |U_1|$ and $|U_0| < \omega$. Then there does not exist a $Cy_\alpha\mathfrak{B}$ and isomorphisms f_0, f_1 such that for $i = 0, 1$ f_i embeds the full $Cy_\alpha\mathfrak{A}_c(\alpha, U_i)$ into $\mathfrak{B}$ and $f_0(D_0) = f_1(D_1)$ where*

D_i is the main diagonal of $\mathfrak{A}(\alpha, U_i)$.

Proof. Suppose there exist such isomorphisms $f_i (i = 0, 1)$ and a $Cy_\alpha\mathfrak{B}$. Let $f_0(D_0) = d = f_1(D_1)$. Since $\mathfrak{A}_c(\alpha, U_0)$ and $\mathfrak{A}_c(\alpha, U_1)$ are simple we may assume that $\mathfrak{B}$ is also simple.

It is easy to verify the following facts about $\mathfrak{A}_c(\alpha, X)$ where $|X| = \kappa$.

(1) There exist κ atoms, say a_i for $i < \kappa$, in $\mathfrak{A}_c(\alpha, X)$ such that $a_i \leqq D^{\mathfrak{A}(\alpha,X)}$ and $D^{\mathfrak{A}(\alpha,X)} = \sum_{i<\kappa} a_i$.

(1′) All atoms of $\mathfrak{A}_c(\alpha, X)$ are of the form $\prod_{i<\alpha} c_{(\alpha\sim\{i\})} a_{\phi(i)}$ for $\phi \in {}^\alpha\kappa$; $a_j = \prod_{i<\alpha} c_{(\alpha\sim\{i\})} a_j$ for $j < \kappa$; and if $j_0, j_1 < \alpha$, $\phi \in {}^\alpha\kappa$ with $\phi(j_0) \neq \phi(j_1)$, then

$$\prod_{i<\alpha} c_{(\alpha\sim\{i\})} a_{\phi(i)} \leqq -D^{\mathfrak{A}(\alpha,X)} .$$

Letting X be U_0 or U_1 and applying the isomorphisms f_0, f_1 we obtain from (1) and (1′) the following:

(2) There exist $\kappa = |U_0|$ atoms, say a_i for $i < \kappa$, in $f_0^*(\mathfrak{A}_c(\alpha, U_0))$ such that $a_i \leqq d$ and $\sum_{i<\kappa} a_i = d$ (since $\kappa < \omega$).

(2′) same as (1′) with $\mathfrak{A}_c(\alpha, X)$ replaced by $f_0^*(\mathfrak{A}(\alpha, U_0))$ and $D^{\mathfrak{A}(\alpha,X)}$ replaced by d.

(3) There exist $\lambda = |U_1|$ atoms, say b_i for $i < \lambda$, in $f_1^*(\mathfrak{A}_c(\alpha, U_1))$ such that $b_i \leqq d$.

(3′) same as (1′) with $\mathfrak{A}_c(\alpha, X)$ replaced by $f_1^*(\mathfrak{A}_c(\alpha, U_1))$ and $D^{\mathfrak{A}(\alpha,X)}$ replaced by d.

By hypothesis $\kappa < \lambda$, so from (2) and (3) we may choose $j < \kappa$, $m, n < \lambda$, $m \neq n$ such that

(4) $a_j \cdot b_m \neq 0$ and $a_i \cdot b_n \neq 0$.

We now show

(5) $a_j \cdot -d \neq 0$.

Let $x = c_{(\alpha\sim\{m\})}(a_j \cdot b_m) \cdot \prod_{i<\alpha, i\neq m} c_{(\alpha\sim\{i\})}(a_j \cdot b_n)$.

From (2′) and (3′) it is clear that $x \leqq a_j \cdot -d$. Suppose $x = 0$.

