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HOMEOMORPHISM GROUPS OF DENDRONS

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Professor J. Dugundji has asked the following question. Is the space of homeomorphisms of a contractible, locally contractible, space necessarily locally contractible? In this paper we answer the question in the negative by showing that the spaces of homeomorphisms of certain dendrons are not locally contractible. Specifically, we will show that any "special" dendron (see Definition 2.3) has a zero dimensional, non-discrete, group of homeomorphisms.

Conventions and notation. If X is a metric continuum, $G(X)$ will denote the group of all homeomorphisms of X, with the following metric topology:

$$d(g, h) = \underset{x \in X}{\operatorname{lub}} \{d(g(x), h(x))\} .$$

The double arrow in $f: A \twoheadrightarrow B$ means *onto.*

By a *dendron* is meant a locally connected metric continuum containing no simple closed curves.

See [3] and [4] for further discussion of dendrons.

2. Homeomorphisms between similar dendrons.

DEFINITION 2.1. *A set B is an order basis for a dendron D if and only if every point x of D has arbitrarily small neighborhoods U_x, with boundaries in B, such that the number of points on* Bd U_x *is less than or equal to the order of x* [3, p. 277].

DEFINITION 2.2. Let D be a dendron, B an order basis for D. Let $\mathscr{W}$: $W_1, \cdots, W_n$ be a cover of D of mesh $< \varepsilon$ with the following properties for each i:

(1) W_i is a continuum which is the closure of an open set,

(2) Bd W_i is a finite subset of B, and

(3) $W_i \cap W_j = \text{Bd } W_i \cap \text{Bd } W_j$ for each $j \neq i$.

Then $\mathscr{W}$ is called a *regular ε-cover with respect to B.*

DEFINITION 2.3. Let S be a nonempty set, finite or infinite, of positive integers greater than or equal to three. Let $\mathscr{D}_S$ be the class of all dendrons with a dense set of cut points of order n, for each $n \in S$, and no cut points of order $k \geqq 3$, if $k \notin S$. An element of $\mathscr{D}_S$ for any such S is called a *special dendron.* Two special dendrons D_1 and D_2 are called *similar* if and only if there exists a class $\mathscr{D}_S$ to which they both belong.

LEMMA 2.1. *Let D be a dendron and let B be an order basis for D containing all the cut points of order greater than or equal to three. Then for each $\varepsilon > 0$, there exists a regular ε-cover with respect to B.*

Proof. See Lemma 4.1 of [2].

LEMMA 2.2. *Let X and Y be dendrons and let $\varepsilon_i \to 0$. Let $\{\mathscr{U}_i\}_{i=1}^{\infty}$ and $\{\mathscr{V}_i\}_{i=1}^{\infty}$ be sequences of closed covers of X and Y respectively, such that*

(1) *$\mathscr{U}_i$ is a regular ε_i-cover of X,*

(2) *$\mathscr{V}_i$ is a regular ε_i-cover of Y,*

(3) *$\mathscr{U}_{i+1}$ is a refinement of $\mathscr{U}_i$; $\mathscr{V}_{i+1}$ is a refinement of $\mathscr{V}_i$,*

(4) *$\mathscr{U}_i$ and $\mathscr{V}_i$ have the same number of elements,*

(5) *The elements of $\mathscr{U}_i$ and $\mathscr{V}_i$ can be so named that $U_{i_j} \cap U_{i_k} \neq \varphi$ if and only if $V_{i_j} \cap V_{i_k} \neq \varphi$,*

(6) *$U_{i+1,j} \subseteqq U_{i,k}$ if and only if $V_{i+1,j} \subseteqq V_{i,k}$.*

Then X and Y are homeomorphic.

Proof. Let $x \in X$. Then there exists a tower of sets $\{U_{i,j_x}\}_{i=1}^{\infty}$ such that $x = \bigcap_{i=1}^{\infty} U_{i,j_x}$. Let $h(x) = \bigcap_{i=1}^{\infty} V_{i,j_x}$. It may be shown, by a standard argument, that $h(x)$ is well-defined, and is a homeomorphism of X onto Y. (See, for example, Theorem 11 of [1].)

