

THE GEOMETRY OF RELATIVISTIC n PARTICLE INTERACTIONS

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In this paper we investigate several mathematical versions of what might be understood by the term "relativistic n particle interactions."

There has been substantial interest in such interactions in recent years. (See for example [4, 5, 6, 8, 9].)

The present treatment involves the idea that for each observer (more concretely, each space-like hyperplane in space-time) there is a separate phase space. There is no natural geometric way to identify these phase spaces with each other. Once this situation is clarified, the central problem emerges of organizing (in the sense of categorical algebra) the various correspondences which do arise between phase spaces. It turns out that there are really two conceptually distinct classes of such correspondences, the *kinematic* and the *dynamic*.

We feel that the interplay of these concepts is fully understood only when the coordinate-free language is used. It might be remarked that this point of view led us to discover a class of nontrivial relativistically invariant n -particle interaction.¹

As remarked already, we present several mathematical versions of of what might be understood by the term ' n particle interaction'. One motive for this is to enable us to define what is meant by the relativistic invariance of an interaction, rather than limit the discussion to such invariant interactions *ab initio*.

Presuming that some of these versions do reflect the ideas underlying various discussions in the literature, another motive is to show to what extent one can pass from one to the other.

The most obvious version of an n -particle interaction is that in which an interaction is characterized by the class of n -tuples of world lines to which it gives rise.

The second version places its emphasis on the dynamic correspondence between the phase spaces of all pairs of Lorentz observers (more precisely, space-like hyperplanes) to which the interaction gives rise. In this case there arises a family of transformations $U(g)$, g ranging over the Poincaré group, in the phase space of any one observer. It turns out that relativistic invariance is characterized by the functional equation

¹ These of course cannot be described in terms of usual Hamiltonian formalism [4, 9].

$$U(gg') = U(g)U(g') .$$

In any case, an interaction in the second sense gives rise to n -tuples of world lines and in fact to an interaction in the first sense (and conversely). Invariance, if any, is passed from one to the other.

In the third version we concentrate on what takes place in the phase space of one observer, and discover conditions ensuring that this local (in a sense) action can be extended to produce the type of dynamic correspondence between all the phase spaces as required by the second version. This third version, in as much as it involves only one (arbitrarily chosen!) phase space, is susceptible of an infinitesimalization in which actions of groups are replaced by actions of Lie algebras, and so forth.

In these remarks, 'phase space' is not limited to its usual meaning. When so limited, and the relevant transformations are contact transformations, we have the special case of Hamiltonian interactions. When coordinates are introduced, the zero interaction theorem of Currie, Jordan and Sudarshan as well as Leutwyler fits in here and says that if the interaction is invariant, then the world lines are straight lines.

2. Interactions in terms of world lines. The purpose of this section is to define the concept of relativistic interaction (2.7 below). This requires recalling several well-known definitions, propositions, and lemmas.

Some of the latter will be proved, mainly in order to illustrate the manner in which the concepts defined do reflect the familiar intuitive notions.

An affine Lorentzian space \mathcal{M} , more fully $\{\mathcal{M}, \mathcal{M}, (,), S, C^+\}$ is a system consisting of a point set \mathcal{M} (the space time manifold), a four-dimensional real vector space \mathcal{M} on which there is defined a bilinear form $(,)$ of signature 1, 3 and a map²

$$(2.0.1) \quad S: \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$$

such that for p, q , and r in \mathcal{M}

$$(2.0.2) \quad S(p, q) + S(q, r) = S(p, r) .$$

This requirement is easily remembered if we denote $s(p, q)$ by $\mathbf{p} - \mathbf{q}$, as we often will. We will sometimes need to mention the maps $S_q: \mathcal{M} \rightarrow \mathcal{M}$ each of which is defined by $S_q(p) = \mathbf{p} - \mathbf{q}$. In particular we require that for each q in \mathcal{M} , S_q is a one-to-one mapping of \mathcal{M} onto \mathcal{M} (i.e., a *bijection*). C is the set of ("time-like") vectors in \mathcal{M} described by $\{\mathbf{v}: \mathbf{v} \in \mathcal{M}, (\mathbf{v}, \mathbf{v}) > 0\}$. The specification of the affine

² Thus our treatment is that of Eberlein [10, p. 425, Example 2].

Lorentz space is completed by selecting one of the two components of C which is denoted by C^+ .

Now \mathcal{M} has an obvious manifold structure, and any one of the S_q carries this structure back to \mathcal{M} giving \mathcal{M} a manifold structure (which is the same for all q). The collection of vectors in \mathcal{M} (the tangent bundle) can be naturally identified with $\mathcal{M} \times \mathcal{M}$, and we shall refer to the latter as the tangent bundle of \mathcal{M} .

DEFINITION 2.1. A *space-like hyperplane* in \mathcal{M} is a subset of \mathcal{M} of the form $\{p: p \in \mathcal{M}, (\mathbf{p} - \mathbf{p}_0, \mathbf{t}_0) = 0\}$ for some fixed p_0, \mathbf{t}_0 where $p_0 \in \mathcal{M}$ and $\mathbf{t}_0 \in \mathcal{M}$ where $(\mathbf{t}_0, \mathbf{t}_0) = 1$.

We denote such a set by $\sigma(p_0, \mathbf{t}_0)$, but confine this notation only to the case in which $\mathbf{t}_0 \in C^+$. Thus $\sigma(p_1, \mathbf{t}_1)$ coincides with $\sigma(p_2, \mathbf{t}_2)$ precisely when $\mathbf{t}_1 = \mathbf{t}_2$ and $p_1 \in \sigma(p_2, \mathbf{t}_2)$ or $p_2 \in \sigma(p_1, \mathbf{t}_1)$.

We may speak of $\sigma(p_0, \mathbf{t}_0)$ as the space-like hyperplane passing through p_0 and perpendicular to \mathbf{t}_0 . The totality of all these $\sigma(p_0, \mathbf{t}_0)$ we will denote by \mathcal{S} .

DEFINITION 2.2. Let W be a one-dimensional C^∞ submanifold of \mathcal{M} . Then W shall be called *time-like* if every nonzero vector tangent to W is time-like. In symbols, let i be the inclusion map of W into \mathcal{M} . Then $(di)_p$ maps the tangent space W_p of W at p . We have identified \mathcal{M}_p with \mathcal{M} and hence the definition requires that

$$(di)_p(W_p) \subset \{0\} \cup C$$

for each p on W . We call W a *world line* if

$$(2.2.1) \quad W \text{ is a time-like submanifold}$$

and

$$(2.2.2) \quad W \text{ intersects each } \sigma \text{ belonging to } \mathcal{S} \text{ in exactly one point}^3.$$

It is convenient to observe the following characterization. W is a world line if 2.2.1 holds and also

³ A characterization (which we will not prove because we will not need it) of world lines is the following. A world line W is a connected time-like submanifold of \mathcal{M} which is complete in the Riemannian structure induced on W , and conversely. The structure here mentioned is the Lorentz structure inherited by W from \mathcal{M} which is, however, Riemannian (i.e., positive definite) because W is time-like. In this paper, 'world line' has not extra dynamical signification it has in [2] and [3].

(2.2.3) *There is a t_0 in C^+ such that W meets in exactly one point all those space-like hyperplanes perpendicular to t_0 .*

Of course 2.2.2 implies 2.2.3.

Now we will consider the Riemannian structure of space-like hyperplanes. Each σ in \mathcal{S} is in fact an *affine Euclidean space*, a concept defined just as the affine Lorentz space was defined above, except that

- (a) the inner product has to be positive definite and
- (b) C is connected and hence may be ignored in the definition.

To fix the notation, let $\sigma \in \mathcal{S}$ be of the form $\sigma(p, t)$ where t is a unit vector in C^+ as agreed above. Let σ be the set

$$\{u: u \in \mathcal{M}, (u, t) = 0\}.$$

For u, v in σ define $(u, v)_\sigma = -(u, v)$. Using the original S of \mathcal{M} (restricted to σ without change of notation) we arrive at $\{\sigma, \sigma, (\cdot, \cdot), S\}$ which evidently satisfies the definition of affine Euclidean space.

The vector space σ is naturally isomorphic to the tangent space of σ at each point, whence arises a natural identification of $\sigma \times \sigma$ with the tangent bundle of σ . Let $B(\sigma)$ denote the open unit ball of σ , that is the class of vectors of length less than 1. The unit ball bundle of σ can and shall be identified with $\sigma \times B(\sigma)$.

PROPOSITION 2.3. *Let W be world line intersecting σ in the point p . Then there is exactly one vector which*

(2.3.1) *is tangent to W at p*

(2.3.2) *is of the form $v_\sigma^w + t$ where $\sigma = \sigma(p, t)$ and $v_\sigma^w \in \sigma$.*

(2.3.3) *Moreover, $v_\sigma^w \in B(\sigma)$.*

Proof. Select any nonzero tangent w to W at p . Then w has the form $\lambda t + v$, where $v \in \sigma$, because the vector space generated by σ and t is four-dimensional. Now $\lambda \neq 0$ because w is time-like. Hence we may select w such that $w = t + v_\sigma^w$. This last vector is evidently unique for otherwise we would have a nonzero tangent to W at p lying in σ . To show (2.3.3), we observe that, since w is time-like (writing v for v_σ^w),

$$\begin{aligned} 0 < (w, w) &= (t + v, t + v) = (t, t) + 2(t, v) + (v, v) \\ &= 1 + (v, v) = 1 - (v, v)_\sigma \end{aligned}$$

whence $(v, v)_\sigma < 1$, as asserted.

We emphasize: \mathbf{v}_σ^W is not tangent to W , nor does it depend only on W and p . We call it the

$$(2.3.4) \quad \text{velocity of } W \text{ relative to } \sigma .$$

As its interpretation will require, the magnitude $(\mathbf{v}_\sigma^W, \mathbf{v}_\sigma^W)_\sigma^{1/2}$ is always less than 1.

We come now to the central notion of a relativistic symmetry (of \mathcal{M}). To begin with, let \mathcal{L}_0 be the component of the identity of the Lie group of automorphisms of $\{\mathcal{M}, (\cdot, \cdot)\}$. \mathcal{L}_0 is, of course, the restricted Lorentz group. The transformations \mathbf{g} of \mathcal{L}_0 act in \mathcal{M} , not \mathcal{M} .

DEFINITION 2.4. Let g be a one-to-one mapping of \mathcal{M} onto itself for which there exists a \mathbf{g} in \mathcal{L}_0 such that

$$(2.4.1) \quad S(g(p), g(q)) = \mathbf{g}(S(p, q)) \quad \text{for all } p, q \text{ in } \mathcal{M} .$$

Then g shall be called a *relativistic symmetry*.

Given g , there is evidently only one \mathbf{g} such that (2.4.1) holds. We denote it by $\psi(g)$. We denote the group of relativistic symmetries by \mathcal{P} , the restricted Poincaré group.