Then

$$\begin{aligned} 0 &= c_{(\alpha\sim\{m\})} x \\ &= c_{(\alpha\sim\{m\})}(a_j \cdot b_m) \cdot c_{(\alpha\sim\{m\})}[\textstyle\prod_{i<\alpha, i\neq m} c_{(\alpha\sim\{i\})}(a_j \cdot b_n)] \\ &= c_{(\alpha\sim\{m\})}(a_j \cdot b_m) \cdot \textstyle\prod_{i<\alpha, i\neq m} c_{(\alpha)}(a_j \cdot b_n) . \end{aligned}$$

Since $\mathfrak{B}$ is simple and $a_j \cdot b_n \neq 0$ by (4), $c_{(\alpha)}(a_j \cdot b_n) = 1$. Hence

$$c_{(\alpha\sim\{m\})}(a_j \cdot b_m) = 0$$

contradicting (4). Thus $x \neq 0$ and (5) holds. However, (5) contradicts $a_j \leqq d$ of (2); hence Lemma 3 is proven.

Proof of Theorem 1. Suppose K is as in the statement of the theorem and the amalgamation property holds in K. Then there exist $\mathfrak{B} \in K$, isomorphisms f_0 and f_1 such that the diagram

$$\begin{tikzcd} \mathfrak{A}(U_0) \arrow[r, "f_0"] & \mathfrak{B} \\ \mathfrak{M} \arrow[u, "g_0"] \arrow[r, "g_1"] & \mathfrak{A}(U_1) \arrow[u, "f_1"'] \end{tikzcd}$$

commutes. By properties of terms, the diagram

$$\begin{tikzcd} \mathfrak{A}_c(\alpha, U_0) \arrow[r, "f_0"] & \tau^*\mathfrak{B} \\ \tau^*\mathfrak{M} \arrow[u, "g_0"] \arrow[r, "g_1"] & \mathfrak{A}_c(\alpha, U_1) \arrow[u, "f_1"'] \end{tikzcd}$$

commutes. By (A) (iii), $f_0(D_0) = f_1(D_1)$. This situation is impossible by Lemma 3.

Proof of Theorem 2. Let K be as in the statement of the theorem. Suppose there is a $\mathfrak{B} \in K$ and monomorphisms f_0 and f_1 such that $f_i\colon \mathfrak{A}(U_i) \rightarrowtail \mathfrak{B}$ for $i = 0, 1$. Then $f_i\colon \mathfrak{A}(\alpha, U_i) \rightarrowtail \tau^*\mathfrak{B}$ for $i = 0, 1$. Suppose $\tau_{\kappa\lambda}$ is the term in the language of K which defines $d_{\kappa\lambda}$. Then for the cylindric monomorphisms f_i, $f_i(D_{\kappa\lambda}) = {}^0\bar{\tau}^{\mathfrak{B}}_{\kappa\lambda}$ for all $i = 0, 1$ and $\kappa, \lambda < \alpha$. Thus $f_0(D_0) = \prod_{\kappa,\lambda<\alpha} {}^0\bar{\tau}^{\mathfrak{B}}_{\kappa\lambda} = f_1(D_1)$ where D_i is the main diagonal of $\mathfrak{A}(\alpha, U_i)$. Such an embedding of the cylindrification part is impossible by Lemma 3; thus the theorem holds.

Except for the claim dealing with the class of projective algebras the following corollaries are immediate. That Theorems 1 and 2 apply to the classes $(RRA)RA$ of (representable) relation algebras follows from the relationship between RA's and CA_2's found on p. 135 of Jónsson and Tarski [9]. For the class of an RA, see Definition 4.34 of Jónsson and Tarski [9]; for information concerning polyadic and polyadic equality algebras see Halmos [5] and for information on projective algebras see Everett and Ulam [4].