LEMMA 2.3. *Let D be a dendron and let $\mathscr{U}$: $U_1, U_2, \cdots, U_n$ be a regular ε-cover. Let P be the set of points of intersection of the various elements of the cover; that is, $p \in P$ if and only if there exist i, j such that $U_i \cap U_j = p$. Then the elements of P can be so named that $D - \{p_1\}$ has exactly one component which meets $P - \{p_1\}$, if this set is not empty.*

Proof. Let p_1' be any element of P. If there exist at least two components of $D - \{p_1'\}$ which contain elements of $P - \{p_1'\}$, let V_1 be one of these components, and consider the points of P in V_1. Let p_2' be any such point of P. Now consider $D - \{p_2'\}$. One component of this set contains p_1'. If no other components contain points of P, then let $p_2' = p_1$, and we are through. If there are other components containing points of P, let V_2 be such a component. We note that $p_1' \notin V_2$. Let p_3' be any element of P in V_2. Consider $D - \{p_3'\}$. We note that p_1' and p_2' are in the same component of this set, since p_2' separates p_1' from p_3. If this is the only component containing points of P, let $p_3' = p_1$, and we are through. If there are other components containing points of P, we continue the above process. This process can be continued only a finite number of times, since P is finite. Therefore, at the last stage we have the required p_1.

LEMMA 2.4. *Let X and Y be similar dendrons, and let $\varepsilon > 0$. Let A and B be order bases for X and Y respectively, each containing all the cut points of order greater than or equal to three. Then X and Y admit regular ε-covers $\mathscr{U}$ and $\mathscr{V}$ with respect to A and B respectively, such that:*

(1) *$\mathscr{U}$ and $\mathscr{V}$ have the same number of elements, and*

(2) *the elements of $\mathscr{U}$ and $\mathscr{V}$ can be so named that $U_i \cap U_j \neq \varphi$ if and only if $V_i \cap V_j \neq \varphi$.*

Proof. Let $\mathscr{U}'$: $U_1', U_2', \cdots, U_r'$ be a regular ε-cover of X with respect to A. We may assume, without loss of generality, that the interiors of the elements of $\mathscr{U}'$ are connected. Let y_0 be an endpoint of Y, and let W_1 be a connected neighborhood of y_0, of diameter less than ε. Let y_1 be a point of order two in W_1. Then y_1 is a point of order two in Y, and one of the two components of $Y - \{y_1\}$ is of diameter less than ε. Call this component W_2. Let W_2' be the other.

We will put a copy of $\mathscr{U}'$ in $\bar{W}_2$ and a copy of a refinement of $\bar{W}_2'$ in an appropriate element of $\mathscr{U}'$ to obtain the isomorphic covers $\mathscr{U}$ and $\mathscr{V}$ of the theorem. We do this in the following way. Let P be the set of points of X which are points of intersection of the elements of $\mathscr{U}'$; that is, $p \in P$ if and only if there exists U_i' and U_j' such that $U_i' \cap U_j' = p$. By Lemma 2.3, there exists $p_1 \in P$ such that $X - \{p_1\}$ contains exactly one component meeting $P - \{p_1\}$. Let k_1 be the order of p_1. Let q_1 be a point of W_2 of order k_1. There are $(n_1 - 1)$ components of $X - \{p_1\}$ which do not contain elements of $P - \{p_1\}$. Let φ_1 assign the closures of these in any one-to-one manner to the closures of the $(n_1 - 1)$ components of $W_2 - \{q_1\}$ which do not contain y_1. Let O_1 and O_1' be the remaining components of $X - \{p_1\}$ and $\bar{W}_2 - \{q_1\}$ respectively, and let φ_1 assign $\bar{O}_1$ to $\bar{O}_1'$.