PROPOSITION 2.4.2. The map $\psi: \mathcal{P} \rightarrow \mathcal{L}_0$ is a homomorphism whose image is all of \mathcal{L}_0 . The kernel \mathcal{T} (the “translations”) of ψ is the image of \mathcal{M} under a homomorphism

$$\varphi: \mathcal{M} \longrightarrow \mathcal{T}$$

which is one-to-one and satisfies (in terms of 2.01)

$$(2.4.3) \quad S_p(\varphi(\mathbf{u})(q)) = S_p(q) + \mathbf{u}$$

for each $\mathbf{u} \in \mathcal{M}$ and every p, q in \mathcal{M} .

Proof. From (2.4.1) one sees at once that ψ is a homomorphism. Moreover, given g , select q in \mathcal{M} and use the formula

$$g(p) = S_q^{-1}(S_q(p))$$

which defines a $g \in \mathcal{P}$ for which $\psi(g) = \mathbf{g}$ (and incidentally, for which $g(q) = q$). Thus $\psi(\mathcal{P}) = \mathcal{L}_0$.

We define φ by letting $\varphi(\mathbf{u})(q) = S_q^{-1}(\mathbf{u})$. Denote the right side by r , so $\mathbf{u} = S_q(r)$. Interchanging p and r in (2.01) gives us $S_q(r) = S_q(r) + S_q(q)$ or $S_p(r) = S_p(q) + \mathbf{u}$. This proves (2.4.3).

Denote $\varphi(\mathbf{u})(q)$ by q' and $\varphi(\mathbf{u})(r)$ by r' . Write down Equation (2.4.3) for q and for r and subtract. This yields

$$(2.4.4) \quad S_p(q') - S_p(r') = S_p(q) - S_p(r) .$$

From (2.0.1) we see that the right side is $S(q, p)$. A similar thing is true for the left side, whence (2.4.1) holds (with p, q replaced by q, r) and g being the identity. Thus $\varphi(u)$ lies in the kernel of ψ .

On the other hand suppose g does lie in that kernel. Then we obtain (2.4.4) at once where $q' = g(q)$ and $r' = g(r)$. From this we obtain (2.4.3) with $u = S_p(r') - S_p(r)$, and conclude that $g = \varphi(u)$.

Finally, (2.4.3) shows that φ is one-to-one.

Having thus established (2.4.2) we may identify \mathcal{M} with the group $\varphi(\mathcal{M})$. In other words we identify each "translation" in \mathcal{M} with a vector in the vector space \mathcal{M} in which the Lorentz group is defined.

Return to a space-like section $\sigma = \sigma(p, t)$ where, of course, t is a unit vector of \mathcal{M} . Evidently $\{\tau t: -\infty < \tau < \infty\}$ is a 1-parameter subgroup of \mathcal{M} and thus of \mathcal{P} . It shall be called the *flow of time perpendicular to σ* . It enables us to describe how an element g of \mathcal{P} transforms one space-like section into another.

LEMMA 2.5. *Let σ be a space-like section, $\sigma = \sigma(p, t)$. Then $g(\sigma) = \sigma(g(p), \psi(g)t)$, for each g in \mathcal{P} .*

Proof. Suppose $q \in \sigma(p, t)$. Then $S(q, p) \perp t$. Thus

$$\psi(g)(Sq, p) \perp \psi(g)t .$$

By (2.4.1), $S(g(q), g(p)) \perp \psi(g)t$. This says that $g(q) \in \sigma(g(p), \psi(g)t)$. This shows one inclusion, and the reversal of this argument shows the other, and completes the proof of 2.5.

PROPOSITION 2.5.1. *The action of \mathcal{P} on \mathcal{S} given in 2.5 is transitive.*

Proof. To achieve $g(\sigma(p, t)) = \sigma(q, u)$ we first select g_1 so that $\psi(g_1)t = u$. This g_1 exists because \mathcal{L}_0 acts transitively on the unit vectors in C^+ . Thus $g_1(\sigma(p, t)) = \sigma(p_1, u)$. Now let g_2 be the translation (by) $S(q, p_1)$. $g = g_2g_1$ will do.

This proof calls attention to the subgroup of \mathcal{P} of those g such that $g(\sigma) = \sigma$. (This does not require that each point of σ be fixed.) This group, to be called \mathcal{E}_σ is readily identified with the proper Euclidean group of the Euclidean space $\{\sigma, \sigma, (\cdot), S\}$ and has thus the structure of a Lie group. Using any desired space-like section σ , form the \mathcal{C}^∞ manifold of left cosets $\mathcal{P}/\mathcal{E}_\sigma$. This is in natural one-to-one correspondence with \mathcal{S} because

- (1) \mathcal{E}_σ is the subgroup of stability of σ and
- (2) \mathcal{P} acts transitively on \mathcal{S} .

Thus \mathcal{S} can be given a C^∞ structure compatible with the action of \mathcal{P} . This structure is actually independent of the choice of σ .

We turn now to the action of \mathcal{P} on world lines.

PROPOSITION 2.5.2. *Let W be a world line and let g belong to \mathcal{P} . Then $g(W)$ is also a world line.*

Proof. $g(W)$ is of course meant to designate the set of points $g(p), p \in W$. Now $g(W)$ is surely a 1-dimensional submanifold. Any tangent v of $g(W)$ is the image under $\psi(g)$ of a tangent u of W . Now $u \in C$ and $\psi(g) \in \mathcal{L}_0$ so $v \in C$. Finally, how often does $g(W)$ meet σ ? Evidently as often as W meets $g^{-1}(\sigma)$, which is, once. Thus 2.5.2 is proved.

The central concept of this paper can now be defined. Let \mathcal{W} be the class of all world lines. Then $\mathcal{W} \times \dots \times \mathcal{W}$ (the n -fold Cartesian product, also denoted by \mathcal{W}^n) is the class of ordered n -tuples of world lines. Suppose that \mathcal{I} is a subset of \mathcal{W}^n

$$(2.6.0) \quad \mathcal{I} \subset \mathcal{W}^n = \mathcal{W} \times \dots \times \mathcal{W}$$

having the following property: given $\sigma \in \mathcal{S}$ and n points of σ :

$$(2.6.1) \quad (p_1, p_2, \dots, p_n) \in \sigma^n$$

and also n vectors length less than 1:

$$(2.6.2) \quad (v_1, \dots, v_n) \in B(\sigma)^n$$

then there is exactly one element (W_1, \dots, W_n) in \mathcal{I} such that, for each i

$$(2.6.3) \quad \text{the intersection of } W_i \text{ with } \sigma \text{ is } p_i \text{ and (see 2.3.4)}$$

$$(2.6.4) \quad \text{the velocity of } W_i \text{ relative to } \sigma \text{ is } v_i .$$

Then we call \mathcal{I} a

$$(2.7) \quad \text{second-order } n \text{ particle relativistic interaction .}$$

In this paper we abbreviate this to *interaction*.

There is nothing in this definition that requires \mathcal{I} to be relativistically invariant. Indeed, we will call \mathcal{I}

$$(2.8) \quad \text{relativistically invariant}$$

if, whenever $(W_1, \dots, W_n) \in \mathcal{I}$ then $(g(W_1), \dots, g(W_n)) \in \mathcal{I}$ for every $g \in \mathcal{P}$.

Examples of such invariant interactions easy to make.

Let us call a world line W a *free* world line if it is an ordinary straight line \mathcal{M} (indefinitely extended in both directions) with a tangent (or direction) vector belonging to C (i.e., which is time-like).

2.9. The *free interaction* $\mathcal{I}_{0,n}$ shall consist of all n -tuples (W_1, \dots, W_n) where each W_i is a free world line.

This is evidently an invariant interaction.

The Definition 2.7 obviously does allow $n = 1$, although then the word “interaction” is misleading. However, an invariant relativistic one particle “interaction” has to be $\mathcal{I}_{0,1}$ that is to say, free.

3. **Interactions in terms of functors.** The following observation about an interaction \mathcal{I} forms the basis for a more general concept, to which the present section is devoted.

Let σ be a space-like section (i.e., $\sigma \in \mathcal{S}$), and let

$$B^n(\sigma) = \sigma^n \times B(\sigma)^n .$$

Then each point of $B^n(\sigma)$ is a pair ((2.6.1), (2.6.2)) which determines a (W_1, \dots, W_n) from \mathcal{I} . Now let σ_1 be another element of \mathcal{S} . The (W_1, \dots, W_n) just found determines a point of $B^n(\sigma_1)$. Thus \mathcal{I} determines a map from $B^n(\sigma)$ to $B^n(\sigma_1)$. This map embodies the dynamics defined by \mathcal{I} . We have to fill in more structural elements into this picture in order to be able to define relativistic invariance in such terms, namely the way in which \mathcal{P} , quite apart from \mathcal{I} , also defines a map from $B^n(\sigma)$ to $B^n(\sigma_1)$.

To do this abstractly and conveniently, it seems desirable to use the terms, although hardly any of the theorems, of homological algebra in particular the concepts of categories and functors. The discussion of these in any introductory text (cf. [10]) suffices as a basis for our presentation.

In fact, we will consider two categories \mathcal{K} and \mathcal{D} . The class of *objects* in each shall be the set \mathcal{S} . For \mathcal{K} , the class $\text{Hom}(\sigma_1, \sigma_2)$ shall be the set of elements g of \mathcal{P} for which $\sigma_2 = g(\sigma_1)$ (see 2.5). The operation from $\text{Hom}(\sigma_2, \sigma_3) \times \text{Hom}(\sigma_1, \sigma_2)$ which the definition of category requires shall be just the multiplication in \mathcal{P} : (h, g) to $h \circ g$ or hg . Note that in particular, $\text{Hom}(\sigma, \sigma) = \mathcal{E}_\sigma$. For \mathcal{D} , $\text{Hom}(\sigma_1, \sigma_2)$ shall consist of the single ordered pair (σ_1, σ_2) and the multiplication is to be defined by $(\sigma_2, \sigma_3) \circ (\sigma_1, \sigma_2) = (\sigma_1, \sigma_3)$.

\mathcal{K} and \mathcal{D} are categories.⁴ They shall be called the *kinematic* and the *dynamic* categories, respectively.

Let \mathcal{M} denote the category of \mathcal{C}^∞ manifolds and their \mathcal{C}^∞

⁴ In fact, each is isomorphic to *concrete* category [10, p. 64]. One takes $\mathcal{U}(\sigma) = \mathcal{K}$ for \mathcal{K} and $\mathcal{U}(\sigma) = \{\sigma\}$ for \mathcal{D} , and morphisms as above.

mappings.

DEFINITION 3.1. A second order, n particle, \mathcal{S} -interaction is a pair of functors κ, δ from \mathcal{H} and \mathcal{D} , respectively, to \mathcal{M} satisfying the conditions 3.1.1–3.1.6.

