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A UNIQUENESS THEOREM FOR WEAK SOLUTIONS OF SYMMETRIC QUASILINEAR HYPERBOLIC SYSTEMS

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The essentially bounded measurable (vector) function $u(x, t) = (u_1(x, t), \cdots, u_r(x, t))$ is called a weak solution of the initial-value problem for the system

$$\frac{\partial u}{\partial t} + \frac{\partial \mathscr{A}(x, t, u)}{\partial x} = 0$$

in the upper half-plane $t \geqq 0$ if it satisfies the usual integral identity (defining "weak") together with the condition that, given a compact set D in $t \geqq 0$, there exists a function $K(t) \in L^1_{\text{loc}}([0, \infty))$ such that

$$\frac{u_i(x_1, t) - u_i(x_2, t)}{x_1 - x_2} \leqq K(t)$$

holds a.e. for $x_1, x_2 \in D$ and $0 < t < \infty$. It is shown that, if the matrix $\partial \mathscr{A}/\partial u$ is symmetric and positive definite (a convexity condition), then weak solutions are uniquely determined by their initial conditions.

In [1] O. A. Oleǐnik established a uniqueness theorem for a rather general class of weak solutions of a quasilinear equation of the form

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(x, t, u)}{\partial x} + \psi(x, t, u) = 0$$

where the function $\varphi(x, t, u)$ was subject to a convexity condition in u, namely, $\varphi_{uu} \geqq 0$. The purpose of this note is to generalize Oleǐniks uniqueness result (in the case $\psi \equiv 0$) to certain quasilinear systems which are subject to a symmetry condition (assumption III below) as well as a convexity condition (assumption IV below). In the case of one equation our uniqueness result is slightly less general than Oleǐniks in that she does not require the function $K(t)$ occurring in (2) to be locally integrable on $[0, \infty)$. The method is the standard variation of Holmgren's method which is employed by Oleǐnik and others, except that we work with mean square rather than sup-norm estimates. Oleǐnik [2] has also established a uniqueness result for a special system of two equations which, however, is not symmetric. Rozhdestvenskii [3] has established a uniqueness theorem for piecewise smooth solutions of certain quasilinear systems but his methods are entirely different from those employed here.

2. In $D = \{(x,t): -\infty < x < \infty, 0 \leqq t < \infty\}$ we consider the quasi-

linear system of r equations

$$\frac{\partial u}{\partial t} + \frac{\partial \mathscr{A}(x, t, u)}{\partial x} = 0 \tag{1}$$

for the (vector) function $u(x, t) = (u_1(x, t), \cdots, u_r(x, t))$ where

$$\mathscr{A}(x, t, u) = (a_1(x, t, u), \cdots, a_r(x, t, u)) .$$

The following assumptions will be made:

I. The functions $a_i(x, t, u)$ possess derivatives $\partial a_i/\partial u_j$, $\partial^2 a_i/\partial x \partial u_j$ and $\partial^2 a_i/\partial u_j \partial u_k$ which are bounded subsets of (x, t, u)-space.

II. Let

$$\frac{\partial a_i(x, t, u)}{\partial u_j} = a_{ij}(x, t, u) .$$

Then, if u is bounded, i.e., $\sum u_i^2 \leqq M^2$, there exists a constant c, depending only on M, such that

$$-c \sum_{i=1}^{r} \xi_i^2 \leqq \sum_{i,j=1}^{r} a_{ij}(x, t, u)\xi_i \xi_j \leqq c \sum_{i=1}^{r} \xi_i^2$$

for all vectors $\xi = (\xi_1, \cdots, \xi_r)$.

III (Symmetry). For all x, t, and u,

$$a_{ij}(x, t, u) = a_{ji}(x, t, u) \qquad (i, j = 1, \cdots, r)$$

IV (Convexity). For all x, t, and u, and each $k = 1, \cdots, r$, we have

$$\sum_{i,j=1}^{r} \frac{\partial a_{ij}(x, t, u)}{\partial u_k} \xi_i \xi_i \geqq 0$$

for all vectors $\xi = (\xi_1, \cdots, \xi_r)$.