We next break up $\bar{O}_1'$ to match $\bar{O}_1$. $\bar{O}_1$ is a dendron containing p_1 as an endpoint, and $\bar{O}_1'$ is a dendron containing q_1 as an endpoint. $\bar{O}_1$ already has inherited a regular ε-cover by elements of $\mathscr{U}'$. By Lemma 2.3, there exists $p_2 \in P$ such that $\bar{O}_1 - \{p_2\}$ has exactly one component which contains points of $P - \{p_1, p_2\}$, if this set is not empty. Let k_2 be the order of p_2. Let q_2 be a point of $\bar{O}_1'$ of order k_2. Exactly one component of $\bar{O}_1 - \{p_2\}$ contains p_1. Let φ_2 assign this component to the component of $\bar{O}_1' - \{q_2\}$ which contains q_1. Let φ_2 map the closures of the other components arbitrarily in a one-to-one manner to the closures of the components of $\bar{O}_1' - \{q_2\}$ which do not contain q_1. We will call the component of $\bar{O}_1 - \{p_2\}$ which contains points of $P - \{p_1, p_2\}$ by the name O_2, and we will call $\varphi_2(\bar{O}_2)$ by the name $\bar{O}_2'$.

We note that O_2 has at most two boundary points in P. We next break up $\bar{O}_2'$. By Lemma 2.3, there exists $p_3 \in O_2$ such that $\bar{O}_2 - \{p_3\}$ has exactly one component which meets $P - \{p_1, p_2, p_3\}$, if this set is not empty. Let k_3 be the order of p_3. We choose a point q_3 in $\bar{O}_2'$ such that (1) the order of q_3 is k_3, and (2) q_i separates q_j from q_k if and only if p_i separates p_j from p_k, $i, j, k = 1, 2, 3$. In case p_i does not separate p_j from p_k, $i, j, k = 1,2, 3$, let A_1 be the unique arc from p_1 to p_2, and let A_1' be the unique arc from p_3 to A_1. Let k_3' be the order of point $A_1 \cap A_1'$. In this case, we additionally require that q_3 have the corresponding property; that is, if B_1 is the arc from q_1 to q_2 and B_1' is the arc from q_3 to B_1 then the order of the point $B_1 \cap B_1'$ is also k_3'. Now $\bar{O}_2 - \{p_3\}$ has exactly one component which meets $P - \{p_1, p_2, p_3\}$, if this set is not empty. Look at the boundary points of this component, and let φ_3 take this component to the component of $\bar{O}_2' - \{q_3\}$ with the corresponding boundary points. Let φ_3 map the closures of the remaining components arbitrarily to the closures of the remaining components of $\bar{O}_2' - \{q_3\}$, except that if one of the components of $\bar{O}_2 - \{p_3\}$ has p_1 (or p_2) as an additional boundary point, let φ_3 take the closure of this component to the closure of the component of $\bar{O}_2' - \{q_3\}$ with q_1 (or q_2) as an additional boundary point.

We continue this process inductively, choosing q_n so that q_n separates q_i from q_j, $(i, j < n)$, if and only if p_n separates p_i from p_j. Further, we ask that if the component of $X - \{p_1, p_2, \cdots, p_{n-1}\}$ which contains p_n has p_i $(i < n)$ as a boundary point, then the component of $W_2 - \{q_1, q_2, \cdots q_{n-1}\}$ which contains q_n has q_i as a boundary point. We also require that the order of q_n equal the order of p_n. In case p_n does not separate any pair $p_i p_j$, $1 \leqq i, j \leqq n - 1$, let A_n be the union of the arcs joining the pairs of points p_i, p_j for $1 \leqq i, j \leqq n - 1$. Let A_n' be the arc joining p_n to A_n. Let the order of the point $A_n \cap A_n'$ be k_n'. In this case we additionally require that q_n have the corresponding property; that is, if B_n is the union of the arcs joining q_i and q_j, $1 \leqq i, j \leqq n - 1$, and if B_n' is the arc from q_n to B_n. Then (a) the order of the point $B_n \cap B_n'$ is k_n', and (b) A_n' and B_n' are isomorphic trees, the isomorphism determined by $p_i \rightarrow q_i$. We define φ_n from the set of closures of components of $\bar{O}_{n-1} - \{p_n\}$ to the set of closures of components of $\bar{O}_{n-1}' - \{q_n\}$ so that if $\varphi_n(\bar{O}) = \bar{O}'$ and p_i is a boundary point of $\bar{O}$, then q_i is a boundary point of $\bar{O}'$.

After a finite number of steps, we use the last p_i in P. Let φ be the one-to-one function determined by $\varphi_1, \varphi_2, \cdots, \varphi_k$, where k is the number of points in P, and $\varphi(U') = \varphi_j(U')$ for some j such that U' contains no points of P in the jth step of the above process, and where $U' \in \mathscr{U}'$.

