ON THE BOUNDARY CORRESPONDENCE OF QUASICONFORMAL MAPPINGS OF DOMAINS BOUNDED BY QUASICIRCLES

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A Jordan curve $\mathscr{L}$ is a quasicircle if there exists a constant $C(1 \leq C < \infty)$ such that the cross ratio (z_1, z_2, z_3, z_4) of any four points $z_1, \cdots, z_4$ in order on $\mathscr{L}$ satisfies $|(z_1, z_2, z_3, z_4)| \leq C$. It is shown that the boundary correspondence f induced by a quasiconformal homeomorphism of two Jordan domains bounded by quasicircles satisfies $|(w_1, w_2, w_3, w_4)| \leq A |(z_1, z_2, z_3, z_4)|^\alpha$, $(A \geq 1, 0 < \alpha \leq 1)$ where $w_k = f(z_k)$ and the points are in order on the boundary. A converse to this result is proved and estimates are computed in each direction.

DEFINITION 1. A Jordan curve $\mathscr{L}$ is called a C-quasicircle $(1 \leq C < \infty)$ if

$$(1) \qquad\qquad |(z_1, z_2, z_3, z_4)| < C$$

for any four points $z_1, \cdots, z_4$ in order on $\mathscr{L}$.

Since cross ratios are invariant under Möbius transformations we will assume without loss of generality from now on that $\mathscr{L}$ contains ∞ whence we may assume $z_4 = \infty$ so that (1) becomes $|(z_1, z_2, z_3)| \leq C$ or more graphically,

$$(1)' \qquad\qquad |z_1 - z_2| \leq C |z_1 - z_3| .$$

This definition and the observation of the importance of these curves to the theory of quasiconformal mappings are due to Ahlfors [1].

DEFINITION 2. (a) We will say that a homeomorphism f of a C-quasicircle $\mathscr{L}$ onto a Jordan curve $\mathscr{L}'$ is (A, α)-upper quasisymmetric, $A \geq 1, 0 < \alpha \leq 1$ $(A, \alpha$ constants) if

$$(2) \qquad\qquad |(w_1, w_2, w_3, w_4)| \leq A |(z_1, z_2, z_3, z_4)|^\alpha$$

for any four points z_1, z_2, z_3, z_4 in order on $\mathscr{L}$ and where $w_k = f(z_k)$, $k = 1, 2, 3, 4$.
  (b) If we replace (2) by

$$(3) \qquad\qquad B |(z_1, z_2, z_3, z_4)|^\beta \leq |w_1, w_2, w_3, w_4)|$$

under the same conditions and for some constants B and $\beta, \beta \geq 1$, $0 < B \leq 1$ then we will call f (B, α)-lower quasisymmetric.

Again in what follows and without loss of generality we will assume that $\infty \in \mathscr{L}, \mathscr{L}'$ and that $f(\infty) = \infty$ in which case (2) and (3) may be replaced by

$$(2)' \qquad\qquad | (w_1, w_2, w_3) | \leqq A | (z_1, z_2, z_3) |^\alpha$$

and

$$(3)' \qquad\qquad B | (z_1, z_2, z_3) |^\beta \leqq | (w_1, w_2, w_3) |$$

respectively and where $(z_1, z_2, z_3) = (z_1, z_2, z_3, \infty)$, etc.

The term quasisymmetric is adapted from the term as applied to real functions of a real variable. We have

DEFINITION 3. f is called ρ-quasisymmetric or simply quasisymmetric if

$$(4) \qquad 1/\rho \leqq (f(x + t) - f(x))/(f(x) - f(x - t)) \leqq \rho$$

for some constant $\rho \geqq 1$ and all x and for all $t \neq 0$.

In Theorem 3.1 we will easily show that for the real case, $(2)'$, $(3)'$ and (4) are equivalent.

REMARKS. The following statements are rather immediate consequences of the above definitions.