DEFINITION. Let $\psi(x)$ be an essentially bounded measurable function defined on $-\infty < x < \infty$. An essentially bounded measurable function $u(x, t)$ is called a weak solution of (1) in D with initial conditions $\psi(x)$ if,

(a) for every test function $\varphi(x, t)$ which is continuously differentiable with compact support in the (x, t)-plane we have

$$\int_D \left[\left\langle u, \frac{\partial \varphi}{\partial t}\right\rangle + \left\langle A(t, x, u), \frac{\partial \varphi}{\partial x}\right\rangle\right] dxdt + \int_{-\infty}^{\infty} \langle \varphi(x, 0), \psi(x)\rangle dx = 0 \tag{2}$$

where $\langle , \rangle$ is the inner product in Euclidean r-space;

(b) given any compact subset of D there is a corresponding function $K(t) \in L^1_{\text{loc}}([0, \infty))$ such that

$$\frac{u_i(x_1, t) - u_i(x_2, t)}{x_1 - x_2} \leqq K(t) \tag{3}$$

$(i = 1, \cdots, r)$ holds a.e. for x_1 and x_2 in the compact subset, and $0 < t < \infty$.

THEOREM. *Weak solutions of* (1) *are uniquely determined by their initial conditions.*

Proof. Let $u_1(x, t)$ and $u_2(x, t)$ be two weak solutions of (1) with the same initial conditions $\psi(x)$. We will show that, if $F(x, t) = (F_1(x, t), \cdots, F_r(x, t))$ is any smooth (vector) function with compact support contained in $t > 0$, then

$$\int_D \langle u_1 - u_2, F\rangle dxdt = 0 ,$$

thus proving that $u_1 = u_2$ a.e. in D.

Let ω^n be the usual Gaussian averaging kernel with support contained in the sphere $x^2 + t^2 \leqq 1/n^2$. Given a function $\varphi(x, t) \in L^2_{\mathrm{loc}}(D)$ we define the averaged function $\varphi^n(x, t)$ by convolution; $\varphi^n = \varphi * \omega^n$. By a familiar argument we see that $u^n_{i,k} \rightarrow u_{ik}$ $(i = 1, 2$ and $k = 1, \cdots, r)$ in mean square on compact subsets of D. From (3) it follows (see [1]) that

$$\frac{\partial u^n_{i,k}}{\partial x} \leqq K(t) \quad (i = 1, 2 \text{ and } k = 1, \cdots, r) \tag{4}$$

on compact subsets of D.

We now define the functions

$$\alpha_{ij}(x, t) = \int_0^1 a_{ij}(x, t, \tau u_1 + (1 - \tau)u_2)d\tau$$
$$\alpha^n_{ij}(x, t) = \int_0^1 a_{ij}(x, t, \tau u^n_1 + (1 - \tau)u^n_2)d\tau$$

$(i, j = 1, \cdots, r$ and $n = 1, 2, \cdots)$ and the associated matrices $A(x, t) = (\alpha_{ij}(x, t))$ and $A^n(x, t) = (\alpha^n_{ij}(x, t))$.

It is immediate that

$$\mathscr{A}(x, t, u_1) - \mathscr{A}(x, t, u_2) = A(x, t)(u_1 - u_2) .$$

Also

$$| a^n_{ij}(x, t) - \alpha_{ij}(x, t) | \leqq \text{const.} [| u^n_1 - u_1 | + | u^n_2 - u_2 |]$$

on compact subsets of D, from which it follows that $a^n_{ij} \rightarrow \alpha_{ij}$ in mean square on compact subsets of D. From II we see that

$$(5) \qquad -c\langle \xi, \xi\rangle \leqq \langle A^n(x, t)\xi, \xi\rangle \leqq c\langle \xi, \xi\rangle$$

for some constant $c > 0$ and all real vectors ξ. Finally we note that

$$\frac{\partial \alpha_{ij}^n}{\partial x} = \int_0^1 \Big\{ \frac{\partial a_{ij}}{\partial x}(x, t, \tau u_1^n + (1-\tau)u_2^n) + \sum_{k=1}^{r} \frac{\partial a_{ij}}{\partial u_k}(x, t, \tau u_1^n + (1-\tau)u_2^n)\Big[\frac{\tau \partial u_{1,k}^n}{\partial x} + (1-\tau)\frac{\partial u_{2,k}^n}{\partial x}\Big]\Big\} dt\ .$$

Using I, IV and (4) it follows that

$$\Big\langle \frac{\partial A^n}{\partial x}\xi, \xi \Big\rangle \leqq K_1(t)\langle \xi, \xi\rangle$$

on compact subsets of D for every vector ξ, where $K_1(t) \in L^1_{\text{loc}}([0, \infty))$.