3.1.1. For each $\sigma \in \mathcal{S}$, $\kappa(\sigma)$ is a \mathcal{C}^∞ fibre space over $\sigma^n = \sigma \times \dots \times \sigma$, and $n \geq 1$.

One should think of the points of $\kappa(\sigma)$ as being the states of the n particle system in question, for the observer associated with σ .

Let π_σ denote the projection of $\kappa(\sigma)$ on σ^n . Note that for $g \in \mathcal{P}$ one has $g: \sigma \rightarrow g(\sigma)$ and this induces a map $g \times \dots \times g: \sigma^n \rightarrow g(\sigma)^n = g(\sigma) \times \dots \times g(\sigma)$.

3.1.2. For each σ and $g \in \mathcal{P}$ there shall be commutativity of the diagram

$$\begin{array}{ccc} \kappa(\sigma) & \xrightarrow{\kappa(g)} & \kappa(g(\sigma)) \\ \pi_\sigma \downarrow & & \downarrow \pi_{g(\sigma)} \\ \sigma^n & \xrightarrow{g \times \dots \times g} & g(\sigma)^n . \end{array}$$

(3.1.3) For each σ , $\delta(\sigma) = \kappa(\sigma)$.

Now let $x \in \kappa(\sigma)$, and $g \in \mathcal{P}$. Then $\delta(\sigma, g^{-1}(\sigma))(x)$ is an element of $\kappa(g^{-1}(\sigma))$ whence

(3.1.3.1) $U_\sigma(g)(x) = \kappa(g)\delta(\sigma, g^{-1}(\sigma))(x)$

is an element of $\kappa(\sigma)$ again. Thus for fixed σ the map (3.1.3.1) defines a map⁵

(3.1.3.2) $U_\sigma: \mathcal{P} \times \kappa(\sigma) \longrightarrow \kappa(\sigma)$.

Concerning these constructions we impose a *regularity condition*.

3.1.4. U_σ shall be a \mathcal{C}^∞ map for each $\sigma \in \mathcal{S}$.

Whenever we have a product space $A_1 \times \dots \times A_n$ we denote the Cartesian projection on the j -th factor by π^j . The symbol π^j will be used without further suffixes whether the product in question is σ^n or $g(\sigma)^n$, etc.

We formulate a *world line condition*.

3.1.5. Suppose σ and σ' belong to \mathcal{S} and that they intersect. Suppose $\pi^j(\pi_\sigma(x))$ lies in σ' for some $j(1 \leq j \leq n)$ and $x \in \kappa(\sigma)$. Then

⁵ See 4.4.

$\pi^j(\pi_\sigma(x))$ shall coincide with $\pi^j(\pi_\sigma(\delta(\sigma, \sigma')(x)))$.

Finally, we impose a condition that the system shall be describable in terms of *second order* equations for each Lorentz frame.

3.1.6. Let $\sigma = \sigma(p, t)$ be any element of \mathcal{S} , and let $x \in \kappa(\sigma)$. For each real τ define (using (3.1.3.1) with $g = \tau t$)

$$\varphi(x, \tau) = \pi_\sigma(U_\sigma(-\tau t)(x)) .$$

As τ varies, $\pi^j(\varphi(x, \tau))$ describes a path in σ whose position for $\tau = 0$ is $\pi^j(\pi_\sigma(x))$ and whose velocity for $\tau = 0$,

$$(3.1.6.1) \quad \frac{d}{d\tau}[\pi^j(\varphi(x, \tau))]_{\tau=0}$$

is an element v^j of σ . Define

$$\Phi_\sigma: \kappa(\sigma) \longrightarrow \sigma^n \times \sigma^n ,$$

by

$$\Phi_\sigma(x) = (\pi^1(\pi_\sigma(x)), \dots, \pi^n(\pi_\sigma(x)), v^1, \dots, v^n) .$$

We require Φ_σ to be a \mathcal{C}^∞ homeomorphism $\kappa(\sigma)$ onto $\sigma^n \times B(\sigma^n)$.

Note that 3.1.6 insures that $\dim \kappa(\sigma)$ is $6n$.

Any functor κ satisfying 3.1.1 and 3.1.2 will be called a *kinematic* functor. A pair κ, δ satisfying 3.1.1-3.1.6 shall be called the *kinematic* and *dynamic* (respectively) functors of the system which they define. The simple structure of the category \mathcal{S} implies that

(3.1.7) *there is exactly one dynamic map $\delta(\sigma, \sigma')$ from $\kappa(\sigma)$ to $\kappa(\sigma')$,*

$$(3.1.8) \quad \delta(\sigma, \sigma) = 1 \quad (\text{the identity map}) ,$$

and

$$(3.1.9) \quad \delta(\sigma_1, \sigma_3) = \delta(\sigma_2, \sigma_3) \circ \delta(\sigma_1, \sigma_2) .$$

Of these, (3.1.7) is the *principle of determinacy*.

Turning to kinematic functors we remark that these involve purely geometric mappings. To anyone acquainted with fiber bundles, the mappings $\kappa(g)$ for $g \in \mathcal{S}$ will be automatically suggested when the objects $\kappa(\sigma)$ have been identified. The following example support this.

3.2. The *standard kinematic functor* κ_s . For each $\sigma \in \mathcal{S}$, $\kappa_s(\sigma) = \sigma^n \times B(\sigma^n)$. For $g \in \mathcal{S}$ let $\kappa_s(g)$ be the map that sends

$$(p_1, \dots, p_n, v_1, \dots, v_n)$$

to

$$(gp_1, \dots, gp_n), \psi(g)v_1, \dots, \psi(g)v_n .$$

This functor corresponds to the conception that a state is a list of n points and n velocities, the speeds being less than 1. If on the other hand we conceive a state to be a list of n points and n momenta (usually called covectors) we come to the next example. It involves the dual linear space σ^* of the vector space σ .

3.3. The *phase space functor* κ_{S^*} . For each $\sigma \in \mathcal{S}$, $\kappa_{S^*}(\sigma)$ is $\sigma^n \times (\sigma^*)^n$ while for $g \in \mathcal{S}$,

$$\begin{aligned} \kappa_{S^*}(g)(p_1, \dots, p_n, \alpha', \dots, \alpha^n) \\ = (gp_1, \dots, gp_n, [\psi(g)^*]^{-1}\alpha', \dots, (\psi(g)^*)^{-1}\alpha^n) , \end{aligned}$$

where $\psi(g)^*: \sigma^* \rightarrow \sigma^*$ is the transpose of $\psi(g): \sigma \rightarrow \sigma$.

The notion of Hamiltonian \mathcal{S} -interaction which we will now define is based on the second type of kinematics. We must recall that $\kappa_{S^*}(\sigma)$, being the cotangent bundle of σ^n has an invariantly defined symplectic structure giving rise to (or ‘consisting of’, if preferred) a Poisson bracket $\{, \}_{\sigma^n}$. We call *contact transformation* any \mathcal{C}^∞ map between two cotangent bundles which preserves all bracket relations $f = \{f_1, f_2\}$. In these terms, we will say that an \mathcal{S} -interaction κ, δ is a

$$(3.4) \quad \text{Hamiltonian } \mathcal{S}\text{-interaction}$$

only if $\kappa = \kappa_{S^*}$, while $\delta(\sigma, \sigma')$ is a contact transformation for each pair σ, σ' of members of \mathcal{S} . (See also 3.9 below.)

We return now to the remarks made at the beginning of this section, which were intended to motivate the concept of \mathcal{S} -interaction. We will now show that an \mathcal{S} -interaction does have associated with it an interaction of type (2.7). Ultimately we establish also the converse namely in the presence of certain mild regularity conditions, an interaction of type (2.7) really defines an \mathcal{S} -interaction.

THEOREM 3.5. *Let (κ, δ) be an \mathcal{S} -interaction. Select $\sigma \in \mathcal{S}$ and an x from $k(\sigma)$. Say $\sigma = (p, \mathbf{t})$. For $-\infty < \tau < \infty$ let σ_τ be the parallel translate $(\tau\mathbf{t})(\sigma)$ of σ , and define w_1, \dots, w_n by*

$$w_j(x, \tau) = \pi^j(\pi_{\sigma_\tau}(\delta(\sigma, \sigma_\tau)(x))) .$$

For a fixed x let $W_j(x)$ be the set of all $w_j(x, \tau)$, $\tau \in \mathbf{R}$. Then

$$(3.5.1) \quad \text{under the manifold structure defined by the parameter } \tau, W_j(x) \text{ is a world line in } \mathcal{M} ,$$

(3.5.2) *the collection of n tuples $(W_1(x), \dots, W_n(x))$
for all $x \in \kappa(\sigma)$ is an interaction
 $\mathcal{L}(\kappa, \delta)$, which depends only on κ, δ .*

Proof. The first thing to observe is that $w_j(x, \tau)$ is $\tau t[\pi^j(\varphi(x, \tau))]$ where φ is defined in 3.1.6 in terms of (3.1.3.2). By the regularity condition 3.1.4 we are assured that $w_j(x, \tau)$ depends in a \mathcal{C}^∞ way on τ , whence we may consider the vector $dw_j(x, \tau)/d\tau$. Its component perpendicular to σ is t_\perp . Thus the vector itself is never zero, which ensures that $W_j(x)$ is a submanifold of \mathcal{M} .

We now examine the vector more closely to see that it is timelike. Using (3.1.9) one can easily verify that

$$w_j(x, \tau + \varepsilon) = \bar{w}_j(\bar{x}, \varepsilon)$$

where $\bar{x} = \delta(\sigma, \sigma_\perp)x$ and the \bar{w}_j is just like w_j except that σ has been replaced by σ_\perp . Therefore

$$w_j(x, \tau + \varepsilon) = \varepsilon t[\pi^j(\bar{\varphi}(\bar{x}, \varepsilon)] .$$

At this point it is useful to remain aware of the fact $\varepsilon t[\pi^j(\bar{\varphi}(\bar{x}, \varepsilon)]$ is the displacement of $\pi^j(\bar{\varphi}(\bar{x}, \varepsilon)$, a point of σ_\perp , by a pure translation of magnitude ε in the direction t which is perpendicular to σ_\perp . Therefore the vector

$$\left. \frac{d}{d\varepsilon} w_j(x, \tau + \varepsilon) \right|_{\varepsilon=0}$$

is a sum $t + \bar{v}^j$ where t is perpendicular to σ_\perp and \bar{v}^j lies in σ_\perp and is the vector (3.1.6.1) with σ replaced by σ_\perp . Since 3.1.6 applies to σ_\perp , we must have \bar{v}^j inside the unit ball $B(\sigma_\perp)$. Now t is a time-like unit vector, while \bar{v}^j is perpendicular to it and of length less than 1. Hence the sum is time-like. An appeal to (2.2.3) now completes the proof of (3.5.1).