1. If f is an (A, α)-upper quasisymmetric mapping of a C-quasicircle $\mathscr{L}$ then $f(\mathscr{L})$ is an AC^α-quasicircle.

2. If f is an (A, α)-upper quasisymmetric map of Jordan curves then f^{-1} is an $(A^{-1/\alpha}, 1/\alpha)$-lower quasisymmetric map of the same curves in reverse order.

3. If f_1 and f_2 are (A_1, α_1) and (A_2, α_2)-upper quasisymmetric mappings respectively then the composition $f_1 \circ f_2$ is $(A_1 A_2^{\alpha_1}, \alpha_1 \alpha_2)$-upper quasisymmetric.

Similar remarks hold for lower quasisymmetric functions. The following lemmas are fundamental to this paper.

The first lemma incorporates the main results of the paper of Beurling and Ahlfors [3] where also originated Definition 3 above.

LEMMA 1. (a) *The boundary correspondence f induced by a K-quasiconformal F of the upper half plane onto itself with $F(\infty) = \infty$ is ρ-quasisymmetric.*

(b) *If f is ρ-quasisymmetric then there exists a K-quasiconformal mapping F of the upper half plane onto itself such that F restricted to the real line is f.*

Let $\mathscr{F}$ denote the class of quasiconformal self homeomorphisms F

of the upper half plane with $F(\infty) = \infty$. Set $\rho(F) = \operatorname{Inf}\{\rho \mid$ the restriction of $F \in \mathscr{F}$ to the real line is ρ-quasisymmetric$\}$ and $\Phi(K) = \operatorname{Sup}\{\rho(F) \mid F \in \mathscr{F}$ and F is K-quasiconformal$\}$. $\Phi(K)$ is given explicitly in [2, p. 65]; we have the convenient estimate

$$(5) \qquad \rho \leqq \Phi(K) \leqq \min\{16^{-1}e^{\pi K}, e^{(K-1)/.2284}\}$$

from [2, p. 65] and [3] for the ρ of (a) of Lemma 1.

Let $K(f) = \inf\{K \mid$ there exists a K quasiconformal extension $F \in \mathscr{F}$ of $f\}$ and $\Psi(\rho) = \sup\{K(f) \mid f$ is ρ-quasisymmetric$\}$. The construction of F in Lemma 1(b) can be carried out so that $K \leqq \min\{\rho^2, 8\rho\}$ (c.f. [3] and [5]). Thus we have the estimate

$$(6) \qquad \Psi(\rho) \leqq \min\{\rho^2, 8\rho\}\ .$$

The exact value of $\Psi(\rho)$ is not known.

LEMMA 2. *Let $\mathscr{D}$ be a Jordan domain with C-quasicircular boundary $\mathscr{L}$ and let $w = g(z)$ be a K_g-quasiconformal mapping of $\mathscr{D}$ onto the upper half plane. Then there exists a K-quasiconformal mapping G of the plane onto itself such that $G|_{\mathscr{D}} = g$ with*

$$(7) \qquad K = K_g\Psi(M),\ M = \Phi(L),\ L = \pi(1 + C^5e^{2\pi})^2$$

where Ψ and Φ are the functions defined above.

Here it is clear that K is a functions only of K_g and C and we will denote this K sometimes by $K(K_g, C)$ and abbreviate this by $K(C)$ when g is conformal ($K_g = 1$).

Lemma 2 is the reflection principle of Ahlfors [1]. We also have the following.