We now construct for each $n = 1, 2, \cdots$ the vector function $\varphi^n(x, t)$ satisfying the linear system

$$\frac{\partial \varphi^n}{\partial t} + A^n(x, t)\frac{\partial \varphi^n}{\partial x} = F(x, t)$$

and vanishing on $t = T$, where the support of F is assumed to be below $t = T$. This is achieved by solving the system

$$\frac{\partial \tilde{\varphi}^n}{\partial t} - A^n(x, T - t)\frac{\partial \tilde{\varphi}^n}{\partial x} = F(x, T - t)$$

for the vector function $\tilde{\varphi}^n(x, t)$ in D, with the initial conditions $\tilde{\varphi}^n(x, 0) = 0$, and then putting $\varphi^n(x, t) = \tilde{\varphi}(x, T - t)$. The classical existence theory guarantees that $\varphi^n(x, t)$ exists, is smooth, and, by (5), has support contained in a compact set which is independent of n, and so is a legitimate test function.

Using (2) we obtain

$$\int_D \Big\langle u_1 - u_2, \frac{\partial \varphi^n}{\partial t}\Big\rangle dxdt = -\int \Big\langle \mathscr{A}(x, t, u_1) - \mathscr{A}(x, t, u_2), \frac{\partial \varphi^n}{\partial x}\Big\rangle dxdt$$
$$= -\int_D \Big\langle A(x, t)(u_1 - u_2), \frac{\partial \varphi^n}{\partial x}\Big\rangle dxdt\ .$$

Thus

$$(6) \qquad \int_D \langle u_1 - u_2, F\rangle dxdt = \int_D \Big\langle u_1 - u_2, (A^n - A)\frac{\partial \varphi^n}{\partial x}\Big\rangle dxdt\ .$$

Using the facts that (i) the supports of the φ^n lie in a fixed compact subset of D, (ii) the u_i are essentially bounded and (iii) the coefficients of A^n converge in the mean square on compact subsets of D to the coefficients of A, we see immediately that the right hand side of (6) approaches zero as $n \to \infty$, as long as the mean square norms of the

$\partial\varphi_i^r/\partial x$ (on compact subsets of D) are uniformly bounded. The proof will be completed by establishing this fact.

Let $\partial\tilde{\varphi}^n/\partial x = v^n$, $A^n(x, T-t) = \tilde{A}^n(x,t)$ and $F(x, T-t) = \tilde{F}(x,t)$. Then v^n satisfies the equation

$$\frac{\partial v^n}{\partial t} - \tilde{A}^n\frac{\partial v^n}{\partial x} - \frac{\partial \tilde{A}^n}{\partial x}v^n = \frac{\partial \tilde{F}}{\partial x}$$

in $0 \leqq t \leqq T$, and the initial conditions $v^n(x, 0) = 0$. We may suppose that the supports of the $v^n(n = 1, 2, \cdots)$ in $0 \leqq t \leqq T$ are all strictly contained in some interval $a < x < b$. Then

$$\frac{\partial}{\partial t}\langle v^n, v^n\rangle - \frac{\partial}{\partial x}\langle\tilde{A}^n v^n, v^n\rangle = 2\left\langle\frac{\partial\tilde{F}}{\partial x}, v^n\right\rangle + \left\langle\frac{\partial\tilde{A}^n}{\partial x}v^n, v^n\right\rangle .$$

Using Green's formula

$$\int_a^b\langle v^n(x, t), v^n(x, t)\rangle dx \leqq \int_0^t\int_a^b 2\left\langle\frac{\partial\tilde{F}}{\partial x}, v^n\right\rangle dxdt + \int_0^t\int_a^b\left\langle\frac{\partial\tilde{A}^n}{\partial x}v^n, v^n\right\rangle dxdt$$

$$\leqq \int_0^t\int_a^b\left\langle\frac{\partial\tilde{F}}{\partial x}\frac{\partial\tilde{F}}{\partial x}\right\rangle dxdt + \int_0^t(1 + K_1(s))\left[\int_a^b\langle v(x, s), v(x, s)\rangle dx\right]ds$$

from which it follows by Gronwall's Lemma that

$$\int_0^T\int_a^b\langle v^n, v^n\rangle dxdt \leqq \text{constant} ,$$

the constant depending on the L^2-norm of $\partial\tilde{F}/\partial x$ and $\int_0^T K_1(t)dt$, but not on n. This completes the proof.

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