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# TORSION IN BBSO

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**The cohomology of BBSO, the classifying space for the stable Grassmanian BSO, is shown to have torsion of order precisely $2^r$ for each natural number $r$. Moreover, the elements of order $2^r$ appear in a pattern of striking simplicity.**

Many of the stable Lie groups and homogeneous spaces have torsion at most of order 2 [1, 3, 5]. There is one such space, however, with interesting torsion of higher order. This is $BBSO = SU/\mathrm{Spin}$ which is of interest in connection with Bott periodicity and in connection with the J-homomorphism [4, 7]. By the notation $SU/\mathrm{Spin}$ we mean that $BBSO$ can be regarded as the fibre of $B\,\mathrm{Spin} \to BSU$ or that, up to homotopy, there is a fibration

$$SU \to BBSO \to B\,\mathrm{Spin}$$

induced from the universal $SU$ bundle by $B\,\mathrm{Spin} \to BSU$. The mod 2 cohomology $H^*(BBSO; Z_2)$ has been computed by Clough [4]. The purpose of this paper is to compute enough of $H^*(BBSO; Z)$ to obtain the mod 2 Bockstein spectral sequence [2] of $BBSO$.

Given a ring $R$, we shall denote by $R[x_i \mid i \in I]$ the polynomial ring on generators $x_i$ indexed by elements of a set $I$. The set $I$ will often be described by an equation or inequality in which case $i$ is to be understood to be a natural number. Similarly $E(x_i \mid i \in I)$ will denote the exterior algebra on generators $x_i$. In this case, we will need only $R = Z_2$.

Let us recall the results on $B\,\mathrm{Spin}$ as given by Thomas [6] and on $BBSO$ as given by Clough [4].

$$H^*(B\,\mathrm{Spin}; Z_2) \approx Z_2[w_i \mid i \neq 2^j + 1]$$

where $w_i$ is (the image of) the Stiefel-Whitney class $w_i$.

$$H^*(B\,\mathrm{Spin}; Z) \approx Z[Q_i \mid i > 0] \oplus \hat{T}$$

where $2\hat{T} = 0$ and $Q_i \in H^{4i}$.

$$H^*(BBSO; Z_2) \approx E(e_i \mid i \geqq 3)$$

where $e_i \in H^i$ and is the image of $w_i$ if $i \neq 2^j + 1$ while $e_{2^j+1}$ maps to an indecomposable element in $H^*(SU; Z_2)$.

Now let ${}_\beta E_r$ denote the mod 2 Bockstein spectral sequence of $BBSO$ [2]. In particular, ${}_\beta E_2 = \mathrm{Ker}\, Sq^1/\mathrm{Im}\, Sq^1$. Now $Sq^1 w_{2i} = w_{2i+1}$ in $BSO$ and $Sq^1 w_{2i+1} = 0$ while $Sq^1 e_{2^j} = 0$ in $B\,\mathrm{Spin}$. We will see that

$e_{2^j+1}$ can be chosen to have $Sq^1 e_{2^j+1} = 0$ except for $Sq^1 e_3 = e_4$. Thus

$$_\beta E_2 = E(e_3 e_4,\ e_{2^2+i},\ v_{4i+1} \mid i > 0)$$

where $v_{4i+1} = e_{2i} e_{2i+1}$ except $v_{2^j+1} = e_{2^j+1};\ j > 1$.

THEOREM 1.

$$_\beta E_r \approx E(e_3 e_4 \cdots e_{2^r},\ e_{2^r+i},\ v_{4i+1} \mid i > 0)$$

*and* $d_r(e_3 \cdots e_{2^r}) = e_{2^r+1}$ *modulo decomposable elements.*

To prove Theorem 1, we will exhibit torsion of order $2^r$ for all $r$.