LEMMA 3. *Let $\mathscr{L}$ be a C quasicircle containing ∞ and let z_1, z_2, z_3 be any three points in order on $\mathscr{L}$. Further, let $\mathscr{A}$ denote the arc $\overparen{z_1, z_2}$ and $\mathscr{B}$ the arc $\overparen{z_3, \infty}$ on $\mathscr{L}$. Then the module M of the ring domain $\mathscr{R}$ consisting of the entire plane with boundaries $\mathscr{A}$ and $\mathscr{B}$ satisfies the inequality*

$$(8) \qquad M < \frac{1}{2\pi} \log 24C \left| \frac{z_3 - z_1}{z_2 - z_1} \right|\ .$$

If in addition,

$$(9) \qquad |z_2 - z_1| < \frac{1}{4}C^{-2}|z_3 - z_2|$$

then we have the lower estimate

(10)
$$\frac{1}{2\pi} \log \frac{1}{4C^2} \left| \frac{z_3 - z_1}{z_2 - z_1} \right| \leqq M \ .$$

In particular if $\mathscr{L}$ is the real line and $\mathscr{A} = (-1, 0), \mathscr{B} = (r, \infty)$ then the module $M(r)$ or $\mathscr{R} = \mathscr{R}_r$ satisfies

(11)
$$\frac{1}{2\pi} \log (1 + r) \leqq M(r) \leqq \frac{1}{2\pi} \log 24(1 + r) \ .$$

Proof. The module of a ring domain $\mathscr{R}$ (homeomorph of an open annulus) is $(2\pi)^{-1} \log (r_2/r_1)$ where r_1 and r_2 are the inner and outer radii respectively of an annulus conformally equivalent to $\mathscr{R}$. From Künzi [4, p. 7]

(12)
$$M \leqq \frac{1}{2\pi} \log \Theta \left(\left| \frac{z_3 - z_2}{z_2 - z_1} \right| \right)$$

where $(2\pi)^{-1} \log \Theta(r)$ is the module of the ring domain $\mathscr{R}_r$ with boundaries $(-1, 0)$ and $(r, \infty), r > 0$. From the known inequality ([4, p. 8]),

(13)
$$\Theta(r) < 16r + 8 \ ,$$

we have

$$M \leqq \frac{1}{2\pi} \log \left(16 \left| \frac{z_3 - z_2}{z_2 - z_1} \right| + 8 \right) \leqq \frac{1}{2\pi} \log 24C \left| \frac{z_3 - z_1}{z_2 - z_1} \right|$$

where in the last step, quasicircularity was used twice. We have proved (8).

Now assume (9). In this case, $z \in \mathscr{A}$ implies by quasicircularity that $|z - z_2|, |z - z_1| \leqq C |z_2 - z_1|$ whence

$$\left| z - \frac{1}{2}(z_2 + z_1) \right| \leqq \frac{1}{2}(|z - z_2| + |z - z_1|)$$

$$\leqq C |z_2 - z_1| \ .$$

Let $r_1 = C |z_2 - z_1|$. Now if $z \in \mathscr{B}$ then $|z - z_2| \geqq C^{-1} |z_3 - z_2|$ so that

$$\left| z - \frac{1}{2}(z_2 + z_1) \right| > |z - z_2| - \frac{1}{2} |z_2 - z_1|$$

$$\geqq \frac{1}{C} |z_3 - z_2| - \frac{1}{2C^2} |z_3 - z_2| \geqq \frac{1}{2C} |z_3 - z_2|$$

where in the last step we have used (9). Set $r_2 = 2C^{-1} |z_3 - z_2|$, then (9) assures us that $r_2 > r_1$. We now have an annulus $\mathscr{A}$ of center $2^{-1}(z_2 + z_1)$ and of inner and outer radii respectively r_2 and r_1 contained in the ring domain $\mathscr{R}$. By the comparison theorem for modules we have

(14)
$$M > \frac{1}{2\pi} \log \frac{r_2}{r_1} = \frac{1}{2\pi^2} \log \frac{1}{2C^2} \left| \frac{z_3 - z_2}{z_2 - z_1} \right|$$
$$> \frac{1}{2\pi} \log \frac{1}{2C^2} \left| \frac{z_3 - z_1}{z_2 - z_1} \right|$$

where in the last step we have used the inequality

$$|z_3 - z_2| \geqq |z_3 - z_1| - |z_2 - z_1| \geqq |z_3 - z_1| - C^{-2}|z_3 - z_2|$$
$$\geqq |z_3 - z_1| - |z_3 - z_2| .$$

This proves (10).