Let $W(\sigma)$ be the class of all n tuples $(W_1(x), \dots, W_n(x))$ obtainable by letting x range over $\kappa(\sigma)$. We will next show that

$$(3.5.3) \quad W(\sigma) = W(\sigma')$$

whenever $\sigma, \sigma' \in \mathcal{S}$. We can be more explicit and show that

$$(3.5.4) \quad w_j(x, \tau) = w_j(\delta(\sigma, \sigma')x, \tau_j) \quad \text{for some } \tau_1, \dots, \tau_n$$

whenever τ, σ, σ' and $x \in \kappa(\sigma)$ are given.

Select a translate $\sigma_j = \sigma'_{\tau_j}$ of σ' which goes through the point

$$w_j(x, \tau) = \pi^j(\pi_{\sigma_\perp} \delta(\sigma, \sigma_\perp)x) .$$

By applying the world line condition to σ_i and σ_j we obtain

$$\begin{aligned} w_j(x, \tau) &= \pi^j(\pi_{\sigma_j} \delta(\sigma_i, \sigma_j) \delta(\sigma, \sigma_i) x) \\ &= \pi^j(\pi_{\sigma_j} \delta(\sigma, \sigma_j) x) \\ &= \pi^j(\pi_{\sigma_j} \delta(\sigma', \sigma_j) \delta(\sigma, \sigma') x) \\ &= w_j(\delta(\sigma, \sigma') x, \tau_j), \end{aligned}$$

which is (3.5.4). Thus (3.5.3) is also established.

Accordingly we may denote $W(\sigma)$ by $\mathcal{I}(\kappa, \delta)$. It remains to show that it is an interaction (see 2.7). To do so, consider a $\sigma \in \mathcal{I}$. We use (3.5.3) in noting that $\mathcal{I}(\kappa, \delta)$ is $W(\sigma)$ for that σ . The initial conditions (2.6.1), (2.6.2) are mapped, via the inverse of the map Φ_σ of 3.1.6 into a point x of $\kappa(\sigma)$. As will be expected, $(W_1(x), \dots, W_n(x))$ is a set of world lines satisfying (2.6.3), (2.6.4). This can be readily verified from (3.1.6.1) and the observation already used before, that $w_j(x, \tau) = \tau t[\pi^j \varphi(x, \tau)]$. Thus this x provides a set of world lines of the desired sort. No other element y of $\kappa(\sigma)$ will do because then the initial conditions $\Phi_\sigma(y)$ would be different.

This concludes the proof of 3.5.

Now we consider the converse. Suppose \mathcal{I} is an interaction. We are going to make an \mathcal{I} -interaction out of \mathcal{I} . For the kinematic functor we choose (3.2), so that $\kappa(\sigma)$ is $\sigma^n \times B(\sigma)^n$. For the dynamic functor δ , we proceed as follows. Suppose σ and σ' are elements of \mathcal{I} . Suppose

$$(3.5.5) \quad x = (p_1, \dots, p_n, v_1, \dots, v_n)$$

and

$$(3.5.6) \quad x' = (p'_1, \dots, p'_n, v'_1, \dots, v'_n)$$

are elements of $\kappa(\sigma)$ and $\kappa(\sigma')$ respectively. We will say that they are $\delta_{\mathcal{I}}(\sigma, \sigma')$ -related if there is an element (W_1, \dots, W_n) in \mathcal{I} such that (2.6.3) and (2.6.4) hold for σ as well as for σ' . From (2.7) it follows that $\delta_{\mathcal{I}}(\sigma, \sigma')$ is a one-to-one mapping of $\kappa(\sigma)$ onto $\kappa(\sigma')$. Moreover, (3.1.7)–(3.1.9) hold. Even (3.1.6) holds. However, the differentiability requirements on \mathcal{I} are not nearly enough to ensure such global properties as (3.1.4), or even to ensure that $\delta(\sigma, \sigma')$ is a \mathcal{C}^∞ map. Accordingly we make a definition.

3.6. An interaction \mathcal{I} will be called *regular* only if

$$(3.6.1) \quad \delta_{\mathcal{I}}(\sigma, \sigma') \text{ is a } \mathcal{C}^\infty \text{ homeomorphism for each pair } \sigma, \sigma' \text{ of elements of } \mathcal{I},$$

and

$$(3.6.2) \quad (\kappa_\sigma, \delta_{\mathcal{I}}) \text{ satisfies (3.1.4).}$$

THEOREM 3.7. *Let \mathcal{F} be a regular interaction. Then $(\kappa_S, \delta_{\mathcal{F}})$ is an \mathcal{S} -interaction.*

Proof. (3.1.1), (3.1.2) hold for κ_S . (3.1.3) holds because of (3.6.1) and (3.1.4) because of (3.6.2). (3.1.6) is obvious. There remains only (3.1.5), whose description as the world line condition seems to make it obvious. $\pi^j(\pi_\sigma(x))$ is where the j -th curve in some (W_1, \dots, W_n) in \mathcal{F} hits σ . $\pi^j(\pi_{\sigma'}(x))$ is where it hits σ' . Thus (3.1.5) is indeed obvious.

PROPOSITION 3.8. *Let \mathcal{F} be a regular interaction. Form $(\kappa_S, \delta_{\mathcal{F}})$. Then the interaction $\mathcal{F}(\kappa_S, \delta_{\mathcal{F}})$ (defined in 3.5) coincides with \mathcal{F} .*

The proof requires no techniques not already exhibited and hence may be omitted.

We come to a final definition for this section, whose natural place was between 3.2 and 3.4 but which has been postponed in order that we can use some notation needed in any case. In an interaction (κ, δ) in which $\kappa = \kappa_S$ and therefore $\kappa(\sigma)$ is the space of positions and velocities, one would expect that if we start with a point (say (3.5.5) for example) that the velocity of the j -th corresponding path (3.1.6.1) should be v_j . Of course this is not implied in ‘standard kinematic functor’ as the definition does not involve any \mathcal{S} -interaction. Consequently the following satisfies a real need, and forms a natural completion of the three concepts 3.2, 3.3, 3.4.

DEFINITION 3.9. Let (κ, δ) be an \mathcal{S} -interaction in which κ is κ_S (see 3.2) and for which Φ_σ (in condition (3.1.6.1)) is the identity map for each σ . Then we may call (κ, δ) a *standard \mathcal{S} -interaction*.

4. Invariant interactions.

DEFINITION 4.1. An interaction \mathcal{F} will be called *invariant* (more fully, Lorentz invariant, relativistically invariant, or perhaps better Poincaré invariant) only if for each g in \mathcal{P} and for $(W_1, \dots, W_n) \in \mathcal{F}$ one has also $(gW_1, \dots, gW_n) \in \mathcal{F}$ (cf. [2, p. 1346]).

What property of $(\kappa_S, \delta_{\mathcal{F}})$ is the counterpart of 4.1?

THEOREM 4.2. *An interaction \mathcal{F} is invariant if and only if for all σ, σ' in \mathcal{S} and g in \mathcal{P} the following diagram is commutative:*

$$\begin{array}{ccc}
 \kappa_S(\sigma) & \xrightarrow{\delta_{\mathcal{F}}(\sigma, \sigma')} & \kappa_S(\sigma') \\
 \kappa_S(g) \Big| & & \Big| \kappa_S(g) \\
 \kappa_S(g\sigma) & \xrightarrow{\delta_{\mathcal{F}}(g\delta, g\delta')} & \kappa_S(g\sigma') .
 \end{array}$$

Proof. Assume that \mathcal{I} is invariant. Assume that x in $\kappa_S(\sigma)$ and x' in $\kappa_S(\sigma')$ are $\delta_{\mathcal{I}}$ related. This means that the statements made about (3.5.5), (3.5.6) hold in the present context. Now $(gW_1, \dots, gW_n) \in \mathcal{I}$ so that

$$(4.2.1) \quad (g(p_1), \dots, g(p_n), \psi(g)(v_1), \dots, \psi(g)(v_n))$$

and

$$(4.2.2) \quad (g(p'_1), \dots, g(p'_n), \psi(g)(v'_1), \dots, \psi(g)(v'_n))$$

are also $\delta_{\mathcal{I}}$ related. The commutativity of the diagram becomes apparent when it is recognized from 3.2, that (4.2.1) is $\kappa_S(g)x$ and (4.2.2) is $\kappa_S(g)x'$.

The converse argument, from the commutativity of the diagram deducing the invariance of \mathcal{I} is an immediate consequence of 3.8 and Proposition 4.4 below. Hence we may regard 4.2 as established.

DEFINITION 4.3. An \mathcal{I} -interaction (κ, δ) will be called *invariant* only if for all σ, σ' in \mathcal{S} and g in \mathcal{P} the following diagram is commutative [3, 2.7, 2.8]

$$\begin{array}{ccc} \kappa(\sigma) & \xrightarrow{\delta(\sigma, \sigma')} & \kappa(\sigma') \\ \kappa(g) \Big| & & \Big| \kappa(g) \\ \kappa(g\sigma) & \xrightarrow{\delta(g\sigma, g\sigma')} & \kappa(g\sigma') \end{array} .$$

PROPOSITION 4.4. *Let (κ, δ) be an invariant \mathcal{I} -interaction. Then $\mathcal{I}(\kappa, \delta)$ is also invariant.*

Proof. Copy down the diagram of 4.3 with $\sigma' = \sigma_{\tau}$. It is important that then $g\sigma'$ is $(g\sigma)_{\tau}$. The commutativity now says that

$$\delta(g\sigma, (g\sigma)_{\tau})\kappa(g) = \kappa(g)\delta(\sigma, \sigma_{\tau}) .$$

Evaluate these mappings on an x of $\kappa(\sigma)$, and to the result apply $\pi(g\sigma)_{\tau}$. Taking into account the general formula

$$w(x, \tau) \equiv (w_1(x, \tau), \dots, w_n(x, \tau)) = \pi_{\sigma_{\tau}}(\delta(\sigma, \sigma_{\tau})x)$$

applied to the special case where σ is replaced by $g\sigma$ and x by $\kappa(g)x$, we obtain on the left hand side,

$$w(\kappa(g)x, \tau) .$$

On the right side we obtain

$$\pi_{(g\sigma)_{\tau}}(\kappa(g)\delta(\sigma, \sigma_{\tau})x) .$$

Using the diagram of 3.1.2 this can be rewritten as $g^n \pi_{\sigma_\tau} \delta(\sigma, \sigma_\tau)x$, where we have used again that $g^{-1}((g\sigma)_\tau)$ is σ_τ . Thus the right side is $g^n w(x, \tau)$. This shows that

$$(w_1(\kappa(g)x, \tau), \dots, w_n(\kappa(g)x, \tau)) \text{ is } (gw_1(x, \tau), \dots, gw_n(x, \tau)) .$$

Now the former of these belongs to $W(g\sigma)$ and thus to $\mathcal{F}(\kappa, \delta)$, its transform under g is also in $\mathcal{F}(\kappa, \delta)$. This is, of course, the assertion of 4.4, according to 4.1.

