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A NOTE ON RECURSIVELY DEFINED ORTHOGONAL POLYNOMIALS

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Let $\{a_i\}_{i=0}^{\infty}$ and $\{b_i\}_{i=0}^{\infty}$ be real sequences and suppose the b_i,s are all positive. Define a sequence of polynomials $\{P_i(x)\}_{i=0}^{\infty}$ as follows: $P_0(x) = 1$, $P_1(x) = (x - a_0)/b_0$, and for $n \geqq 1$

$$(*) \qquad b_n P_{n+1}(x) = (x - a_n)P_n(x) - b_{n-1}P_{n-1}(x) .$$

Favard showed that the polynomials $\{P_i(x)\}$ are orthonormal with respect to a bounded increasing function ψ defined on $(-\infty, +\infty)$. This note generalizes recent constructive results which deal with connections between the two sequences $\{a_i\}$ and $\{b_i\}$ and the spectrum of ψ. (The spectrum of ψ is the set $S(\psi) = \{\lambda : \psi(\lambda + \varepsilon) - \psi(\lambda - \varepsilon) > 0$ for all $\varepsilon > 0\}$.) It is shown that if $b_i \to 0$ then every limit point of the sequence $\{a_i\}$ is in $S(\psi)$.

2. Preliminaries. In order to use theorems from functional analysis, consider the space $\mathscr{L}^2(\psi) = \{f : \int_{-\infty}^{+\infty} f^2 d\psi < \infty\}$. This is a Hilbert space where the inner product is gived by $(f, g) = \int f g \, d\psi$ and where we identify all functions which agree on $S(\psi)$. In [2], (p. 215), Carleman showed that the condition $\sum 1/\sqrt{b_i} = \infty$ implies that when ψ is normalized to be continuous from the left and to have $\psi(-\infty) = 0$, $\psi(+) = 1$, then it is unique. In [6], M. Riesz showed that if ψ is essentially unique then Parseval's relation holds for the orthonormal set $\{P_i\}$ in the space $\mathscr{L}^2(\psi)$. Hence the set $\{P_i\}$ is dense in this space.

We now make the assumption that $\lim b_i = 0$. Combining the Carleman result and the Riesz result we see that ψ is essentially unique and the polynomials $\{P_i\}$ are a dense set in $\mathscr{L}^2(\psi)$. Using this information we define an operator A on a dense subset of $\mathscr{L}^2(\psi)$. The domain of A is the set of all functions f which are in $\mathscr{L}^2(\psi)$ and for which xf is also in $\mathscr{L}^2(\psi)$. We take A to be the self-adjoint operator defined by $(Af)(x) = xf(x)$. By inspection of $(*)$ we see that for $i = 1, 2, 3, \cdots$ we have

$$(**) \qquad A(P_i) = b_{i-1}P_{i-1} + a_iP_i + b_iP_{i+1} .$$

We call A the operator associated with the sequences $\{a_i\}$ and $\{b_i\}$.

3. Theorems. Let $\sigma(A)$ be the spectrum of the operator A, i.e., all points λ where $A - \lambda I$ does not have a bounded inverse. Then we have the following:

LEMMA. $\sigma(A) \subset S(\psi)$.

Proof. Let $\lambda \in \sigma(A)$. Since A is self-adjoint, λ is in the approximate point spectrum of A. Hence there exists a sequence $\{f_n\}$ in the domain of A satisfying $\|f_n\| = 1$, $n = 1, 2, \cdots$, and $\|(A - \lambda)f_n\| \to 0$ as $n \to \infty$. Now by the definition of the norm in $\mathscr{L}^2(\psi)$ this means $\int_{-\infty}^{+\infty} f_n^2 d\psi = 1$, $n = 1, 2, \cdots$, and $\int_{-\infty}^{+\infty} (x - \lambda)^2 f_n^2 d\psi \to 0$ as $n \to \infty$. Now suppose $\lambda \notin S(\psi)$. Then there exists $\varepsilon > 0$ such that

$$\psi(\lambda + \varepsilon) - \psi(\lambda - \varepsilon) = 0 .$$

Thus ψ has no mass in the interval $[\lambda - \varepsilon, \lambda + \varepsilon]$, and we have

$$\int_{-\infty}^{\lambda-\varepsilon} f_n^2 d\psi + \int_{\lambda+\varepsilon}^{+\infty} f_n^2 d\psi = 1 , \qquad n = 1, 2, \cdots ,$$

and

$$\int_{+\infty}^{\lambda-\varepsilon} (x - \lambda)^2 f_n^2 d\psi + \int_{\lambda+\varepsilon}^{+\infty} (x - \lambda)^2 f_n^2 \psi \to 0 \quad \text{as} \quad n \to \infty .$$

But these are contradictory since

$$\int_{-\infty}^{\lambda-\varepsilon} (x - \lambda)^2 f_n^2 d\psi + \int_{\lambda+\varepsilon}^{+\infty} (x - \lambda)^2 f_n^2 d\psi \\ \geqq \varepsilon^2\left[\int_{-\infty}^{\lambda-\varepsilon} f_n^2 d\psi + \int_{\lambda+\varepsilon}^{+\infty} f_n^2 d\psi\right] = \varepsilon^2 .$$

This completes the proof.

We are now ready for our result about $S(\psi)$. It is motivated by the results in [5] where we constructed ψ in the case where $b_i \to 0$ and $\{a_i\}$ has only a finite number of limit points.

THEOREM. *Let the sequence of polynomials $\{P_i\}_0^\infty$ be recursively defined by $(*)$ and assume $b_i > 0$ for each i and $b_i \to 0$. Then each limit point of the sequence $\{a_i\}$ is a point of the spectrum of the associated distribution function ψ.*

Proof. From the above lemma it suffices to show that each limit point of the sequence $\{a_i\}$ is in $\sigma(A)$. Thus let λ be a limit point of $\{a_i\}$ and suppose $\{a_{i(n)}\}$ is a subsequence converging to λ. Next let $f_n(x) = P_{i(n)}(x)$, $n = 1, 2, 3, \cdots$. By the defining relation $(*)$ and by the definition of A, we have

$$\begin{aligned} \|(A - \lambda)f_n\|^2 &= \|(x - \lambda)P_{i(n)}\|^2 \\ &= \int_{-\infty}^{+\infty} (b_{i(n)-1}P_{i(n)-1} + (a_{i(n)} - \lambda)P_{i(n)} + b_{i(n)}P_{i(n)+1})^2 d\psi \\ &= b_{i(n)-1}^2 + (a_{i(n)} - \lambda)^2 + b_{i(n)}^2 . \end{aligned}$$

Now $b_i \longrightarrow 0$ and $a_{i(n)} \longrightarrow \lambda$, so we see $||(A - \lambda)f_n||^2 \longrightarrow 0$ as $n \longrightarrow \infty$. Moreover $||f_n|| = ||P_{i(n)}|| = 1$, so $\lambda \in \sigma(A)$ and the proof is complete.

REMARK. If we choose the a_i's to be dense in the real line, for example any enumeration of the rationals, then for every set of b_i's satisfying $b_i \longrightarrow 0$ we have $S(\psi) = (-\infty, +\infty)$.

CONJECTURE. The converse of the above theorem does not hold since in [5] our construction exhibited points of $S(\psi)$ which were not limit points of $\{a_i\}$. However each limit point of $S(\psi)$ was a limit point of $\{a_i\}$. So it seems reasonable to conjecture that when $b_i \longrightarrow 0$, λ is a limit point of $S(\psi)$ if and only if λ is a limit point of $\{a_i\}$.

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