This right hand side of (11) results from letting $z_1 = -1, z_2 = 0,$ $z_3 = 1$ and $C = 1$. The right hand side of (11) follows by applying the comparison theorems for modules to the annulus of inner radius $1/2$, outer radius $r + 1/2$ and center $(-1/2, 0)$ contained in $\mathscr{A}$.

We are now able to relate the types of quasisymmetry previously defined.

THEOREM 1. (a) *If the real function f is (A, α)-upper (or (B, β)-lower) quasisymmetric on the real line with $f(\infty) = \infty$ then f is ρ-quasisymmetric and ρ depends only on A and α (or B and β).*

(b) *If f is ρ-quasisymmetric then f is (A, α)-upper and (B, β)-lower quasisymmetric. A, α, B and β and depend only on ρ.*

Proof. For (a) it is easier to suppose first that f is (B, β)-lower quasisymmetric. If we let $z_3 = x + t, z_2 = x, z_1 = x - t$ and

$$Q = (f(x + t) - f(x))/(f(x) - f(x - t))$$

then from (3)', $B2^{-\beta} \leqq 1/(1 + Q)$ whence $Q \leqq 2^\beta/B - 1$ and we may take

(15)
$$\rho = 2^\beta/B - 1$$

in Definition 3 (it is clearly sufficient to demonstrate only half of the inequality (4)). To obtain ρ-quasisymmetry from upper quasisymmetry directly is not easy. However if f is (A, α)-upper quasisymmetric then f^{-1} is $(A^{-1/\alpha}, 1/\alpha)$-lower quasisymmetric which in turn is ρ-quasisymmetric which ρ given in (15) and $B = A^{-1/\alpha}, \beta = 1/\alpha$. By Lemma 1 (b), f^{-1} may be extended to a $\Psi(\rho)$-quasiconformal mapping F^{-1} of the upper half plane onto itself, F is also $\Psi(\rho)$ quasisymmetric ([2] or [6]) and its restriction f by Lemma 1 (a) is ρ'-quasisymmetric with $\rho' = \Phi(\Psi(\rho))$. ρ' clearly depends only on A, α and estimates on ρ' are easily recovered from (5), (6) and (15).

To prove (b) we observe by Lemma 1 (b) that f has a K-quasiconformal extension F to the upper half plane with $K = \Psi(\rho)$. F may be normalized by a Möbius transformation so that $F(0) = 0, F(1) = 1$.

The quasisymmetry (of any type) and the quasiconformality of F are un-affected by this.   Let $r = (z_3 - z_2)/(z_2 - z_1)$, and $r' = (w_3 - w_2)/(w_2 - w_1)$ where $w_k = f(z_k)$, $k = 1, 2, 3$; then $r, r' > 0$.   Further let M_z be the module of the ring domain R_r and M_w the module of $R_{r'} = F(R_r)$ (R_r is defined in the proof of Lemma 3).   By (11) of Lemma 3,

$$\left(\frac{1}{2\pi}\right) \log \left(\frac{(z_3 - z_1)}{(z_2 - z_1)}\right) \leqq M_z$$

and

$$M_w \leqq \left(\frac{1}{2\pi}\right) \log \left(24\frac{(w_3 - w_1)}{(w_2 - w_1)}\right)$$

and by quasiconformality, $M_z \leqq KM_w$.   Combining these three inequalities and simplifying gives us

$$\frac{w_2 - w_1}{w_3 - w_1} \leqq 24\left(\frac{z_2 - z_1}{z_3 - z_1}\right)^{1/K}$$

whence f is (A, α)-upper quasisymmetric with $A = 24$ and $\alpha = 1/K = 1/\Psi(\rho)$.   Lower quasisymmetry is established in a similar fashion.