THEOREM 4.5. *An \mathcal{S} -interaction (κ, δ) is invariant if and only if for every $\sigma \in \mathcal{S}$ and any two, g, h in \mathcal{S} one has (see (3.1.3.1))*

$$(4.5.1) \quad U_\sigma(gh) = U_\sigma(g)U_\sigma(h) .$$

Proof. Upon inserting h for g in the diagram, and replacing σ by $h^{-1}\sigma$ and σ' by $h^{-1}g^{-1}\sigma$, one obtains

$$\kappa(h)\delta(h^{-1}\sigma, h^{-1}g^{-1}\sigma) = \delta(\sigma, g^{-1}\sigma)\kappa(h) .$$

Now let us expand.

$$\begin{aligned} U_\sigma(gh) &= \kappa(gh)\delta(\sigma, h^{-1}g^{-1}\sigma) \\ &= \kappa(g)\kappa(h)\delta(h^{-1}\sigma, h^{-1}g^{-1}\sigma)\delta(\sigma, h^{-1}\sigma) \\ &= \kappa(g)\delta(\sigma, g^{-1}\sigma)\kappa(h)\delta(\sigma, h^{-1}\sigma) \\ &= U_\sigma(g)U_\sigma(h) . \end{aligned}$$

This establishes the ‘only if’.

Retracing these steps gives the commuting of the diagram at least for pairs σ, σ' which are related in the manner of $h^{-1}\sigma, h^{-1}g^{-1}\sigma$. By 2.5.1, this takes care of every possible pair σ, σ' and 4.5 is proved.

A mapping of the type (3.1.3.2) for which formula (4.5.1) holds is technically called an *action* of the group in question, here denoted by \mathcal{S} . Thus 4.5 says that if (κ, δ) is invariant, then the corresponding U_σ is an action. Whether we have invariance or not, when U_σ is restricted to \mathcal{E}_σ , we do have an action—this is nothing more than the functorial property of κ . The action of \mathcal{E}_σ in $\kappa(\sigma)$ is generally nontrivial.

PROPOSITION 4.5.2. *When (κ, δ) is invariant for each $\sigma \in \mathcal{S}$, U_σ defines an action of \mathcal{S} in $\kappa(\sigma)$ which is an extension of the action of \mathcal{E}_σ in $\kappa(\sigma)$.*

This proposition (just proved) says roughly, “kinematic functor + invariant dynamics yields action of \mathcal{S} extending the action of \mathcal{E}_σ .”

We now prove that kinematic functor + action of \mathcal{P} extending action of \mathcal{E}_σ gives rise to an invariant dynamics. It is to be noted that we require this action for only one σ .

THEOREM 4.6. *Let κ be a kinematic functor. Let $\sigma \in \mathcal{S}$ be fixed. Suppose there is a \mathcal{C}^∞ action*

$$(4.6.1) \quad V: \mathcal{P} \times \kappa(\sigma) \longrightarrow \kappa(\sigma)$$

which extends the action of \mathcal{E}_σ :

$$(4.6.2) \quad g \in \mathcal{E}_\sigma \text{ implies } V(g) = \kappa(g).$$

Suppose also that

$$(4.6.3) \quad \begin{aligned} &\text{if } x \in \kappa(\sigma) \text{ and } g(\pi^j(\pi_\sigma x)) \in \sigma \text{ then} \\ &g(\pi^j(\pi_\sigma x)) = \pi^j(\pi_\sigma V(g)x). \end{aligned}$$

The final condition involves the t for which $\sigma = \sigma(p, t)$. Suppose that the mapping $\Phi_V: \kappa(\sigma) \rightarrow \sigma^n \times \sigma^n$ defined by

$$\Phi_V(x) = (\pi^1(\pi_\sigma x), \dots, \pi^n(\pi_\sigma x), v^1, \dots, v^n)$$

where

$$(4.6.4) \quad v^j = \frac{d}{d\tau} \{ \pi^j(\pi_\sigma(V(-\tau t)x)) \}_{t=0},$$

is a \mathcal{C}^∞ homeomorphism onto $\sigma^n \times B(\sigma^n)$. Then there exists a functor δ such that (κ, δ) is an invariant \mathcal{S} -interaction with $U_\sigma = V$.

Proof. Suppose $g\sigma = h\sigma$ for $g, h \in \mathcal{P}$. Then $\kappa(h^{-1}g) = V(h^{-1}g)$ by (4.6.2). From (4.6.1) we obtain $\kappa(g)V(g^{-1}) = \kappa(h)V(h^{-1})$. Thus $\delta(\sigma, \sigma') = \kappa(g)V(g^{-1})$ for any g such that $g\sigma = \sigma'$ defines $\delta(\sigma, \sigma')$ unambiguously for every $\sigma' \in \mathcal{S}$. For any other pair σ_1, σ_2 from \mathcal{S} define $\delta(\sigma_1, \sigma_2)$ as $\delta(\sigma, \sigma_2)\delta(\sigma, \sigma_1)^{-1}$. This assures 3.1.7-3.1.9 and makes δ a functor on \mathcal{S} .

We consider now the condition 3.1.4. We have it of course for the special σ . For $\sigma' = h\sigma$ we obtain from 3.1.3.1

$$U_{\sigma'}(g) = \kappa(h)V(h^{-1}gh)\kappa(h^{-1}).$$

From this, 3.1.4 is evident.

The two unused hypotheses (4.6.3), (4.6.4) are needed only to establish the world line and second order conditions (3.1.5), (3.1.6). In fact, (4.6.4) is itself the explicit assertion of (3.1.6) for the case of the special σ of our theorem. To derive from it (3.1.6) for other space-like sections requires only the use of the functorial properties of κ , and we may omit it. We turn to the world line condition.

Suppose the σ and σ' of (3.1.5) are $g\sigma$ and $h\sigma$. We may take the x there to be $\kappa(g)y$, $y \in \kappa(\sigma)$. So the hypothesis of (3.1.5) is that $\pi^j(\pi_{g\sigma}\kappa(g)y) \in h\sigma$. Using 3.1.2 we obtain $h^{-1}(\pi^j\pi_\sigma y) \in \sigma$. From (4.6.3) we obtain

$$h^{-1}g(\pi^j\pi_\sigma y) = \pi^j\pi_\sigma V(h^{-1}g)y$$

or

$$(4.6.5) \quad g(\pi^j\pi_\sigma y) = h\pi^j\pi_\sigma V(h^{-1}g)y .$$

Now 3.1.5 requires us to show that

$$(4.6.6) \quad \pi^j(\pi_{g\sigma}\kappa(g)y) = \pi^j(\pi_{h\sigma}\delta(g\sigma, h\sigma)\kappa(g)y) .$$

From 3.1.2 it follows that the left side in (4.6.6) is the left side in (4.6.5). We evaluate the right side of (4.6.6). Now

$$\delta(g\sigma, h\sigma) = \delta(\sigma, h\sigma)\delta(g\sigma, \sigma) = \kappa(h)V(h^{-1})V(g)\kappa(g^{-1}) ,$$

so the right side is

$$\pi^j(\pi_{h\sigma}\kappa(h)V(h^{-1}g)y)$$

which, by 3.1.2, is exactly the right side of (4.6.5). Thus 3.1.5 is established. It is clear that $U_\sigma = V$, and by 4.5 the (κ, δ) is invariant. Our proof of 4.6 is thus complete.

On the basis of 4.6, we are justified in making a definition as follows.

DEFINITION 4.7. Let κ be a kinematic functor, and let V be a \mathcal{C}^∞ action satisfying (4.6.1)-(4.6.4), relative to some $\sigma \in \mathcal{S}$. Then (κ, V) may be called an *invariant σ -interaction*.

Note that we do not define ‘ σ -interaction’ and ‘invariant’ separately. The reason is that without the property $V(gh) = V(g)V(h)$, which insures invariance, one cannot even define a noninvariant dynamic functor in terms of κ and V .

Now we want to characterize those invariant σ -interactions which give rise to invariant standard \mathcal{S} -interactions, (3.1.9). In order to facilitate the application of such a characterization we will examine the form taken by (4.6.3) and (4.6.4) when the special properties of κ_s (3.2) are taken into account.

PROPOSITION 4.7.1. *If $\kappa = \kappa_s$ then (4.6.3) is equivalent to: Suppose*

$$V(g)(p_1, \dots, p_n, v_1, \dots, v_n) = (q_1, \dots, q_n, w_1, \dots, w_n) .$$

In case $gp_j \in \sigma$ for some j then $gp_j = q_j$, for that j .

As to (4.6.4), the problem is not merely to satisfy it, but to insure that Φ_v is the identity map, as 3.9 will unavoidably require. What it comes down to is obviously the following. For

$$(4.7.2) \quad x = (p_1, \dots, p_n, v_1 \dots v_n) \in \sigma^n \times B(\sigma)^n$$

where $\sigma = \sigma(p, t)$, if $V(\tau t)(x) = (p_1(\tau), \dots, p_n(\tau), \dots)$ then

$$\left. \frac{d}{d\tau} p_j(\tau) \right|_{\tau=0} = -v_j .$$

DEFINITION 4.8. Suppose (κ, V) is an invariant σ -interaction in which $\kappa = \kappa_S$ while (4.7.1), (4.7.2) hold. Then (κ, V) may be called a *standard invariant σ -interaction*.

DEFINITION 4.8.1. Suppose (κ, V) is an invariant σ -interaction in which $\kappa = \kappa_{S^*}$ (see 3.3) and $V(g)$ for each $g \in \mathcal{S}$ is a contact transformation. Then we may say that (κ, V) is a *Hamiltonian invariant σ -interaction*.

THEOREM 4.9. Let $\sigma \in \mathcal{S}$ and suppose (κ, V) is either a standard or Hamiltonian σ -interaction. Then there exists an invariant \mathcal{S} -interaction (κ, δ) such that $V = U_\sigma$ and which is standard or Hamiltonian, respectively.

The proof for the standard case is perfectly clear. In the Hamiltonian case one needs only to observe that (cf. 3.3)

$$(4.9.1) \quad \begin{aligned} \kappa_{S^*}(g) \text{ is a contact transformation, for} \\ \text{each } g \in \mathcal{S} . \end{aligned}$$

This is a classical theorem.⁶ [1, 14.16, 14.17, 14.30] and has nothing to do with the special properties of g .

In the same way one can also prove the following.

PROPOSITION 4.9.2. Let $\sigma \in \mathcal{S}$. Let (κ, δ) be an \mathcal{S} -interaction. Then it is either standard invariant or Hamiltonian invariant if and only if (κ, U_σ) is a standard invariant or Hamiltonian invariant σ -interaction, respectively.