COROLLARY 4. *The amalgamation property fails for the following classes of algebras where* $1 < \alpha < \omega$: CA_α, RCA_α, Cy_α, RCy_α, *representable polyadic and polyadic equality algebras of dimension* α, *polyadic and polyadic equality algebras of dimension* α, RA, RRA, *and projective algebras.*[1]

The following proof that Theorem 1 applies to the class K of

[1] This result concerning projective algebras was pointed out by the referee.

projective algebras is due to the referee. Define τ_0 as $P_0 1 \square P_1 v_0$, τ_1 as $P_0 v_0 \square P_1 1$, and τ natural for the Boolean operations. Then τ is an e.d.e. of K into Cy_2. Suppose $U_0 \subset U_1$, $|U_0| = 3$, $a \in U_0$ and let $\mathfrak{A}(U_j)$ be the projective algebra of all subsets of ${}^2U_j$ with atom $\{\langle a, a\rangle\}$. The Boolean algebra with the following atoms is easily seen to be a projective subalgebra of $\mathfrak{A}(U_j)$:

$$f_{0j} = \{\langle a, a\rangle\}\ , \qquad f_{1j} = \{\langle u, u\rangle : u \in U_j,\ u \neq a\}\ ,$$
$$f_{2j} = \{\langle a, v\rangle : v \in U_j,\ v \neq a\}\ , \qquad f_{3j} = \{\langle u, a\rangle : u \in U_j,\ u \neq a\}\ ,$$

and

$$f_{4j} = \{\langle u, v\rangle : u, v \in U_j \sim \{a\},\ u \neq v\}\ .$$

Denote this subalgebra by $\mathfrak{B}_j$. Then obviously there is a projective algebra $\mathfrak{M}$ and isomorphisms g_j onto $\mathfrak{B}_j$ such that $g_1 g_0^{-1} D_0 = D_1$. Thus (A) holds and Theorem 1 applies.

COROLLARY 5. *The embedding property fails for the following classes of algebras where* $1 < \alpha < \omega$: *RCA's and* CA_α*'s of characteristic* 0, (*representable*) *polyadic equality algebras of dimension* α *and characteristic* 0, *and RRA's and RA's of class* 3.

For $1 < \alpha < \omega$ the classes of CA_α's and polyadic equality algebras of dimension α with a fixed nonzero characteristic and the classes of *RA's* with class $\neq 3$ are known to have the embedding property. In fact, these classes have the amalgamation property (cf., Comer [1]). By essentially the same argument given on p. 226 of Halmos [5] the embedding property can be shown to hold for the class of projective algebras, RCy_α, and the class of representable polyadic algebras of dimension α where $1 < \alpha < \omega$. Whether or not this property holds for the class Cy_α where $1 < \alpha < \omega$ and for the class of all polyadic algebras of dimension α where $2 < \alpha < \omega$ still appears to be open.

By examining the proof of Lemma 3 we see that if we restrict ourselves to the category of all complete CA_α's with complete homomorphisms, then the amalgamation and embedding properties fail for all $\alpha > 1$. Similar results hold for the other classes of algebras listed in Corollaries 4 and 5 if we modify the category.

For the sake of completeness we will also consider the amalgamation property for the classes Cy_α and CA_α for $\alpha \leqq 1$ (and hence also for the classes of all polyadic and polyadic equality algebras of dimension $\alpha \leqq 1$).

THEOREM 6. *For* $\alpha \leqq 1$ *the amalgamation property holds for the classes* CA_α *and* Cy_α.

Proof. For $\alpha = 0$ the algebras to be considered are just Boolean algebras so the conclusion follows from Dwinger and Yaqub [3]. For $\alpha = 1$ notice first that we can clearly amalgamate the simple algebras of each of the classes. The amalgamation property for the classes now follows by the same argument as used in Theorem 2.7 of Daigneault [2] for the class of all locally-finite polyadic algebras of infinite dimension.