We now state the main results of this paper.   In the proofs we will assume without loss of generality that $\infty \in \mathscr{L}, \mathscr{L}'$ and that $f(\infty) = \infty$ (hence that $F(\infty) = \infty$).

THEOREM 2.   *Let F be a K-quasiconformal homeomorphism of the plane onto itself then its restriction f to a mapping of a C-quasi-circle $\mathscr{L}$ onto a C'-quasicircle $\mathscr{L}'$ is $(S, 1/K)$-upper quasisymmetric and (T, K)-lower quasisymmetric with*

(16)
$$\begin{aligned} S &= \max \{24C'(4C^2)^{1/K}, (4C'^2 + 1)^{1/K}C\} \\ T &= \min \{(4C^2 + 1)^{-1}C'^{-1}, 4^{-1}C^{-2}(24C)^{-K}\} . \end{aligned}$$

We will denote the right hand sides by $S(C, C', K)$ and $T(C, C', K)$ respectively.

THEOREM 3.   (a)   *Let f be an (A, α)-upper quasisymmetric home-omorphism of a C-quasicircle $\mathscr{L}$ onto a C'-quasicircle $\mathscr{L}'$ then there is an extension of f to a K-quasiconformal mapping F of the plane onto itself ($f = F|_{\mathscr{L}}$) with*

(17) $K = K(C)\Psi(\rho(A_0, \alpha_0))K(C')$

where $K(C)$ is defined in Lemma 3.2 and

$$\begin{aligned} A_0 &= S(C', 1, K(C')) \cdot A^{1/K(C')} \cdot S(1, C, K(C))^{\alpha/K(C')} , \\ \alpha_0 &= \alpha/K(C) \cdot K(C') \end{aligned}$$

and where the function Ψ has been defined previously.

(b) *Let f be a (B, β)-lower quasisymmetric homeomorphism of a C-quasicircle $\mathscr{L}$ onto a C'-quasicircle $\mathscr{L}'$ then there is an extension of f to a K-quasiconformal mapping F of the plane onto itself $(f = F|_{\mathscr{L}})$ with*

(18) $$K = K(C)\Psi(\sigma(B_0, \beta_0))K(C')$$

where

$$B_0 = T(C', 1, K(C'))^{\beta K(C')}B^{K(C')}T(1, C, K(C)) \,,$$
$$\beta_0 = \beta \cdot K(C) \cdot K(C') \,.$$

The function $\rho(A, \alpha)$ is the ρ of Theorem 1 (a) and $\sigma(B, \beta)$ is similarly defined.

Proof of Theorem 2. Let z_1, z_2, z_3 be three points in order on $\mathscr{L}$ and $w_k = f(z_k)$, $k = 1, 2, 3$. Let M_z be the modulus of the ring domain $\mathscr{R}_z$ obtained by removing the arcs z_1, z_2 and z_3, ∞ from $\mathscr{L}$. Let $R_w = F(\mathscr{R}_z)$ and M_w be the module of $\mathscr{R}_w$. First we note that

(19) $$K^{-1}M_z \leqq M_w \leqq KM_z \,.$$

We consider the two cases

1. $|w_2 - w_1| \leqq 1/4C^{-2}|w_3 - w_2|$. Here we use the right hand side of (19) and (10) of Lemma 3 to discover that

$$\frac{1}{2\pi} \log \frac{1}{4C'^2}\left|\frac{w_3 - w_1}{w_2 - w_1}\right| \leqq M_w$$

$$\leqq KM_z \leqq \frac{K}{2\pi} \log 24C\left|\frac{z_3 - z_1}{z_2 - z_1}\right|$$

which implies that

(20) $$\frac{1}{4C'^2(24C)^K}\left|\frac{z_2 - z_1}{z_3 - z_1}\right|^K \leqq \left|\frac{w_2 - w_1}{w_3 - w_1}\right| \,.$$