Let $\varphi(\bar{U}_i') = \bar{V}_i'$ in $\bar{W}_2$, and call this cover of $\bar{W}_2$ by the name $\mathscr{V}'$. Let $\bar{V}_s'$ be the element of $\mathscr{V}'$ which contains y_1. Consider $\varphi^{-1}(\bar{V}_s') = \bar{U}_s'$. If $\bar{U}_s'$ is an "end element" of $\mathscr{U}'$, that is, the boundary of $\bar{U}_s'$ contains exactly one point in P, let x_1 be any point of order two in $\bar{U}_s'$, and let W_2'' be the component of $X - \{x_1\}$ which does not contain the points of P.

If U_s' is not an end element, let $Q = \{q_i \mid q_i$ corresponds to p_i in $P\}$, and let B the finite tree which is the union of arcs joining the points of Q on the boundary of $\bar{V}_s'$, and let B' be the arc joining y_1 to B. Let $y = B \cap B'$ and let k be the order of y. Note that $k \geqq 3$. Now let A be the finite tree which is the union of the arcs joining the points of P on the boundary of U_s'. A is isomorphic to B. Choose a point $x \in A$, of order k, such that x separates p_i from p_j in A if and only if y separates q_i from q_j in B. Let x_1 be any point of order 2 of a component of $A - \{x\}$ whose only boundary point is x. Let W_2'' be the component of $X - \{x_1\}$ which does not contain points of P.

Now let $\mathscr{E}'$ be a regular ε-cover of W_2'. Let $\mathscr{E}''$ be a copy of $\mathscr{E}'$ in W_2'', using the same procedure as above, and making sure in the first step that the component in W_2' which contains y_1 corresponds to the component in W_2'' which contains x_1. $\mathscr{E}''$ is obtained in a finite number of steps. Let $\psi: \mathscr{E}' \twoheadrightarrow \mathscr{E}''$ be the one-to-one function obtained.

The subdivisions $\mathscr{U}$ and $\mathscr{V}$ of X and Y obtained in this manner are regular ε-covers of X and Y respectively, and are isomorphic; that is, there is a one-to-one function $f: \mathscr{U} \twoheadrightarrow \mathscr{V}$ such that $U_i \cap U_j \neq \varphi$ if and only if $f(U_i) \cap f(U_j) \neq \varphi$. This completes the proof.

THEOREM 2.1. *Let X and Y be two similar dendrons containing endpoints $\{a, b\}$ and $\{c, d\}$ respectively. Then there exists a homeomorphism h, $h: X \twoheadrightarrow Y$ such that $h(a) = c$ and $h(b) = d$.*

Proof. Let $\varepsilon > 0$. We show that there exist regular ε-covers $\mathscr{U}$ and $\mathscr{V}$ for X and Y respectively, and a one-to-one function φ from $\mathscr{U}$ onto $\mathscr{V}$ such that (1) $U_i \cap U_j \neq \varphi$ if and only if

$$\varphi(U_i) \cap \varphi(U_j) \neq \varphi ,$$

and (2) φ takes the element of $\mathscr{U}$ containing "a" to the element of $\mathscr{V}$ containing "c," and the element of $\mathscr{U}$ containing "b" to the element of $\mathscr{V}$ containing "d."

Let x_1 be a point of order two separating a from b in X, and let A_1 and B_1 be the two components of $X - \{x_1\}$. Let y_1 be a point of order two separating c from d in Y, and let C_1 and D_1 be the two components of $Y - \{y_1\}$. Let $\mathscr{U}'$ be a regular ε-cover of X which is the union of a regular ε-cover of A_1 and a regular ε-cover of B_1. Let $\mathscr{V}'$ be a regular ε-cover of Y such that $\mathscr{V}'$ is the union of a regular ε-cover of C_1 and a regular ε-cover of D_1 and such that there exists a one-to-one function φ: $\mathscr{U}' \to \mathscr{V}'$, constructed with care as in Lemma 2.4, so that:

(1) φ takes each element containing x_1 to an element containing y_1;

(2) φ takes each element which is a subset of A_1 to an element which is a subset of C_1, and each element which is a subset of B_1 to an element which is a subset of D_1, and

(3) φ takes the element containing "a" to the element containing "c," and the element containing "b" to the element containing "d."