3. We conclude the paper by establishing in Theorem 8 a sufficient (and obviously necessary) condition for two given algebras of a class K to be embeddable in some algebra of K. Conditions of a different nature may be found in Łos [10], [11]. As a corollary of Theorem 8 we show that the hypothesis $|U_0| < \omega$ is necessary for the conclusion of Lemma 3 and Theorem 2. More precisely:

COROLLARY 7. *If* $|U_0|, |U_1| \geqq \omega$ *and* $1 < \alpha < \omega$, *then we can embed the full set* Cy_α*'s* $\mathfrak{A}_c(\alpha, U_0)$ *and* $\mathfrak{A}_c(\alpha, U_1)$ *in a* Cy_α *such that the main diagonals of* $\mathfrak{A}(\alpha, U_0)$ *and* $\mathfrak{A}(\alpha, U_1)$ *are mapped to the same element.*

We need the following definition. For the general notion of *reduct* consult Tarski [14]; for the notion restricted to cylindric algebras as well as the notion of a *neat embedding* consult Henkin and Tarski [6]. Suppose the similarity type μ is an expansion of the similarity type τ, i.e., $\operatorname{dom} \tau \subseteqq \operatorname{dom} \mu$ and $\mu \mid \operatorname{dom} \tau = \tau$. A class L of algebras with similarity type μ is called a *μ-extension* of a class K of algebras of type τ if every τ-reduct of an algebra in L belongs to K. We donote the τ-reduct of $\mathscr{A} \in L$ by $\mathrm{Rd}_\tau \mathscr{A}$.

THEOREM 8. *Suppose K is a class of similar algebras of type τ and $\mathfrak{A}, \mathfrak{B} \in K$. A sufficient (and obviously necessary) condition for there to exist $\mathfrak{C} \in K$ such that $\mathfrak{A}$ and $\mathfrak{B}$ are embeddable in $\mathfrak{C}$ is that there exist a μ-extension K' of K for some expansion μ of τ and algebras $\mathfrak{A}'$ and $\mathfrak{B}'$ in K' such that*

(i) $\mathfrak{A} \rightarrowtail \mathrm{Rd}_\tau \mathfrak{A}'$ *and* $\mathfrak{B} \rightarrowtail \mathrm{Rd}_\tau \mathfrak{B}'$;

(ii) *There exist* $\mathfrak{C}' \in K'$ *such that* $\mathfrak{A}', \mathfrak{B}'$ *are embeddable in* $\mathfrak{C}'$.

Proof. Suppose we are given μ, K', $\mathfrak{A}'$, $\mathfrak{B}'$ as in the statement. By (ii) there is a $\mathfrak{C}' \in K'$ such that $\mathfrak{A}' \rightarrowtail \mathfrak{C}'$ and $\mathfrak{B}' \rightarrowtail \mathfrak{C}'$. By (i) and properties of reducts $\mathfrak{A}$ and $\mathfrak{B}$ are embeddable in $\mathrm{Rd}_\tau \mathfrak{C}'$ which belongs to K since K' is a μ-extension of K.

Proof of Corollary 7. First observe that it suffices to embed the CA_α's $\mathfrak{A}(\alpha, U_0)$ and $\mathfrak{A}(\alpha, U_1)$ in some CA_α. To do this we apply

Theorem 8 with $K = CA_\alpha$ and K' the class of locally-finite CA_ω's. It is clear that K' is a μ-extension of K where μ is the similarity type of K'. $\mathfrak{A}(\alpha, U_0)$ and $\mathfrak{A}(\alpha, U_1)$ can be neatly embedded (cf., Theorem 1.2 of Monk [13]) in locally-finite subalgebras $\mathfrak{A}'$ of $\mathfrak{A}(w, U_0)$ and $\mathfrak{B}'$ of $\mathfrak{A}(\omega, U_1)$, respectively. Hence we have $\mathfrak{A}'$ and $\mathfrak{B}'$ for which 8 (i) holds. Since $|U_0|, |U_1| \geqq \omega$, $\mathfrak{A}'$ and $\mathfrak{B}'$ have isomorphic minimal subalgebras; thus by the amalgamation property for locally-finite CA_ω's condition 8 (ii) holds. The corollary now follows.

Another obvious consequence of Theorem 8 is that if K is a class of similar algebras of type μ which has the embedding property, then the class $\mathrm{Rd}_\tau K = \{\mathrm{Rd}_\tau \mathfrak{A} : \mathfrak{A} \in K\}$ also has the embedding property, where τ is any restriction of μ.

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