THEOREM 2. *In* $H^*(BBSO; Z)$, *we have*

$$2^r Q_{2^r} \neq 0 \quad \text{and} \quad 2^{r+1} Q_{2^r} = 0\ .$$

$H^*(BBSO; Z_2)$. We recall some of Clough's observations on $H^*(BBSO; Z_2)$. We know $H^*(SU; Z_2) = E(y_i \mid i > 1)$ where $y_i \in H^{2i+1}$ transgresses universally to the mod 2 reduction of the Chern class $c_i$ and hence to the image of $w_i^2$ in $B$ Spin. Thus $w_i^2 = 0$ in $BBSO$ for $i \neq 2^j + 1$ and $y_{2^j}$ is the restriction of a class $e_{2^{j+1}+1}$. In particular since $Sq^{2^j}(w_{2^{j-1}+1})^2 = (w_{2^j+1})^2$ we can take $e_{2^j+1}$ to be $Sq^{2^{j-1}} Sq^{2^{j-2}} \cdots Sq^4 Sq^2 e_3$. The class $e_3$ is uniquely determined ($H^3(BBSO; Z_2) \approx Z_2$) and this definition of $e_{2^j+1}$ implies $Sq^1 e_{2^{j+1}+1} = (e_{2^i+1})^2 = 0$ if $e_3^2 = 0$. The only alternative to $e_3^2 = 0$ is $e_3^2 = e_6$; there is no other class in this dimension. Since $Sq^1 w_6 = w_7$ in $B$ Spin and $w_6, w_7$ map to $e_6, e_7$, we have $Sq^1 e_6 = e_7$ but $Sq^1(e_3)^2 = 0$; therefore $e_3^2$ must be zero.

$H^*(BBSO, Z)$. Consider $BBSO$ as the fibre of $B$ Spin $\to BSU$. The latter map factors: $B$ Spin $\xrightarrow{\pi} BSO \to BSU$. Recall that

$$H^*(BSU; Z) = Z[c_i \mid i > 1] \quad \text{and} \quad H^*(BSO; Z) = Z[P_i] \oplus T$$

where $T$ is the torsion ideal, $2T = 0$, $c_{2i+1}$ maps into $T$ and $c_{2i}$ maps to $P_i$. To determine Im $(H^*(B\,\text{Spin}))$ in $H^*(BBSO)$, we need to know $\pi^*[P_i]$ in $H^*(B\,\text{Spin})$.

THEOREM 3 (*Thomas* [6]). *If $i$ is not a power of 2,* $\pi^* P_i = Q_i$. *If* $j = 2^r,\ r > 0,\ \pi^* P_{2j} = 2Q_{2j} + Q_j^2 - \pi^* \Phi_{2j}$. $\pi^* P_1 = 2Q_1$.

LEMMA. $\pi^* \Phi_{2j}$ *maps into* Im $T \subset H^*(BBSO)$.

*Proof.* $H^*(BSO; Z)$ maps onto Im $T$ in $H^*(BBSO)$ since $H^*(BSU)$ maps onto the $Z[P_i]$ part.

Since $\pi^* P_j$ goes to zero in $BBSO$, we have in $H^*(BBSO; Z)$

$$2Q_{2j} = -Q_j^2 + t \qquad \text{where } 2t = 0 \text{ and } j = 2^r .$$
$$2Q_1 = 0 .$$

By iteration we find

$$2^{r+1}Q_{2^r} = \pm 2Q_{2^r}Q_{2^{r-1}} \cdots Q_2(Q_1)^2 = 0 .$$

To determine the order of $Q_{2^i}$ in $BBSO$, consider $\Gamma(u \mid 2u = 0)$, a divided polynomial algebra on a single generator $u$ of dimension 4 and order 2; i.e., additively $\Gamma$ has generators $\gamma_i(u)$ in dimension $4i$ and the multiplication table is $\gamma_i(u)\gamma_j(u) = (i, j)\gamma_{i+j}(u)$ where $(i, j)$ is the binomial coefficient $\{(i + j)!/i!j!\}$.

In particular $i!\gamma_i(u) = u^i$.

We construct a map $f$ from $\mathrm{Im}\,(H^*(B\,\mathrm{Spin}; Z) \rightarrow H^*(BBSO; Z))$ to $\Gamma$ by mapping $\hat{T}$ to zero, $Q_i$ to zero for $i \neq 2^j$ and $Q_{2^j}$ to $-\gamma_2(f(Q_{2^{j-1}}))$ with $f(Q_1) = u$. Since $2Q_{2^j} = -Q_{2^{j-1}}^2 + \pi^*\Phi_{2^j}$, and $\Phi_{2^j}$ goes into $\mathrm{Im}\,\hat{T}$ in $BBSO$, the map $f$ is well defined. Since for any $x$, the order of $\gamma_2(x)$ is twice the order of $x$, we have

$$\mathrm{ord}\, f(Q_{2^j}) = 2\, \mathrm{ord}\, f(Q_{2^{j-1}}) = 2^j\, \mathrm{ord}\, f(Q_1) = 2^{j+1} .$$

Thus the order of $Q_{2^j}$ is at least $2^{j+1}$ and that $2^{j+1}Q_{2^i}$ is in fact zero we have already seen.