5. Reduction and decomposition of interactions. Consider the free interaction (2.9). It is obvious that this is a composition of in-

⁶ Roughly "point transformations always define contact transformation."

dividual ($n = 1$) interactions. This concept will now be made precise.

Let \mathcal{J}_1 be a second-order n_1 particle interaction (see 2.7). Let \mathcal{J}_2 be a second-order n_2 particle interaction. Let \mathcal{J} be the class of all $(W_1, \dots, W_{n_0}, W_{n_1+1}, \dots, W_{n_1+n_2})$ where

$$(W_1, \dots, W_{n_1}) \in \mathcal{J}_1$$

and

$$(W_{n_1+1}, \dots, W_{n_1+n_2}) \in \mathcal{J}_2.$$

PROPOSITION. *This \mathcal{J} is a second-order $n_1 + n_2$ particle interaction.*

Proof. The verification (2.6)–(2.6.4) is elementary.

DEFINITION 5.1. \mathcal{J} may be denoted by $\mathcal{J}_1 \oplus \mathcal{J}_2$ and called the *composition* of \mathcal{J}_1 and \mathcal{J}_2 . Conversely, an interaction \mathcal{J} of this type may be called *decomposable*. This definition may be extended to the case in which the (W_1, \dots, W_{n_1}) are interspersed in some more general way among some permutation of the elements of \mathcal{J}_2 rather than merely placed in front.

DEFINITION 5.2. Let \mathcal{J} be a second-order n particle interaction. Suppose for some $k < n$ there is some subset i_1, \dots, i_k of the integers $1, 2, \dots, n$ such that if we form the class of k -tuples $(W_{i_1}, \dots, W_{i_k})$ which can be lifted out of the n -tuples (W_1, \dots, W_n) in \mathcal{J} then these k -tuples from a second-order k particle interaction \mathcal{J}^1 , then \mathcal{J}^1 may be called a *constituent* of \mathcal{J} , and \mathcal{J} may be called *reducible*.

Evidently a decomposable interaction is reducible. These definitions suggest two questions.

5.2.1. *Is a reducible invariant second order interaction necessarily decomposable?*

5.2.2. *Are there any irreducible invariant second order n particle interactions with $n > 1$?*

We will show that the answers are ‘no’ and ‘yes’, in that order. The second question forms the subject of a paper by Wigner and van Dam [12]. However, it is not clear from their exposition whether the affirmative given there applies to interactions of the specific sort considered by us. We therefore give our own examples. We begin by answering 5.2.1, introducing a concept which will be useful also for answering

⁷ or, the *direct sum*.

5.2.2, to be called a *geodesic helix* in R^k . The case $k = 4$ is the one of interest to us.

Think of R^k as $E + R$ where E is R^{k-1} with the usual Cartesian metric. It is clear what the orthogonal group in E is, which we will denote by $O(E)$ for the moment. We shall be wanting to introduce new Riemannian metrics ρ into E where

$$(5.3) \quad \rho \text{ is } O(E) \text{ invariant ,}$$

(5.3.1) but the geodesics relative to ρ are not merely straight lines .

To be specific, consider the “paraboloid of revolution”

$$S_\varepsilon: x_x = \varepsilon(x_1^2 + \cdots + x_{k-1}^2)$$

in R^k . Here ε is a real parameter. Now E is the hyperplane, $x_k = 0$. There is an obvious projection of S_ε on E , and this projection becomes an isometry if the metric of S_ε induced thereon by the Euclidean metric of R^k is carried down into a metric $\rho(\varepsilon)$ for E . Note that when $\varepsilon \neq 0$, the metric $\rho(\varepsilon)$ satisfies (5.3), (5.3.1), while $\rho(0)$ is just the Cartesian metric of E . Moreover

$$(5.3.2) \quad \rho(\varepsilon) \text{ depends analytically on } \varepsilon .$$

Suppose now that ρ is any Riemannian metric satisfying (5.3). Let $(f_1(\tau), \cdots, f_{k-1}(\tau))$ describe a complete geodesic in E (relative to ρ), so parametrized that

$$(5.3.3) \quad f_1'(\tau)^2 + \cdots + f_{k-1}'(\tau)^2 \text{ is a constant } a^2 .$$

Choose a number b such that a, b are not both zero. Then let Γ be the curve in R^k described by

$$(5.3.4) \quad (f_1(\tau), \cdots, f_{k-1}(\tau), b\tau) .$$

This Γ may be called a *geodesic helix in R^k (relative to ρ)*. It may be called *time-like* if $b/a > 1$.

Now let \mathcal{M} be a general affine Lorentzian space and let T be an affine Lorentzian map of R^4 onto \mathcal{M} , where we have in mind that affine Lorentzian structure in R^4 associated in the familiar way with the quadratic form $-x_1^2 - x_2^2 - x_3^2 + x_4^2$. Let Γ be a geodesic helix in R^4 . Then the curve $W = T(\Gamma)$ may be called a *geodesic helix in \mathcal{M} about the line L* (L being the image under T of the x_4 axis.) When needed the phrase “relative to ρ ” may be affixed, as well as “time-like”, if Γ is time-like.

A special case of a time-like geodesic helix about the line L is a line W parallel to L . This highly desirable limiting case is that at

which $a = 0$ in (5.3.3).

In the rest of this section, the ρ shall be one of the $\rho(\varepsilon)$ already described, so that we are not pretending to make deep assertions about geodesics in general.

PROPOSITION 5.4. *Let (p, t) be a point of $\mathcal{M} \times \mathcal{M}$ (see 2.2), where t is time-like. Let L be any time-like line in \mathcal{M} . Then there is exactly one time-like geodesic helix W about L in \mathcal{M} which passes through p and whose tangent there is t .*

Proof. Select any Lorentzian map T which maps the x_4 axis onto L , and maps E onto a set σ_1 which contains p . Then the inverse image u of t under T is a vector in R^4 which may be resolved into its E component v and an x_4 component b . Let the length of v be a . Now there is exactly one geodesic in E , passing through $T^{-1}(p)$, with tangent v at that point, and it can be parametrized in just one way if (5.3.3) is to hold and the sense of the parameter is to agree with the sense of v . Using the b mentioned we form Γ and let $W = T(\Gamma)$. This W is the desired object. There remains the question of its uniqueness, for the map T is not unique. However, different T are related by $O(E)$ and so (5.3) assures the uniqueness of W .

THEOREM 5.5. *There exists a reducible yet indecomposable invariant second order binary interaction.*

Proof. This interaction shall be called \mathcal{J}_ε and requires first that an ε be chosen, $\varepsilon > 0$ (otherwise the result will be \mathcal{J}_0). With the corresponding $\rho(\varepsilon)$ in mind, we let \mathcal{J}_ε consist of all pairs (W_1, W_2) where W_1 is a time-like straight line in $\mathcal{M} = R^4$ endowed with the usual affine Lorentzian structure, and W_2 is a time-like geodesic helix in \mathcal{M} about W_1 .

We must now test the Definition 2.7. Let σ be given, and also p_1, p_2 in σ and v_1, v_2 in σ , as (2.6.1) and (2.6.2) require us to consider. Supposing that σ is $\sigma(p, t)$ we construct the vectors u_1, u_2 where $u_i = v_i + t$ (compare 2.3). We now abandon the original σ and work with $\sigma_1 = \sigma(p_2, u_1)$, which passes through p_2 , but is perpendicular to u_1 . For W_1 we take the time-like line L through p_2 and tangent to u_2 . Thus there is one and only one pair (W_1, W_2) in \mathcal{J}_ε satisfying the required "initial" conditions.

This \mathcal{J}_ε is surely reducible because the W_1 by themselves form the free 1 particle interaction. If it were decomposable, the same would be true of the W_2 , but some of these (since $\varepsilon \neq 0$) are not straight lines, because of (5.3.1). (See 2.9.) Thus 5.5 is proved and 5.2.1 is answered.

We now proceed to answer 5.2.2.

THEOREM 5.6. *There exists an irreducible invariant second order binary interaction.*

For this example we need to select a \mathcal{C}^∞ real valued function η of a real variable such that

$$(5.6.1) \quad \eta(1) \neq 0, \quad \eta(3) \neq 0$$

$$(5.6.2) \quad \eta \text{ vanishes in a neighborhood of } 2.$$

We will call the interaction \mathcal{I}_η . To describe the pairs (W_1, W_2) in \mathcal{I}_η we have to observe the following.

5.6.3. Let W be a time-like geodesic helix about L and suppose u_1, u_2 are vectors tangent to L and W respectively, normalized to unit vectors and oriented to belong to C_+ . Then the inner product (u_1, u_2) depends only on L and W .

We will denote it by $L \circ W$. Evidently $L \circ W \geq 1$ and for each L one can pick W so that $L \circ W$ is as large as desired.

Now \mathcal{I}_η shall consist of all pairs (W_1, W_2) where either

$$(5.6.4) \quad \begin{aligned} &W_2 \text{ is a time-like geodesic helix about } W_2 \\ &\text{relative to the metric } \rho(\eta(W_1 \circ W_2)) \text{ and } W_2 \circ W_1 \geq 2. \end{aligned}$$

or

$$(5.6.5) \quad \begin{aligned} &W_1 \text{ is a time-like geodesic helix about } W_2 \text{ relative} \\ &\text{to the metric } \rho(\eta(W_2 \cdot W_1)) \text{ and } W_2 \cdot W_1 \leq 2. \end{aligned}$$

These conditions overlap when $W_1 \cdot W_2 = 2$ but then η is 0 and thus (W_1, W_2) is a pair of free lines.

Let us verify that \mathcal{I}_η has the properties announced in 5.6.3. We begin as for 5.5 and calculate u_1 and u_2 . We evaluate $\Delta = (u_1, u_2)$. If $\Delta \leq 2$ we continue as in the case of 5.5. If $\Delta \geq 2$ we make W_1 be the helix about W_2 .

In this way we have insured that some W_1 is not a free world line and some W_2 is not a free world line. Thus \mathcal{I}_η is an example with the properties claimed.

REMARK 5.6.6. In the conventional terminology of physical explanations, one could say that depending on the initial conditions v_1, v_2 , if $v_1 \cdot v_2 \leq \sqrt{5}$ then the first particle ignores the second while if $v_1 \cdot v_2 \geq \sqrt{5}$ then the second ignores the first.

PROPOSITION 5.7. *The interactions \mathcal{I}_ϵ and \mathcal{I}_η are regular (3.6).*

It is to make sure of this that we stipulated (5.3.2) and that we required η to be a \mathcal{C}^∞ function. The proof of 5.7 is purely technical and we will omit it.

