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QUOTIENTS OF THE SPACE OF IRRATIONALS

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It is proved that every metric space which is a continuous image of the irrationals is also a quotient of the irrationals.

In this paper we are concerned with the class $\mathscr{A}$ of all those metric spaces which are continuous images of complete separable metric spaces. The members of $\mathscr{A}$ are generally called "(absolutely) analytic sets" or "A-sets" [9] or "Souslin spaces" [5], and are known to be precisely those metric spaces which are either empty or are continuous images of the space $\boldsymbol{P}$ of irrational numbers[1]. Suppose, then, that $Y \in \mathscr{A}$ and Y is nonempty. There exists a continuous surjection $f: \boldsymbol{P} \to Y$; how "nice" can f be taken to be? In general, f cannot be one-to-one (or Y would have to be absolutely Borel; see [9 p. 487]); nor can f be open or closed (as Y would then be an absolute G_δ; see 3.4 and 3.5 below). However, we shall see that f can always be chosen to be a quotient map. More precisely, we prove the following theorem.

THEOREM 1.1. *Every metrizable space Y which is a continuous image of $\boldsymbol{P}$ is also a quotient of $\boldsymbol{P}$ (under a different map, in general).*

Since the space $\boldsymbol{Q}$ of rational numbers is in $\mathscr{A}$, Theorem 1.1 has the following rather striking consequence:

COROLLARY 1.2. *The space of rationals is a quotient of the space of irrationals.*

The proof of Theorem 1.1 is given in the next section, after which we mention some generalizations, related results and open questions.

2. Proof of Theorem 1.1.

The proof depends on the following characterization of $\boldsymbol{P}$, due to Hausdorff [7].

LEMMA 2.1. *A space X is homeomorphic to $\boldsymbol{P}$ if and only if X is a separable metrizable 0-dimensional absolute G_δ such that no nonempty open subset of X is compact.*

Now let Y be a metrizable space which is a continuous image of $\boldsymbol{P}$, and let us show that Y is a quotient of $\boldsymbol{P}$. Since $\boldsymbol{P}$ is separable,

[1] While the reals are usually denoted by $\boldsymbol{R}$, and the rationals by $\boldsymbol{Q}$ (for quotient), there seems to be no standard symbol for the irrationals. The natural choice would be $\boldsymbol{I}$, but that has been pre-emptied by the unit interval. We therefore propose $\boldsymbol{P}$, which permits the equation $\boldsymbol{P} \cup \boldsymbol{Q} = \boldsymbol{R}$, and which may be thought of as standing for psychotic (=irrational).

Y has a countable base $\{V_n: n \in N\}$. We can choose this base, and the metric d on Y, so that diam $V_n \to 0$, no V_n is empty, and each $y \in Y$ is in V_n for infinitely many n (the V_n need not all be distinct): For if Y is compact we merely choose a sequence of finite open covers whose meshes decrease to 0; if Y is not compact, we imbed it in a compact metric space $\bar{Y}$ and give Y the metric and base it inherits from $\bar{Y}$.

We construct a subspace X of the plane, and a map $f: X \to Y$, as follows. By assumption, there is a continuous surjection $g: \boldsymbol{P} \to Y$. Let $X_0 = \boldsymbol{P} \times \{0\}$. For each $n \in N$ and each integer j, let

$$A_{nj} = g^{-1}(V_n) \cap \left[\frac{j}{n}, \frac{j+1}{n}\right],$$

and let

$$X_{nj} = A_{nj} \times \left\{\frac{1}{n}\right\}.$$

For all $n \in N$, let $J_n = \{j: A_{nj} \neq \emptyset\}$. Finally, let

$$X = X_0 \cup \bigcup \{X_{nj}: n \in N, j \in J_n\}.$$

Note that the sets X_0, X_{nj} are all pairwise disjoint, and that each X_{nj} is open-closed in X.

Let us now define $f: X \to Y$. First define $f_0: X_0 \to Y$ by $f_0(s, 0) = g(s)$. Next, if $n \in N$ and $j \in J_n$, then X_{nj} is homeomorphic to the nonempty open subset A_{nj} of $\boldsymbol{P}$, and hence (from Lemma 2.1) to $\boldsymbol{P}$. But $g^{-1}(V_n)$ is also homeomorphic to $\boldsymbol{P}$, for the same reason. Thus, by composing g with a homeomorphism, we obtain a continuous surjection $f_{nj}: X_{nj} \to V_n$. We now define $f: X \to Y$ by taking

$$f \mid X_0 = f_0 \quad \text{and} \quad f \mid X_{nj} = f_{nj}$$

for all $n \in N$ and $j \in J_n$.

To complete the proof, we shall show that X is homeomorphic to $\boldsymbol{P}$, and that f is a quotient map. Again we use Lemma 2.1. Clearly X is separable metric. It is 0-dimensional by the sum theorem; and each nonempty open subset of X has a closed subset homeomorphic to $\boldsymbol{P}$, and so cannot be compact. We have only to show that X is G_δ in a complete metric space. Now $\boldsymbol{P}$ has a complete metric; hence so has $\boldsymbol{P} \times \{0, 1, 1/2, 1/3, \cdots\}$, and X is obtained from the latter space by removing a closed set from each $\boldsymbol{P} \times \{1/n\}$. Thus X is homeomorphic to $\boldsymbol{P}$.