2. $|w_2 - w_1| \geqq 1/4C^{-2}|w_3 - w_2|$. Then

(21) $$\left|\frac{w_3 - w_1}{w_2 - w_1}\right| \leqq \frac{|w_3 - w_2| + |w_2 - w_1|}{|w_2 - w_1|} \leqq 4C^2 + 1 \,.$$

From quasicircularity $|z_3 - z_1| \geqq (C')^{-1}|z_2 - z_1|$ so that from this and (21),

$$\left|\frac{w_3 - w_1}{w_2 - w_1}\right| \leqq (4C^2 + 1)\frac{C'^K}{C'^K} \leqq (4C^2 + 1)C'^K\left|\frac{z_1 - z_3}{z_2 - z_3}\right|^K$$

whence

$$(22) \qquad \frac{1}{(4C^2 + 1)C'^K} \left| \frac{z_2 - z_1}{z_3 - z_1} \right|^K \leqq \left| \frac{w_2 - w_1}{w_3 - w_1} \right| .$$

Combining (20) and (21) we have the assertion (16) for B and β. The estimates for A and α follow in an entirely analogous manner.

Proof of Theorem 3. Let $\mathscr{D}$ be one of the simply connected domains on the Riemann sphere $\mathscr{S}$ determined by $\mathscr{L}$ and let $\mathscr{D}'$ be the simply connected domain determined by $\mathscr{L}'$ which is chosen so that f is sense preserving with respect to $\mathscr{D}$ and $\mathscr{D}'$. There is a conformal mapping G of $\mathscr{D}$ onto the upper half plane $\mathscr{H}$. G may be extended by Lemma 2 to a $K(C)$-quasiconformal mapping of $\mathscr{S}$ onto itself and mapping $\mathscr{L}$ onto the real line $\mathscr{L}_0$. Let $g = G|_{\mathscr{L}}$. According to Theorem 2 g^{-1} is $(S(1, C, K(C)), K(C)^{-1})$-upper quasisymmetric ($G^{-1}$ is $K(C)$-quasisymmetric and $\mathscr{L}_0$ is a 1-quasicircle). Similarly we may construct a $K(C')$-quasiconformal mapping G' of $\mathscr{S}$ onto itself which is conformal on $\mathscr{D}'$ and whose restriction g' on $\mathscr{L}$ maps $\mathscr{L}$ onto $\mathscr{L}_0$ and g' is $(S(C', 1, K(C')), K(C')^{-1})$-upper quasisymmetric.

According to Remarks 2 and 3 the mapping $h = g' \circ f \circ g^{-1}$ is (A_0, α_0)-upper quasisymmetric with $\alpha_0 = \alpha_0(\alpha, C, C') = \alpha K(C)^{-1}K(C')^{-1}$ and

$$\begin{aligned} A_0 &= A_0(A, \alpha, C, C') \\ &= S(C', 1, K(C')) \cdot A^{1/K(C')} \cdot S(1, C, K(C))^{\alpha/K(C')} . \end{aligned}$$

The mappings constructed above may be chosen so that $h(\infty) = \infty$; in this case h is ρ-quasisymmetric with $\rho = \rho(A_0, \alpha_0)$. Now according to Lemma 1 (b) h has a $\Psi(\rho(A_0, \alpha_0))$ quasiconformal extension $H: \mathscr{S} \to \mathscr{S}$. The desired extension F of f may now be set equal to $G'^{-1} \circ H \circ G$ which is now $K(C)\Psi(\rho(A_0, \alpha))K(C')$-quasiconformal thus finishing the proof of the theorem.

Part (b) of Theorem 3 is proved in a similar manner.

Recently, S. Rickman [6] has obtained results similar to those proved in the last section of this paper. Here, we sacrifice the more general considerations of Rickman to obtain the two sided extension of Theorem 3 and the estimates exhibited in both Theorems 2 and 3.

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