The notions of composition and reduction can be applied to the \mathcal{S} -interactions (see 3.1). Suppose we had two second order \mathcal{S} -interactions (κ, δ) and (κ', δ') where the “particle numbers” are n and n' respectively. Suppose κ' is a subfunctor of κ and δ' is a subfunctor of δ [10, p. 26]. In this case we may say that

$$(5.8) \quad \begin{aligned} (\kappa', \delta') \text{ is a constituent of } (\kappa, \delta) \text{ and} \\ (\kappa, \delta) \text{ is reducible.} \end{aligned}$$

Now suppose (κ_1, δ_1) and (κ_2, δ_2) are two second order \mathcal{S} -interactions as before. Let $\kappa(\sigma)$ be the Cartesian product $\kappa_1(\sigma) \times \kappa_2(\sigma)$ which has a natural projection π_σ on $\sigma^{n_1+n_2}$ which makes it into a fibre space. For g in \mathcal{S} let $\kappa(g) = \kappa_1(g) \times \kappa_2(g)$, and for σ, σ' in \mathcal{S} let $\delta(\sigma, \sigma') = \delta_1(\sigma, \sigma') \times \delta_2(\sigma, \sigma')$. This merely a case of products of functors [10, p. 528].

PROPOSITION 5.9. (κ, δ) satisfies 3.1.1-3.1.6.

The proof is so simple that we omit it. The interaction (κ, δ) may be called

$$(5.9.1) \quad \textit{The product interaction } (\kappa_1, \delta_1) \times (\kappa_2, \delta_2) .$$

If an interaction is a product interaction we may call it *decomposable*.

6. Infinitesimal interactions. Our first task in this section is to obtain infinitesimal counterparts to the properties (4.6.1)-(4.6.4). More fully, if (κ, V) satisfies 4.7, what does (κ, dV) satisfy? The consequences should be as strong as possible, but still of a local character.

To begin with, (4.6.1) implies that for each Z in \mathfrak{p} , the Lie algebra of \mathcal{S} , there is a vector field Z^+ defined by ([7, p. 112]) $Z^+ = dV(Z)$ where

$$(Z^+f)(x) = \lim_{t \rightarrow 0} \frac{f(V(\exp \tau Z)x) - f(x)}{\tau}$$

for each $f \in \mathcal{C}^\infty(\kappa(\sigma))$ and each x in $\kappa(\sigma)$, and

$$(6.0.1) \quad \begin{aligned} dV \text{ is a Lie algebra homomorphism of } \mathfrak{p} \text{ into the Lie} \\ \text{algebra } \mathfrak{X}(\kappa(\sigma)) \text{ of } \mathcal{C}^\infty \text{ vector fields on } \kappa(\sigma) . \end{aligned}$$

Corresponding to (4.6.2) we have

$$(6.0.2) \quad dV|_{e_\sigma} = d\kappa .$$

It will surely be granted that the infinitesimal analogue of (4.6.3) is that if the curve $(\exp \tau Z)(\pi^j(\pi_\sigma x))$ starts off by being tangent to σ , then the (geometric) velocity of that curve with respect to the parameter τ , should be the same, at $\tau = 0$, as the velocity of the curve $\pi^j(\pi_\sigma V(\exp \tau Z)x)$. This analogue is in fact a consequence of (4.6.3), in the presence of the regularity assumption in 4.6. It merely remains to state it in terms of dV .

The velocity of the second curve is

$$d\pi^j(d\pi_\sigma(dV(Z)|_x))$$

where $dV(Z)|_x$ is simply the notation for the vector attached to x by the field $dV(Z)$, while $d\pi_\sigma$ is the projection of vectors in $\kappa(\sigma)$ to vectors in $\sigma \times \dots \times \sigma$ and $d\pi^j$ gives the component in the j -th factor. The velocity of the first curve is

$$dW(Z)|_{\pi^j(\pi_\sigma x)}$$

where we have adopted W as the name of the action of \mathcal{P} in \mathcal{M} :

$$W(g, p) = W(g)(p) = gp .$$

It seems a pity to bring in such a letter, so we merely refer to $dW(Z)$ as the image of Z in \mathcal{M} . We have arrived at the following.

(6.0.3) Suppose that the image of Z in \mathcal{M} is tangent to σ at the point $\pi^j(\pi_\sigma x)$. Then the image of Z in \mathcal{M} at that point coincides with $d\pi^j(d\pi_\sigma(dV(Z)|_x))$.

The final condition, (4.6.4) is itself infinitesimal in form. Let us use T_σ as the generator of the one parameter time-like translation group in question, $\exp \tau T_\sigma = \tau t$, where $\sigma = \sigma(p, t)$.

(6.0.4) The map Φ_V by which

$$\Phi_V(x) = (\pi^1(\pi_\sigma x), \dots, v^1, \dots) ,$$

wherein

$$v^j = d\pi^j(-dV(T_\sigma)|_x) ,$$

is a \mathcal{C}^∞ homeomorphism of $\kappa(\sigma)$ onto $\sigma^n \times B(\sigma)^n$.

DEFINITION 6.1. Let $\sigma \in \mathcal{S}$ and let κ be a kinematic functor. Suppose (κ, V') satisfy 6.0.1-6.0.4 with dV replaced by V' . Then (κ, V') may be called an *infinitesimal invariant (n particle σ -)interaction*.

Our definition was intended to make obvious the following, of course.

PROPOSITION 6.2. *If (κ, V) is an invariant σ -interaction, then (κ, dV) is an infinitesimal invariant interaction.*

In order to define infinitesimal invariant *standard* interactions fairly, we have to bear in mind the comment made above, just prior to (4.7.2). In fact (4.7.2) says

$$(6.2.1) \quad \text{for } x = (p_1, \dots, p_n, v_1, \dots, v_n) \in \kappa_S(\sigma), \sigma = \sigma(p, t),$$

one has

$$(6.2.2) \quad dV(-T_\sigma)|_x = (v_1, \dots, v_n, A_1, \dots, A_n).$$

There is a special name for vector fields on tangent bundles which have this property, more abstractly that

$$d\pi(W|_x) = x.$$

They are called *basic* [11], so we reformulate (6.2.1) as

$$(6.2.3) \quad dV(-T_\sigma) \text{ is a basic vector field in } \sigma^n \times B(\sigma)^n = \kappa_S(\sigma).$$

If an infinitesimal invariant interaction (κ, V') has $\kappa = \kappa_S$, and if (6.3.2) holds (in addition to (6.0.1) to (6.0.4) with V' inserted for dV), then we may call (κ, V') a *standard infinitesimal invariant interaction* or S.I.I.I.

We will now explore this concept by means of coordinates. Choose a Lorentzian coordinate system (t, x^1, x^2, x^3) so that σ is the hyperplane on which $t = 0$. This allows us to make the following identifications.

$$(6.3) \quad \sigma = \{p \in \mathcal{M} : t(p) = 0\}.$$

(6.3.1) \mathcal{M} and \mathcal{M} are identified with \mathbf{R}^4 in such a way that for (a^0, a^1, a^2, a^3) and (b^0, b^1, b^2, b^3) in \mathcal{M} we have $a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3$ as the value of the bilinear form for these vectors.

$$(6.3.2) \quad \sigma \text{ and } \sigma \text{ are identified with } \mathbf{R}^3.$$

$$(6.3.3) \quad \text{A point } (a, b, c) \text{ of } \mathbf{R}^3 \text{ is identified with } (0, a, b, c) \text{ in } \mathbf{R}^4.$$

(6.3.4) \mathcal{P} is the usual Poincaré group, generated by the translations and the restricted Lorentz group.

(6.3.5) The generators of \mathfrak{p} or rather their images in \mathcal{M} (see the remark just prior to (6.0.3)) are these ten:

For T_σ we have $\partial/\partial t$. For the x^a translation we have $\partial/\partial x^a$ ($a = 1, 2, 3$). For the rotations in the x^a, x^b plane we have

$$M_{ab} \equiv x^a \frac{\partial}{\partial x^b} - x^b \frac{\partial}{\partial x^a} .$$

For the rotation in the t, x^a plane we have

$$Z_a \equiv x^a \frac{\partial}{\partial t} + t \frac{\partial}{\partial x^a} .$$

Now the ten vector fields $V'(T_a), \dots, V'(Z_3)$ which we will compute in coordinate form are vector fields in $\sigma^n \times B(\sigma^n)$. The latter is a part of R^{6n} and we will denote the cartesian coordinate system by $(x_1, \dots, x_n, u_1, \dots, u_n)$ where x_λ is the triple $(x_\lambda^1, x_\lambda^2, x_\lambda^3)$ and u_λ is $(u_\lambda^1, u_\lambda^2, u_\lambda^3)$ with $\lambda = 1, \dots, n$. (Thus for example u_4^2 is the y component of the velocity of the fourth particle.) We use the summation convention for Roman indices, from 1 to 3. *For Greek indices, repeated or not, we sum from 1 to n , unless otherwise stated.*

PROPOSITION 6.4. *Let (κ, V') be a standard infinitesimal invariant interaction in the situation just described. Then there is a uniquely determined set*

$$\{A_\lambda^c : a = 1, 2, 3; \lambda = 1, 2, \dots, n\}$$

of $3n$ functions such that

$$(6.4.1) \quad V'\left(\frac{\partial}{\partial t}\right) = -u_\lambda^a \frac{\partial}{\partial x_\lambda^a} - A_\lambda^a \frac{\partial}{\partial u_\lambda^a}$$

$$(6.4.2) \quad V'\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial x_\lambda^a}$$

$$(6.4.3) \quad V'(M_{ab}) = x_\lambda^a \frac{\partial}{\partial x_\lambda^b} - x_\lambda^b \frac{\partial}{\partial x_\lambda^a} + u_\lambda^a \frac{\partial}{\partial u_\lambda^b} - u_\lambda^b \frac{\partial}{\partial u_\lambda^a} .$$

$$(6.4.4) \quad V'(Z_c) = -x_\lambda^a u_\lambda^a \frac{\partial}{\partial x_\lambda^a} + (\delta_c^a - u_\lambda^a u_\lambda^c - x_\lambda^a A_\lambda^c) \frac{\partial}{\partial u_\lambda^a} .$$

Proof. (6.4.1) is nothing but (6.2.3) explicitly written out as in (6.2.2). Next, (6.2.4) and (6.4.3) are consequences of (6.0.2). The terms in u reflect the fact that under linear transformations, velocity components transforms as the coordinates.