To show that f is continuous, it suffices to check continuity at each $(s_0, 0) \in X_0$, since continuity at points of X_{nj} is obvious. Suppose that V is the ε-neighborhood of $f(s_0, 0) = g(s_0)$ in Y. Let W be the $\varepsilon/2$-neighborhood of $g(s_0)$ in Y, and pick $n_0 \in N$ so that diam $V_n < \varepsilon/2$

whenever $n \geqq n_0$. Let

$$U = X \cap \left(g^{-1}(W) \times \left[0, \frac{1}{n_0}\right]\right),$$

If $(s, 0) \in U$, then

$$f(s, 0) = g(s) \in W \subset V .$$

If $(s, 1/n) \in U$, then $g(s) \in W$, $g(s) \in V_n$ and $n \geqq n_0$, so that

$$\begin{aligned} d\left(f\left(s, \frac{1}{n}\right), g(s_0)\right) &\leqq d\left(f\left(s, \frac{1}{n}\right), g(s)\right) + d(g(s), g(s_0)) \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon , \end{aligned}$$

and again $f(s, 1/n) \in V$. Thus $f(U) \subset V$, and f is continuous.

To show that f is a quotient map, we prove the following slightly stronger result (which actually implies that f is "bi-quotient" in the sense of [13]): If $y \in Y$, then there is an element $x \in f^{-1}(y)$ such that $f(U)$ is a neighborhood of y in Y whenever U is a neighborhood of x in X.

In fact, we have only to choose $x = (s, 0)$ in $f^{-1}(y) \cap X_0$ (that is, $s \in g^{-1}(y)$). There are arbitrarily large values of n for which $y \in V_n$, and for each such n there is a unique $j_n \in J_n$ such that $s \in A_{nj_n}$; moreover, if n is large enough then $X_{nj_n} \subset U$ so that

$$y \in V_n = f(X_{nj_n}) \subset f(U) .$$

That completes the proof.

3. Some related results and problems.

3.1. By a similar, though more elaborate, argument one can prove Theorem 1.1 if the hypothesis that Y is (necessarily separable) metric is replaced by the slightly weaker hypothesis that Y has a countable base. (In effect, Y need not be assumed regular.)

3.2. It would *not* suffice, in Theorem 1.1, to assume that Y is *first* countable (instead of metrizable). There exists a continuous image of $\boldsymbol{P}$ which is first countable, regular T_1 and Lindelöf (hence paracompact), but which is not a quotient of any separable metric space; see [12, Example 12.1 and Corollary 11.5]. However, we don't know whether a regular T_1 space which is a continuous image of $\boldsymbol{P}$ and which is also a quotient of some separable metric space (such quotients are characterized in [12, Cor. 11.5]) is always a quotient of $\boldsymbol{P}$.

3.3. Theorem 1.1 and its proof can be generalized to nonseparable metric spaces. If $B(m)$ denotes the "Baire space" of order m (i.e., the product of $\aleph_0$ discrete spaces each of cardinality m), then every metrizable space which is a continuous image of $B(m)$ is also a quotient of $B(m)$. When $m = \aleph_0$, this is precisely Theorem 1.1. The generalization uses a characterization of $B(m)$ similar to Lemma 2.1 (see [15, p. 6]).

3.4. A nonempty separable metric space Y is the image of $\boldsymbol{P}$ under an *open* continuous map if and only if Y has a complete metric (or equivalently is an absolute G_δ). "Only if" follows from a theorem of Hausdorff [6] asserting that every metrizable image of a complete metric space under a continuous open map has a complete metric. "If" was proved by Arhangel'skiĭ [2, Corollary 4.7].

3.5. The assertion in 3.4 also holds if "open" is replaced by "*closed*". "Only if" now follows from a theorem of Vaĭnšteĭn [16] asserting that every metrizable image of a complete metric space under a closed continuous map has a complete metric. "If" is a recent result of R. Engelking [4]; he shows, more generally, that every nonempty complete metric space of weight m is the image of $B(m)$ under closed a continuous map.

3.6. It can be shown, by methods similar to those in § 2, that a space Y will be the image of $\boldsymbol{P}$ under a continuous map which is *both* open and closed, if and only if Y has a complete metric, is separable and zero-dimensional, and has the further property that each nonempty open compact subset has an isolated point (or, equivalently, no open subset of Y is homeomorphic to the Cantor set).

3.7. A continuous map $f: X \rightarrow Y$ is called *compact-covering* if each compact subset of Y is the image of some compact subset of X. For a continuous surjection f of a complete metric space X onto a metric space Y, it is known that if f is open or closed it is compact-covering, and if f is compact-covering it is a quotient map (see [12, Lemma 11.2], [3, § 2 Proposition 18], and [1, Theorem 15] or [11, Corollary 1.2]). We have seen that in Theorem 1.1 the quotient map cannot in general be chosen open or closed; can it always be chosen so that it is compact-covering? As we shall see, the answer is "no". In fact, we conjecture that Y (assumed nonempty separable metric) is a compact-covering image of $\boldsymbol{P}$ if and only if Y has a complete metric. "If" of course follows from 3.4 or 3.5 and can also be proved directly. In the other direction, it is not hard to show that Y (assumed nonempty separable metric) is the image of $\boldsymbol{P}$ under a compact-covering map if and only if the space $\mathscr{K}(Y)$ of nonempty compact subsets of Y, equipped with the Hausdorff metric, is analytic. Now Hurewicz has shown that $\mathscr{K}(\boldsymbol{Q})$ is *not* analytic (in [8]; a simpler proof is in [10]). Thus, if Y is a compact-covering image of $\boldsymbol{P}$, it always has the following properties: It is analytic, and contains no closed (or G_δ) subset homeomorphic to the space $\boldsymbol{Q}$ of rational numbers. These properties suggest that Y ought to be an absolute G_δ, but unfortunately they do not suffice to prove it; Gödel and Novikov [14] have shown that it is (relatively) consistent with the usual axioms of set theory to suppose the contrary[2]. Thus our conjecture remains open.

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[2] We are indebted to Professor C. Kuratowski for calling our attention to this result.