Thus we have $2^r$ torsion for each $r$. From the exact cohomology sequence derived from $0 \rightarrow Z \xrightarrow{2^r} Z \rightarrow Z_{2^r} \rightarrow 0$, we see that $Q_{2^{r-1}} = \beta_{2^r}^{\infty}x_r$ for some class $x_r \in H^*(BBSO; Z_{2^r})$, where $\beta_{2^r}^{\infty}$ is the connecting homomorphism $H^*(\ ; Z_{2^r}) \rightarrow H^{*+1}(\ ; Z)$.

LEMMA. $(\beta_{2^r}^{\infty}x_r)_2 = d_r(x_r)_2$ *where* $(\ )_2$ *means reduction* mod 2.

*Proof.* Recall how $d_r$ is defined: $d_r(x) = (\beta_2^{\infty}(x)/2^{r-1})_2$. From the commutativity of the diagram

$$\begin{array}{ccccc} Z & \xrightarrow{2^r} & Z & \longrightarrow & Z_{2^r} \\ \downarrow{\scriptstyle 2^{r-1}} & & \| & & \downarrow \\ Z & \xrightarrow{2} & Z & \longrightarrow & Z_2 \end{array}$$

it follows that $\beta_2^{\infty} = 2^{r-1}\beta_{2^r}^{\infty}$. In particular, $d_r(x_r)_2 = (Q_{2^{r-1}})_2$. According to Thomas, $(Q_{2^{r-1}})_2 = \pi^*(w_{2^r+1} + \psi_{2^r+1})$ where $\psi_{2^r+1}$ is decomposable. In particular, $(Q_1)_2 = W_4$.

We prove Theorem 2 by induction. Since

$$Sq^1 w_{2i} = w_{2i+1} \quad \text{and} \quad Sq^1 w_{2i+1} = 0 ,$$

we know $Sq^1 e_{2i} = e_{2i+1}$ and $Sq^1 e_{2i+1} = 0$ unless $i = 2^j$. Since we have chosen $e_{2^j+1} = Sq^{2^{j-1}} \cdots Sq^2 e_3$, we have $Sq^1 e_{2^j+1} = (e_{2^{j-1}+1})^2 = 0$ for

$j \geqq 2$. For $j = 1$, we have $Sq^1 e_3 = e_4$ because $e_4 = (Q_1)_2$ which is in the image of $Sq^1$ since $2Q_1 = 0$.

Thus

$$\begin{aligned} {}_\beta E_2 &= \operatorname{Ker} Sq^1/\operatorname{Im} Sq^1 \\ &= E(e_3 e_4) \otimes E(e_{2i} e_{2i+1} \mid 2 < i \neq 2^j) \otimes E(e_{2^j+1}, e_{2^j+1} \mid j \geqq 2) . \end{aligned}$$

Since $d_2(x_2)_2 = (Q_2)_2 = e_8$, we must have $x_2 = e_3 e_4$.

In general $d_r(x_r)_2 = (Q_{2^r-1})_2 = e_{2^r+1}$ modulo decomposables. Now consider $H^*(BBSO; Q)$. Since $H^*(BSO; Q) = Q[P_i]$ with the usual diagonal $m^*(P_i) = \sum_{j+h=i} P_j \otimes P_k$, we have $H^*(BBSO; Q) = E(R_i)$ where $\dim R_i \in H^{4j+1}$. Thus ${}_\beta E_\infty = E(S_{4i+1})$ and the only possibility is

$$\begin{aligned} S_{4i+1} &= e_{2i} e_{2i+1} \quad i \neq 2^j , \\ S_{2^i+1} &= e_{2^i+1} \end{aligned}$$

modulo terms decomposable in terms of the $S_{4i+1}$. This leaves $e_3 e_4 \cdots e_{2^r}$ as the only possibility for $x_r$, i.e., $d_r(e_3 e_4 \cdots e_{2^r}) = e_{2^r+1}$ mod decomposables as claimed.

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