Now let $\varepsilon_i \to 0$. For each ε_i we construct isomorphic covers $\mathscr{U}_i$ and $\mathscr{V}_i$ with properties of $\mathscr{U}$ and $\mathscr{V}$ above, and such that (1) $\mathscr{U}_i$ is a refinement of $\mathscr{U}_{i-1}$ and $\mathscr{V}_i$ is a refinement of $\mathscr{V}_{i-1}$, and (2) $U_{ij} \subseteqq U_{i-1,k}$ if and only if $V_{ij} \subseteqq V_{i-1,k}$. Clearly this may be done by using the care used in the construction of the isomorphic covers of Lemma 2.4, and applying this to each element of $\mathscr{U}_{i-1}$, when constructing $\mathscr{U}_i$. Then it follows from Lemma 2.2 that there is a homeomorphism h, $h: X \twoheadrightarrow Y$ such that $h(a) = b$ and $h(c) = d$.

3. The main theorem.

In this section we show that the space of homeomorphisms of a special dendron is zero dimensional and nowhere discrete. Thus it is not locally contractible.

DEFINITION 3.1. A topological space is called *contractible* if and only if there exists a continuous function $F: X \times I \to X$ such that

(1) $F(x, 0) = x$, and

(2) $F(x, 1)$ is constant,

for all $x \in X$.

DEFINITION 3.2. X is called *locally contractible* if and only if for each $x \in X$ and neighborhood U of x, there are a neighborhood V of x and a continuous function F: $V \times I \to U$ such that

(1) $F(v, 0) = v$, and

(2) $F(v, 1)$ is constant, for some constant in U, all $v \in V$.

REMARK. A dendron is a locally connected continuum. Therefore a component V of any open set U is also open. Further, its closure is again a dendron. It is well-known that a dendron is contractible (See [3]), and thus it is also locally contractible.

THEOREM 3.1 *Let X be a special dendron. Then $G(X)$ is nowhere discrete.*

Proof. It is sufficient to show that the identity is not isolated. Let $\varepsilon > 0$. We show there exists an $h \in G(X)$, $h \neq e$, such that $d(h, e) < \varepsilon$. Let a, b be two endpoints of X, and let U be a connected neighborhood of b of diameter less than ε. Let p_1, p_2, p_3, p_4 be points of U of order two such that p_1 separates a from b, p_2 separates p_1 from b, p_2 separates p_2 from b, and p_4 separates p_3 from b. Thus the unique arc from a to b contains these points in the order $a - p_1 - p_2 - p_3 - p_4 - b$. Exactly one component of $X - \{p_1, p_3\}$ contains p_2. Call the closure of this component A. Exactly one component of $X - \{p_1, p_4\}$ contains p_2 (and p_3). Call the closure of this component A'. Exactly one component of $X - \{p_3\}$ contains p_4 (and b). Call the closure of this component B. Exactly one component of $X - \{p_4\}$ contains b. Call the closure of this component B'. Let h be a homeomorphism of X onto itself which carries A onto A' so that $h(p_1) = p_1$ and $h(p_3) = p_4$, and which carries B to B' so that $h(p_3) = p_4$ and $h(b) = b$. Let h be the identity on $X - (A \cup B)$. By Theorem 2.1, such a homeomorphism exists. Since h moves only points of $A \cup B$, and $A \cup B \subsetneqq U$, and diam $U < \varepsilon$, we know that $d(h, e) < \varepsilon$.

THEOREM 3.2. *Let X be a special dendron. Then $G(X)$ is zero dimensional and is not locally contractible.*

Proof. Since X is special, the cut points of order greater than two are dense in X. Thus it follows from Theorem 4.1 of [2] that $G(X)$ is zero dimensional, and therefore no arcs exist in $G(X)$. But if $G(X)$ were locally contractible, there would exist a continuous function $F: U \times I \rightarrow G(X)$ which is not constant on $\{g\} \times I$, for some $g \in U$, where U is some open set in $G(X)$. But $F | \{g\} \times I$ is a nonconstant continuous function from an arc into $G(X)$, and therefore must contain an arc. This is a contradiction. It follows that $G(X)$ is not locally contractible.

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