The task set by (6.4.4) is to discover the b and B in

$$(6.4.5) \quad V'(Z_c) = b_\lambda^a \frac{\partial}{\partial x_\lambda^a} + B_\lambda^a \frac{\partial}{\partial u_\lambda^a} .$$

Choose a point $p = (p^1, p^2, p^3)$ in σ . There is an element of \mathfrak{p} whose image in \mathcal{M} is $Z_c - p^c(\partial/\partial t)$ i.e.,

$$(x^c - p^c) \frac{\partial}{\partial t} + t \frac{\partial}{\partial x^c}$$

and this is evidently tangent to σ at p . Select a j ($1 \leq j \leq n$) and let x be arbitrary except that it shall have (p^1, p^2, p^3) in the (x_j^1, x_j^2, x_j^3) place, respectively. Thus $\pi^j(\pi_\sigma x) = p$ and (6.0.3) applies. It tells us that

$$d\pi^j \left(V' \left(Z_c - p^c \frac{\partial}{\partial t} \right) \Big|_x \right) = 0,$$

because $Z_c - p^c(\partial/\partial t)$ is in fact 0 at p . In general, for a vector W such as that displayed in (6.4.5), $d\pi^j(d\pi_\sigma(W))$ is $b_j^a(\partial/\partial x_j^a)$, with no sum on j . Thus in the case at hand,

$$b_j^a \frac{\partial}{\partial x_j^a} + p^c u_j^a \frac{\partial}{\partial x_j^a} = 0$$

with no sum on j , at the point x . This implies that

$$b_j^a(x_1, \dots, x_{j-1}, p, x_{j+1}, \dots, x_n, u_1, \dots, u_n) = -p^c u_j^a,$$

from which it follows that the first half of the terms of (6.4.4) have been correctly stated.

We now recall that $[\partial/\partial t, Z_c] = \partial/\partial x^c$. It follows from (6.0.1) that

$$\left[V' \left(\frac{\partial}{\partial t} \right), V'(Z_c) \right] = V' \left(\frac{\partial}{\partial x^c} \right).$$

We invite the reader to insert here (6.4.1), (6.4.2), and what we already know about (6.4.5), namely that the b_i^a are as stated in (6.4.4). We ask the reader to calculate the x_j^d component on each side. The equation which results will be

$$u_j^c u_j^d + A_j^d x_j^c - B_j^d = \delta_i^c$$

with no sum on i . This establishes (6.4.4) and thereby concludes the proof of 6.4.

By computing and equating also the u_j^d components in the commutation relation just considered, we obtain the Currie-Hill relations [4, 8].

COROLLARY 6.5. *In a standard I.I.I., the A 's of (6.4.1) must satisfy for each $j(1 \leq j \leq n)$ and each $d(= 1, 2, 3)$ and each $c(= 1, 2, 3)$ the relation (no sum on j)*

$$\begin{aligned}
 & u_j^d A_j^c + 2u_j^c A_j^d + u_j^a(x_j^c - x_j^c) \frac{\partial A_j^d}{\partial x_j^a} \\
 & + (\delta_i^a - u_c^a u_i^c + (x_j^c - x_j^c) A_i^c) \frac{\partial A_j^d}{\partial u_i^a} = 0 .
 \end{aligned}$$

It should be remarked that in spite of the minus signs in (6.4.1), the A_j^d is the d -th component of the acceleration of the j -th particle. This can be proved by tracing the development back to § 2. However, it is plausible on the face of it because the coefficient of $\partial/\partial x_j^a$ in (6.4.1) is minus the time rate of x_j^a , whence the coefficient of $\partial/\partial u_j^a$ should be minus the time rate of u_j^a . The minus sign would not have appeared had one replaced g by g^{-1} in (3.1.3.1), but then the factors on the right of (4.5.1) would have become interchanged.

The relations of 6.5 are not the only ones that the A 's have to satisfy. Others are obtained by exploiting the other commutation relations of \mathfrak{p} . Several of them again yield the relations of 6.5 while the others merely testify to their *Euclidean* invariance.

We shall now discuss how the well-known zero-interaction theorems of the Hamiltonian theory fit into the framework developed in this paper. In particular let (κ_{S^*}, V) be an (invariant n -particle) Hamiltonian σ -interaction. Let (κ_{S^*}, dV) be the corresponding infinitesimal interaction. $dV(Z)$ is clearly and infinitesimal contact transformation on $\kappa_{S^*}(\sigma)$ for each $Z \in \mathfrak{p}$. Since the set $\mathfrak{C}(\kappa_{S^*}(\sigma))$ of infinitesimal contact transformation on $\kappa_{S^*}(\sigma)$ is a Lie sub-algebra of $\mathfrak{L}(\kappa_{S^*}(\sigma))$, dV , is in fact a representation of \mathfrak{p} in $\mathfrak{C}(\kappa_{S^*}(\sigma))$.

In the usual treatments of infinitesimal Hamiltonian interactions, one usually considers a representation of \mathfrak{p} in the Lie algebra $\mathcal{C}^\infty(\kappa_{S^*}(\sigma))$ where the Lie algebra operation in $\mathcal{C}^\infty(\kappa_{S^*}(\sigma))$ is just the Poisson brackt $\{, \}$. We obtain such a representation from (κ_{S^*}, dV) as follows: Pick a Lorentzian coordinate system (t, x^1, x^2, x^3) for \mathcal{M} such that $\sigma = \{p \in \mathcal{M} : t(p) = 0\}$. This induces a coordinate system $(x_1, \dots, x_n, p^1, \dots, p^n)$ [where $x_\lambda = (x_\lambda^1, x_\lambda^2, x_\lambda^3)$ and $p^\lambda = (p_\lambda^1, p_\lambda^2, p_\lambda^3)$] on $\sigma \times (\sigma^*)^n$. We define the Lie algebra homomorphism

$$\xi: \mathcal{C}^\infty(\kappa_{S^*}(\sigma)) \longrightarrow \mathfrak{C}(\kappa_{S^*}(\sigma))$$

by

$$\xi(f) = \frac{\partial f}{\partial p_i^1} \frac{\partial}{\partial x_i^1} - \frac{\partial f}{\partial x_i^1} \frac{\partial}{\partial p_i^1} .$$

It is known that this is surjective. Pick $H, P_a, K_a, (a = 1, 2, 3), J_{ab}, (1 \leq a < b \leq 3 \in \mathcal{C}^\infty(\kappa_{S^*}(\sigma)))$ such that $\xi(H) = dV(T_\sigma), \xi(P_a) = dV(\partial/\partial x^a), \xi(K_a) = dV(Z_a), (a = 1, 2, 3), \xi(J_{ab}) = dV(M_{ab}), (1 \leq a < b \leq 3)$ where $T_\sigma, \partial/\partial x^a, Z_a, (a = 1, 2, 3), M_{ab}, (1 \leq a < b \leq 3)$, are the ten generators for

\mathfrak{p} which were defined in (6.3.5). This is possible since ξ is surjective. Define

$$dV_\xi: \mathfrak{p} \longrightarrow \mathcal{C}^\infty(\kappa_{S^*}(\sigma))$$

by

$$dV_\xi(T_\sigma) = H, \dots, dV_\xi(M_{ab}) = J_{ab} (1 \leq a < b \leq 3)$$

and extend to all of \mathfrak{p} by linearity. Using the fact that the kernel of ξ is \mathbf{R} (the constant functions on $\kappa_{S^*}(\sigma)$) it is straightforward to check that dV_ξ is in fact a Lie algebra homomorphism. The world line condition (6.0.3) and the properties of $\xi(H)$ assure us that the following important property holds for all n -tuples $\{(t, x_1(t)), \dots, (t, x_n(t))\}$ for world lines for $(K_{S^*}V)$:

$$\frac{d^2 X_\lambda^j}{dt^2} = \{X_\lambda^i, \{X_\lambda^j, H\}\}$$

where

$$X_\lambda^j(x_1, \dots, x_n, p^1, \dots, p^n) = x_\lambda^j: j = 1, 2, 3, \lambda = 1, 2, \dots, n.$$

Finally the world line condition (6.0.3) implies that

$$\{X_\lambda^a, K_b\} = X_\lambda^b \{X_\lambda^a, H\}.$$

We now summarize the preceding discussion. Given a Hamiltonian σ -interaction (K_{S^*}, V) and a Lorentzian coordinate system (t, x^1, x^2, x^3) for \mathcal{M} such that $\sigma = \{p \in \mathcal{M} : t(p) = 0\}$, there exists

$$(6.5) \quad H, P_a, K_a, (a = 1, 2, 3) J_{ab}, (1 \leq a < b \leq 3)$$

in $\mathcal{C}^\infty(K_{S^*}(\sigma))$ such that

$$(6.5.1) \quad \{H, P_a\} = 0, \{H, K_a\} = P_a \quad (a = 1, 2, 3)$$

$$(6.5.2) \quad \{H, J_{ab}\} = 0, \{P_b, K_a\} = \delta_b^a H, \quad \{K_a, K_b\} = J_{ab}, (1 \leq a < b \leq 3)$$

$$(6.5.3) \quad \{K_a, J_{bc}\} = \begin{cases} 0 & \text{if } a, b, c \text{ are distinct} \\ K_c & \text{if } a = b \end{cases}$$

$$(6.5.4) \quad \{X_\lambda^a, K_b\} = X_\lambda^b \{X_\lambda^a, H\} \quad (a, b = 1, 2, 3 \text{ and } \lambda = 1, 2, \dots, n)$$

and such that if

$$(6.5.5) \quad \{(t, x_1(t)), \dots, (t, x_n(t))\}$$

is an n -tuple of world lines $(\kappa_{S^*}(\sigma), V)$, then

$$\frac{d^2 X_\lambda^j}{dt^2} = \{X_\lambda^i, \{X_\lambda^j, H\}\}.$$

Now the zero-interaction theorem of Currie, et al [6, 9], says that (6.5.1), (6.5.2), (6.5.3), and (6.5.4) imply that

$$(6.5.6) \quad \{X_\lambda^j, \{X_\lambda^j, H\}\} = 0$$

for all $j = 1, 2, 3, \lambda = 1, 2, \dots, n$. We translate this conclusion in terms of the framework of this paper (see (3.5.2) and 2.9).

THEOREM 6.6. *If (κ_{S^*}, V) is a Hamiltonian (n -particle invariant) interaction then*

$$\mathcal{I}(\kappa_{S^*}, V) = \mathcal{I}_0 \oplus \dots \oplus \mathcal{I}_0 = \mathcal{I}_{0, n} \quad (n \text{ summands}).$$

Proof. From (6.5.5) and (6.5.6) we conclude

$$\frac{d^2 X_\lambda^j}{dt^2} = 0 (j = 1, 2, 3, \lambda = 1, 2, \dots, n)$$

for any

$$\{(t, x_1(t)), \dots, (t, x_n(t))\} \in \mathcal{I}(\kappa_{S^*}, V).$$

This of course says the world lines are straight lines, which is what was to be proved.

It should be remarked that the answer to the following question is not known: if (κ_{S^*}, V) is a σ -interaction such that $V(tt_0)$ is a contact transformation for each real number t^s , is the associated second order interaction $\mathcal{I}(\kappa_{S^*}, V)$ a zero interaction? Since such interactions have Hamiltonians, they presumably could be quantized and thus if nontrivial examples existed they would certainly be worth investigating.

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^s Note that these conditions do *not* imply that $V(g)$ is a contact transformation for all $g \in \mathcal{S}$.

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